# On the Metric Dimension of Cartesian Products of Graphs *

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**Abstract.** A set of vertices $S$ *resolves* a graph $G$ if every vertex is uniquely determined by its vector of distances to the vertices in $S$. The *metric dimension* of $G$ is the minimum cardinality of a resolving set of $G$. Using bounds on the order of the so called doubly resolving sets, we establish bounds on $G\Box H$ for many examples of $G$ and $H$. One of our main results is a family of graphs $G$ with bounded metric dimension for which the metric dimension of $G\Box G$ is unbounded.

**Keywords.** graph, distance, resolving set, metric dimension, metric basis, cartesian product, Hamming graph, Mastermind, coin weighing

## 1. Introduction

A set of vertices $S$ *resolves* a graph $G$ if every vertex of $G$ is uniquely determined by its vector of distances to the vertices in $S$. This work undertakes a general study of resolving sets in cartesian products of graphs.

All the graphs considered are finite, undirected, simple, and connected. The vertex set and edge set of a graph $G$ are denoted by $V(G)$ and $E(G)$. The distance between vertices $v, w \in V(G)$ is denoted by $d_G(v, w)$, or $d(v, w)$ if the graph $G$ is clear from the context. A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$ if $d(v, x) \neq d(w, x)$. A set of vertices $S \subseteq V(G)$ *resolves* $G$, and $S$ is a *resolving set* of $G$, if every pair of distinct vertices of $G$ are resolved by some vertex in $S$. A resolving set $S$ of $G$

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with the minimum cardinality is a *metric basis* of $G$, and $|S|$ is the *metric dimension* of $G$, denoted by $\beta(G)$.

The *cartesian product* of graphs $G$ and $H$, denoted by $G \square H$, is the graph with vertex set $V(G) \times V(H) := \{(a,v) : a \in V(G), v \in V(H)\}$, where $(a,v)$ is adjacent to $(b,w)$ whenever $a = b$ and $\{v,w\} \in E(H)$, or $v = w$ and $\{a,b\} \in E(G)$. Where there is no confusion the vertex $(a,v)$ of $G \square H$ will be written $av$. Observe that if $G$ and $H$ are connected, then $G \square H$ is connected. In particular, for all vertices $av, bw$ of $G \square H$ we have $d(av,bw) = d_G(a,b) + d_H(v,w)$. Assuming isomorphic graphs are equal, the cartesian product is associative, and $G_1 \square G_2 \square \cdots \square G_d$ is well-defined.

Resolving sets in general graphs were first defined by Harary and Melter [7] and Slater [12], although resolving sets in hypercubes were studied earlier under the guise of a coin weighing problem. Resolving sets have been widely investigated and arise in many diverse areas including network discovery and verification, robot navigation, connected joins in graphs, and strategies for the Mastermind game.

Part of our motivation for studying the metric dimension of cartesian products is that in two of the above-mentioned applications, namely Mastermind and coin weighing, the graphs that arise are in fact cartesian products. These connections are explained in Sections 2 and 6 respectively.

The main contributions of this work are based on the notion of doubly resolving sets, which are introduced in Section 1. We prove that the minimum order of a doubly resolving set in a graph $G$ is tied in a strong sense to $\beta(G \square G)$. Thus doubly resolving sets are essential in the study of metric dimension of cartesian products. We then give a number of examples of bounds on the metric dimension of cartesian products through doubly resolving sets. In particular, Sections 5, 6, 7, 8, and 9 respectively study complete graphs, Hamming graphs, paths and grids, cycles, and trees. One of our main results here is a family of (highly connected) graphs with bounded metric dimension for which the metric dimension of the cartesian product is unbounded. We omitt the proofs because of limited space. See [1] for an extended version of the work.

## 2. Coin Weighing and Hypercubes

The *hypercube* $Q_n$ is the graph whose vertices are the $n$-dimensional binary vectors, where two vertices are adjacent if they differ in exactly one coordinate. It is well known that $Q_n = \underbrace{K_2 \square K_2 \square \cdots \square K_2}_{n}$. It is easily seen that $\beta(Q_n) \leq n$ (see Section 7). The first case when this bound is not tight is $n = 5$. We have determined $\beta(Q_n)$ for small values of $n$ by computer search.

| $n$ | 2 | 3 | 4 | 5 | 6 | 7 | 8 | 9 | 10 | 13 | 15 |
|---|---|---|---|---|---|---|---|---|---|---|---|
| $\beta(Q_n)$ | 2 | 3 | 4 | 4 | 5 | 6 | 6 | $\leq 7$ | $\leq 7$ | $\leq 9$ | $\leq 10$ |

The asymptotic value of $\beta(Q_n)$ turns out to be related to the following coin weighing problem first posed by Söderberg and Shapiro [13]. Given $n$ coins, each with one of two distinct weights, determine the weight of each coin with the minimum number of weighings. We are interested in the static variant of this problem, where the choice of sets of coins to be weighed is determined in advance. Weighing a set $S$ of coins determines how many light (and heavy) coins are in $S$, and no further information. It follows that the minimum number of weighings differs from $\beta(Q_n)$ by at most one. A lower bound on the number of weighings by Erdös and Rényi [4] and an upper bound by Lindstrom [11] imply that $\lim_{n\to\infty} \beta(Q_n) \cdot \dfrac{\log n}{n} = 2$, where, as always in this paper, logarithms are binary. Note that Lindstrom's proof is constructive. He gives a scheme of $2^k - 1$ weighings that suffice for $k \cdot 2^{k-1}$ coins.

## 3. Projections

Let $S$ be a set of vertices in the cartesian product $G \square H$ of graphs $G$ and $H$. The *projection* of $S$ onto $G$ is the set of vertices $a \in V(G)$ for which there exists a vertex $av \in S$. Similarly, the *projection* of $S$ onto $H$ is the set of vertices $v \in V(H)$ for which there exists a vertex $av \in S$. A *column* of $G \square H$ is the set of vertices $\{av : v \in V(H)\}$ for some vertex $a \in V(G)$, and a *row* of $G \square H$ is the set of vertices $\{av : a \in V(G)\}$ for some vertex $v \in V(H)$. Observe that each row induces a copy of $G$, and each column induces a copy of $H$. This terminology is consistent with a representation of $G \square H$ by the points of the $|V(G)| \times |V(H)|$ grid.

**Lemma 1.** *Let $S \subseteq V(G \square H)$ for graphs $G$ and $H$. Then every pair of vertices in a fixed row of $G \square H$ are resolved by $S$ if and only if the projection of $S$ onto $G$ resolves $G$. Similarly, every pair of vertices in a fixed column of $G \square H$ are resolved by $S$ if and only if the projection of $S$ onto $H$ resolves $H$.*

**Corollary 1.** *For all graphs $G$ and $H$, and for every resolving set $S$ of $G \square H$, the projection of $S$ onto $G$ resolves $G$, and the projection of $S$ onto $H$ resolves $H$. In particular, $\beta(G \square H) \geq \max\{\beta(G), \beta(H)\}$.*

## 4. Doubly Resolving Sets

Many of the results that follow are based on the following definitions. Let $G \neq K_1$ be a graph. Two vertices $v, w \in V(G)$ are *doubly resolved* by $x, y \in V(G)$ if $d(v,x) - d(w,x) \neq d(v,y) - d(w,y)$.

A set of vertices $S \subseteq V(G)$ *doubly resolves* $G$, and $S$ is a *doubly resolving set*, if every pair of distinct vertices $v, w \in V(G)$ are doubly resolved by two vertices in $S$. Every graph with at least two vertices has a doubly resolving set. Let $\psi(G)$ denote the minimum cardinality of a doubly resolving set of a graph $G \neq K_1$. Note that if $x, y$ doubly resolves $v, w$ then $d(v,x) - d(w,x) \neq 0$ or $d(v,y) - d(w,y) \neq 0$, and at least one of $x$ and $y$ (singly) resolves $v, w$. Thus a doubly resolving set is also a resolving set, and $\beta(G) \leq \psi(G)$.

**Lemma 2.** *For every graph $G$ with $n \geq 3$ vertices we have $\psi(G) \leq n - 1$.*

Our interest in doubly resolving sets is based on the following upper bound.

**Theorem 1.** *For all graphs $G$ and $H \neq K_1$, $\beta(G \square H) \leq \beta(G) + \psi(H) - 1$.*

The relationship between resolving sets of cartesian products and doubly resolving sets is strengthened by the following lower bound.

**Lemma 3.** *Suppose that $S$ resolves $G \square G$ for some graph $G$. Let $A$ and $B$ be the two projections of $S$ onto $G$. Then $A \cup B$ doubly resolves $G$. In particular, $\beta(G \square G) \geq \frac{1}{2}\psi(G)$.*

Observe that Theorem 1 and Lemma 3 prove that $\beta(G \square G)$ is always within a constant factor of $\psi(G)$. In particular, $\frac{1}{2}\psi(G) \leq \beta(G \square G) \leq \psi(G) + \beta(G) - 1 \leq 2\psi(G) - 1$. Thus doubly resolving sets are essential in the study of the metric dimension of cartesian products.

## 5. Complete Graphs

Let $K_n$ denote the complete graph on $n \geq 1$ vertices. It is well known that for every graph $G$ with $n$ vertices, $\beta(G) = n - 1$ if and only if $G = K_n$.

**Lemma 4.** *For all $n \geq 2$ we have $\psi(K_n) = \max\{n-1, 2\}$.*

**Lemma 5.** *For all $n \geq 1$ we have $\beta(K_n \square G) \leq \beta(G) + \max\{n-2, 1\}$.*

**Lemma 6.** *For every graph $G$ and $n \geq 1$, $\beta(K_n \square G) \leq \max\{n-1, 2 \cdot \beta(G)\}$.*

**Theorem 2.** *For every graph $G$ and for all $n \geq 2 \cdot \beta(G) + 1$ we have $\beta(K_n \square G) = n - 1$.*

## 6. Mastermind and Hamming Graphs

*Mastermind* is a game for two players, the *code setter* and the *code breaker*. The code setter chooses a secret vector $s = [s_1, s_2, \ldots, s_n] \in \{1, 2, \ldots, k\}^n$. The task of the code breaker is to infer the secret vector by a series of questions, each a vector $t = [t_1, t_2, \ldots, t_n] \in \{1, 2, \ldots, k\}^n$. The code setter answers with two integers, first being the number of positions in which the secret vector and the question agree, denoted by $a(s,t) = |\{i : s_i = t_i, 1 \leq i \leq n\}|$. The second integer $b(s,t)$ is the maximum of $a(\tilde{s}, t)$, where $\tilde{s}$ ranges over all permutations of $s$.

In the commercial version of the game, $n = 4$ and $k = 6$. The secret vector and each question is represented by four pegs each coloured with one of six colours. Each answer is represented by $a(s,t)$ black pegs, and $b(s,t) - a(s,t)$ white pegs. Knuth [10] showed that four questions suffice to determine $s$ in this case. Here the code breaker may determine each question in response to the previous answers. *Static mastermind* is the variation in which all the questions must be supplied at once. Let $g(n,k)$ denote the maximum, taken over all vectors $s$, of the minimum number of questions required to determine $s$ in this static setting.

The *Hamming graph* $H_{n,k}$ is the cartesian product of cliques $H_{n,k} = \underbrace{K_k \square K_k \square \cdots \square K_k}_{n}$. Note that the hypercube $Q_n = H_{n,2}$. The vertices of $H_{n,k}$ can be thought of as vectors in $\{1, 2, \ldots, k\}^n$, with two vertices being adjacent if they differ in precisely one coordinate. Thus the distance $d_H(v,w)$ between two vertices $v$ and $w$ is the number of coordinates in which their vectors differ. That is, $d_H(v,w) = n - a(v,w)$.

Suppose for the time being that we remove the second integer $b(s,t)$ from the answers given by the code setter in the static mastermind game. Let $f(n,k)$ denote the maximum, taken over all vectors $s$, of the minimum number of questions required to determine $s$ without $b(s,t)$ in the answers. For the code breaker to correctly infer the secret vector $s$ from a set of questions $T$, $s$ must be uniquely determined by the values $\{a(s,t) : t \in T\}$. Equivalently, for any two vertices $v$ and $w$ of $H_{n,k}$, there is a $t \in T$ for which $a(v,t) \neq a(w,t)$; that is, the distances $d_H(v,t) \neq d_H(w,t)$. Hence the secret vector can be inferred if and only if $T$ resolves $H_{n,k}$. Thus $g(n,k) \leq f(n,k) = \beta(H_{n,k})$. Chvátal [3] proved the upper bound $\beta(H_{n,k}) = f(n,k) \leq (2+\epsilon)n \dfrac{1 + 2\log k}{\log n - \log k}$ for large $n > n(\epsilon)$ and small $k < n^{1-\epsilon}$. For $k \in \{3, 4\}$, improvements to the constant in the above upper bound are stated without proof by Kabatianski et al. [8]. They also state that a "straightforward generalisation" of the lower bound on $\beta(Q_n)$ by Erdös and Rényi [4] gives for large $n$, $\beta(H_{n,k}) \geq g(n,k) \geq (2+o(1))\dfrac{n \log k}{\log n}$.

Here we study $\beta(H_{n,k})$ for large values of $k$ rather than for large values of $n$. A similar approach is take by Goddard [5, 6] for static Mastermind, who proved that $g(2,k) = \left\lceil \frac{2}{3}k \right\rceil$ and $g(3,k) = k-1$. Our contribution is to determine the exact value of $\beta(H_{2,k})$. We show that for all $k \geq 1$, $\beta(H_{2,k}) = \left\lfloor \frac{2}{3}(2k-1) \right\rfloor$. This equation is a special case (with $m = n = k$) of the following more general result.

**Theorem 3.** *For all $n \geq m \geq 1$ we have*

$$\beta(K_n \square K_m) = \begin{cases} \left\lfloor \frac{2}{3}(n+m-1) \right\rfloor & \text{, if } m \leq n \leq 2m-1 \\ n-1 & \text{, if } n \geq 2m-1. \end{cases}$$

## 7. Paths and Grids

Let $P_n$ denote the path on $n \geq 1$ vertices. Khuller et al. [9] and Chartrand et al. [2] proved that a graph $G$ with $n$ vertices has dimension 1 if and only if $G = P_n$. Thus, by Theorem 2, for all $n \geq 3$, $\beta(K_n \square P_m) = n-1$. Minimum doubly resolving sets in paths are easily characterised.

**Lemma 7.** *For all $n \geq 2$ we have $\psi(P_n) = 2$. Moreover, the two endpoints of $P_n$ are in every doubly resolving set of $P_n$.*

**Lemma 8.** *If $\beta(G \square H) = 2$, then $G$ or $H$ is a path.*

**Lemma 9.** *Every graph $G$ satisfies $\beta(G) \leq \beta(G \square P_n) \leq \beta(G) + 1$.*

An $n$-dimensional *grid* is a cartesian product of paths $P_{m_1} \square \cdots \square P_{m_n}$. Lemma 9 implies that $\beta(P_{m_1} \square \cdots \square P_{m_n}) \leq n$ for all $m_1 \geq 1, \ldots, m_n \geq 1$, as proved by Khuller et al. [9], who also claimed that $\beta(P_{m_1} \square \cdots \square P_{m_n}) = n$. They wrote "we leave it for the reader to see why $n$ is a lower bound". This claim is false if every $m_i = 2$ and $n$ is large since, as discussed in Section 2, $\beta(P_2 \square \cdots \square P_2)$ tends to $2n/\log n$.

## 8. Cycles

Let $C_n$ denote the cycle on $n \geq 3$ vertices. Two vertices $v$ and $w$ of $C_n$ are *antipodal* if $d(v,w) = \frac{n}{2}$. Note that no two vertices are antipodal in an odd cycle. It is well known that $\beta(C_n) = 2$ for all $n \geq 3$ (see [9]). Moreover, two vertices resolve $C_n$ if and only if they are not antipodal.

**Lemma 10.** *For all $n \geq 3$, $\psi(C_n)$ is 2 if $n$ is odd, and 3 if $n$ is even.*

**Theorem 4.** *For every graph $G$ and for all $n \geq 3$, $\beta(G \square C_n) = 2$ if and only if $G$ is a path and $n$ is odd.*

**Theorem 5.** *For all $m \geq 2$ and $n \geq 3$ we have that $\beta(P_m \square C_n)$ is 2 if $n$ is odd, and 3 if $n$ is even.*

**Theorem 6.** *For all $m, n \geq 3$ we have that $\beta(C_m \square C_n)$ is 3 if $m$ or $n$ is odd, and 4 otherwise.*

**Theorem 7.** *For all $n \geq 1$ and $m \geq 3$ we have that $\beta(K_n \square C_m)$ is: (i) 2, if $n = 1$, or $n = 2$ and $m$ is odd; (ii) 3, if $n = 2$ and $m$ is even, or $n = 3$, or if $n = 4$ and $m$ is even; (iii) 4, if $n = 4$ and $m$ is odd; (iv) $n - 1$, if $n \geq 5$.*

## 9. Trees

Let $v$ be a vertex of a tree $T$. Let $\ell_v$ be the number of components of $T \setminus v$ that are (possibly edgeless) paths. Khuller et al. [9] and Chartrand et al. [2] proved that for every tree $T$ that is not a path, $\beta(T) = \sum_{v \in V(T)} \max\{\ell_v - 1, 0\}$. A *leaf* of a graph is a vertex of degree one. The following result for doubly resolving sets in trees is a generalisation of for paths.

**Lemma 11.** *The set of leaves $L$ is the unique minimum doubly resolving set for a tree $T$, and $\psi(T) = |L|$.*

**Lemma 12.** *Every graph $G$ with $k \geq 2$ leaves satisfies $\beta(G \square G) \geq k$.*

The following result implies that $\psi$ is not bounded by any function of metric dimension.

**Theorem 8.** *For every integer $n \geq 4$ there is a tree $B_n$ with $\beta(B_n) = 2$ and $n = \psi(B_n) \leq \beta(B_n \square B_n) \leq n + 1$.*

**Theorem 9.** *For all $k \geq 1$ and $n \geq 2$ there is a $k$-connected graph $G_{n,k}$ for which $\beta(G_{n,k}) \leq 2k$ and $\beta(G_{n,k} \square G_{n,k}) \geq n$.*

We conclude that for all $k \geq 1$, there is no function $f$ such that $\beta(G \square H) \leq f(\beta(G), \beta(H))$ for all $k$-connected graphs $G$ and $H$.

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