EXTREMAL GRAPH THEORY FOR METRIC DIMENSION AND DIAMETER

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Abstract. A set of vertices S *resolves* a connected graph G if every vertex is uniquely determined by its vector of distances to the vertices in S. The *metric dimension* of G is the minimum cardinality of a resolving set of G. Let $\mathcal{G}_{\beta,D}$ be the set of graphs with metric dimension β and diameter D. It is well-known that the minimum order of a graph in $\mathcal{G}_{\beta,D}$ is exactly $\beta+D$. The first contribution of this paper is to characterise the graphs in $\mathcal{G}_{\beta,D}$ with order $\beta+D$ for all values of β and D. Such a characterisation was previously only known for $D\leq 2$ or $\beta\leq 1$. The second contribution is to determine the maximum order of a graph in $\mathcal{G}_{\beta,D}$ for all values of D and β. Only a weak upper bound was previously known.

1. Introduction

Let G be a connected graph[1]. A vertex $x\in V(G)$ *resolves*[2] a pair of vertices $v,w\in V(G)$ if $\text{dist}(v,x)\neq\text{dist}(w,x)$. A set of vertices $S\subseteq V(G)$ *resolves* G, and S is a *resolving set* of G, if every pair of distinct vertices of G are resolved by some vertex in S. Informally, S resolves G if every vertex of G is uniquely determined by its vector of distances to the

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[1]Graphs in this paper are finite, undirected, and simple. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$. For vertices $v,w\in V(G)$, we write $v\sim w$ if $vw\in E(G)$, and $v\not\sim w$ if $vw\notin E(G)$. For $S\subseteq V(G)$, let $G[S]$ be the subgraph of G induced by S. That is, $V(G[S])=S$ and $E(G[S])=\{vw\in E(G):v\in S,w\in S\})$. For $S\subseteq V(G)$, let $G\setminus S$ be the graph $G[V(G)\setminus S]$. For $v\in V(G)$, let $G\setminus v$ be the graph $G\setminus\{v\}$. Suppose that G is connected. The *distance* between vertices $v,w\in V(G)$, denoted by $\text{dist}_G(v,w)$, is the length (that is, the number of edges) in a shortest path between v and w in G. The *eccentricity* of a vertex v in G is $\text{ecc}_G(v):=\max\{\text{dist}_G(v,w):w\in V(G)\}$. We drop the subscript G from these notations if the graph G is clear from the context. The *diameter* of G is $\text{diam}(G):=\max\{\text{dist}(v,w):v,w\in V(G)\}=\max\{\text{ecc}(v):v\in V(G)\}$. For integers $a\leq b$, let $[a,b]:=\{a,a+1,\dots,b\}$.

[2]It will be convenient to also use the following definitions for a connected graph G. A vertex $x\in V(G)$ *resolves* a set of vertices $T\subseteq V(G)$ if x resolves every pair of distinct vertices in T. A set of vertices $S\subseteq V(G)$ *resolves* a set of vertices $T\subseteq V(G)$ if for every pair of distinct vertices $v,w\in T$, there exists a vertex $x\in S$ that resolves v,w.