LECTURE 23

Geodesic Convexity
and
Cartesian Products of Graphs

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- **Cartesian product of graphs**
- General results
- Domination parameters
- Independency parameters

CARTESIAN PRODUCT OF GRAPHS

⇝ $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ connected graphs.

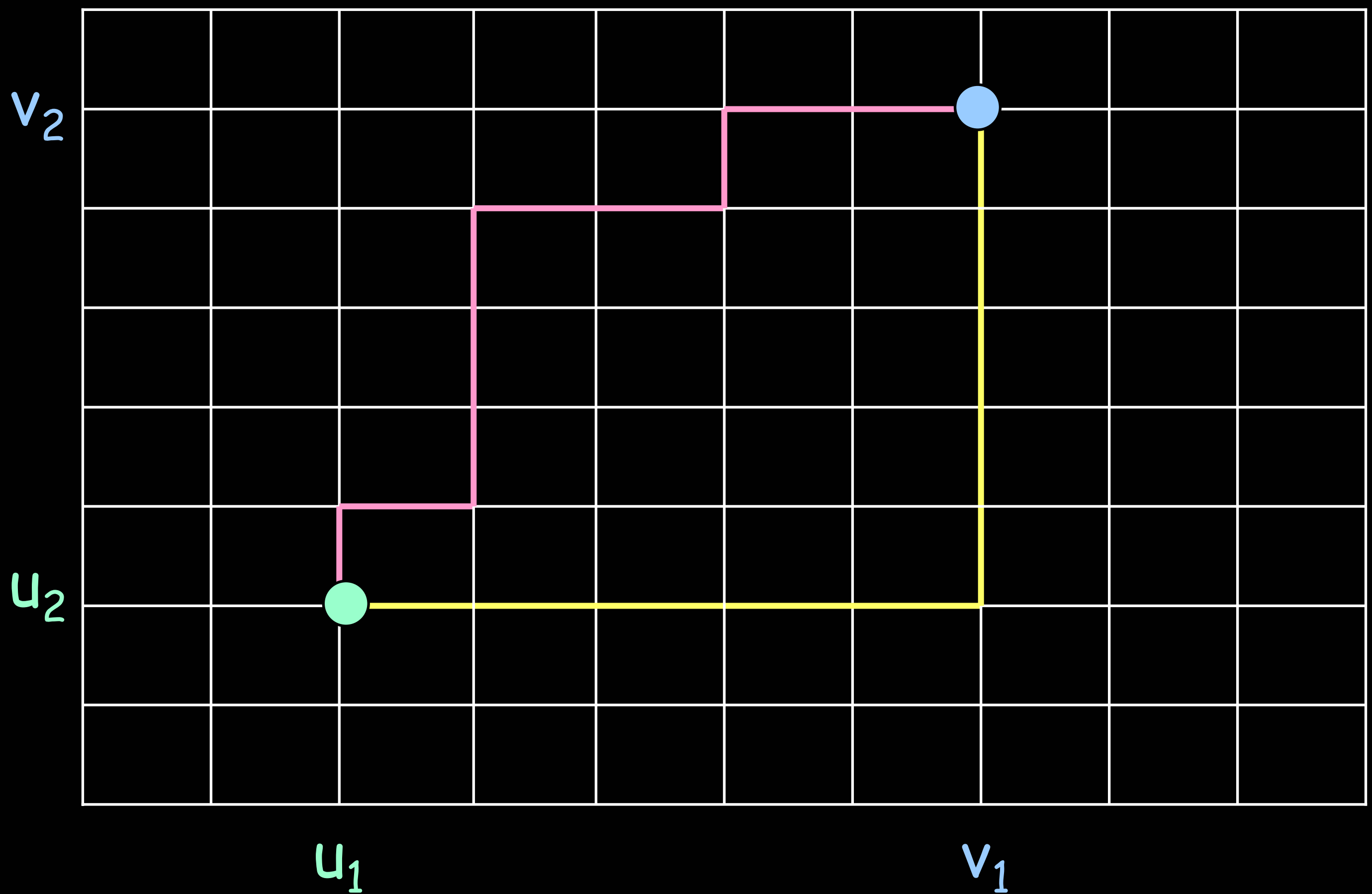
⇝ $(V_1, \mathcal{C}_1)$, $(V_2, \mathcal{C}_2)$ convexity spaces.

► Cartesian product: $G = G_1 \Box G_2 = (V, E)$, where:

- $V = V_1 \times V_2$
- $uv = (u_1, u_2)(v_1, v_2) \in E \;\Leftrightarrow\; \begin{cases} u_1 = v_1 \text{ and } u_2v_2 \in E_2 \\ \text{or} \\ u_1v_1 \in E_1 \text{ and } u_2 = v_2 \end{cases}$

★ $d_G((u_1, u_2), (v_1, v_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$

★ if P is a $u - v$ geodesic in $G_1 \Box G_2$, then $P_i = \pi_i(P)$ induces a $u_i - v_i$ geodesic in G_i, $i \in \{1, 2\}$.



$$d((u_1, u_2),(v_1, v_2)) = d_1(u_1, v_1) + d_2(u_2, v_2)$$

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CONVEX PRODUCT SPACE

⇝ $(V_1, \mathcal{C}_1)$, $(V_2, \mathcal{C}_2)$ convexity spaces.

- **Convex product space**: $(V, \mathcal{C})$, where:
  - $V = V_1 \times V_2$
  - $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1,\ B \in \mathcal{C}_2\}$

⇝ $S \subseteq V$, $S_1 = \pi_1(S)$, $S_2 = \pi_2(S)$

- $[S]_{\mathcal{C}} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$.

Let $(X, \mathcal{C}) = (V_1 \times V_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$.

If $S \subseteq X$, $S_1 = \pi_1(S)$ and $S_2 = \pi_2(S)$, then $[S]_{\mathcal{C}} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$.

Proof.

Clearly, the set $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$ is an element of $\mathcal{C}$.

A convex set containing S must be of the form $A \times B$ with $A \in \mathcal{C}_1,\ B \in \mathcal{C}_2$, where $S_1 \subseteq A$ and $S_2 \subseteq B$ in order for $S \subseteq A \times B$.

Thus $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$ is the smallest element of $\mathcal{C}$ containing S, so it is $[S]_{\mathcal{C}}$.

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GEODESIC CONVEXITY AND CARTESIAN PRODUCT

⇝ $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ connected graphs.

⇝ $(V_1, \mathcal{C}_1)$, $(V_2, \mathcal{C}_2)$ g-convexity spaces (i.e., $\mathcal{C}_i = (\mathcal{C}_i)_g$, $i = 1, 2$).

⇝ $\mathcal{C}_g$ is the g-convexity of $G = G_1 \Box G_2 = (V, E)$.

⇝ $S \subseteq V$, $S_1 = \pi_1(S)$, $S_2 = \pi_2(S)$

★ $a = (a_1, a_2), b = (b_1, b_2) \in V(G)$: $I[a, b] = I[a_1, b_1] \times I[a_2, b_2]$.

★ $I[S] \subseteq I[S_1] \times I[S_2]$.

If $a=(a_1,a_2), b=(b_1,b_2)\in V(G)$ then, $I[a,b]=I[a_1,b_1]\times I[a_2,b_2]$.

Proof. Consider any $x=(x_1,x_2)\in I[a,b]$, along an $a-b$ geodesic. Then since $d(a,b)=d(a_1,b_1)+d(a_2,b_2)$, it must be for $i=1,2$ that x_i is on an a_i-b_i geodesic, so $x\in I[a_1,b_1]\times I[a_2,b_2]$.

Conversely, consider any $x=(x_1,x_2)\in I[a_1,b_1]\times I[a_2,b_2]$. Then there exists an a_1-b_1 geodesic which follows a path P from a_1 to x_1 and from there on to b_1 via a path Q, and likewise an a_2-b_2 geodesic which follows a path R through x_2.

Then the path from $a=(a_1,a_2)$ to (x_1,a_2) (formed as in P while holding the second entry a_2 fixed) followed by the path from there to (x_1,x_2) and then on to x_1,b_2 (formed as in R while holding the first entry x_1 fixed) followed by the path from there to (b_1,b_2) (formed as in Q while holding the second entry b_1 fixed) yields an $a-b$ geodesic containing x. ∎

GEODESIC CONVEXITY AND PRODUCT CONVEXITY

⇝ $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ connected graphs.

⇝ $(V_1, \mathcal{C}_1)$, $(V_2, \mathcal{C}_2)$ g-convexity spaces (i.e., $\mathcal{C}_i = (\mathcal{C}_i)_g$, $i = 1, 2$).

⇝ $\mathcal{C}_g$ is the g-convexity of $G = G_1 \square G_2 = (V, E)$.

⇝ $S \subseteq V$, $S_1 = \pi_1(S)$, $S_2 = \pi_2(S)$

★ $[S]_{\mathcal{C}_g} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$.

★ $\mathcal{C}_g = \mathcal{C}_1 \oplus \mathcal{C}_2$.

If $S \subseteq V = V_1 \times V_2$ then, $[S]_{\mathcal{C}_g} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$.

Proof.

$\pi_1([S]_{\mathcal{C}}) \in \mathcal{C}_1$: Take $(x,y),(x',y') \in [S]_{\mathcal{C}}$. Take an $x - x'$ geodesic in G_1. Take a vertex z in this geodesic. As $d((x,y),(x',y')) = d_{G_1}(x,x') + d_{G_2}(y,y')$, there exists an $(x,y) - (x',y')$ geodesic that goes through (z,y) along the way to (x',y) before continuing on to (x',y'). Since $[S]_{\mathcal{C}} \in \mathcal{C}$ we have that $(z,y) \in [S]_{\mathcal{C}}$, so $z \in \pi_1([S]_{\mathcal{C}})$.

Therefore $[S_i]_{\mathcal{C}_i} \subseteq \pi_i([S]_{\mathcal{C}})$ for $i = 1,2$, since $\pi_i([S]_{\mathcal{C}}) \in \mathcal{C}_i$ and $S_i \subseteq \pi_i([S]_{\mathcal{C}})$.

Next we show that $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \subseteq [S]_{\mathcal{C}}$. Consider any $(x,y) \in [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$. There exist x', y' for which $(x,y') \in [S]_{\mathcal{C}}$ and $(x',y) \in [S]_{\mathcal{C}}$, since $[S_i]_{\mathcal{C}_i} \subseteq \pi_i([S]_{\mathcal{C}})$. There is an $(x,y') - (x',y)$ geodesic that passes through (x,y), so $(x,y) \in [S]_{\mathcal{C}}$. Therefore $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \subseteq [S]_{\mathcal{C}}$.

It is easy to see that $S \subseteq [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \subseteq [S]_{\mathcal{C}}$ where $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \in \mathcal{C}$. Yet $[S]_{\mathcal{C}}$ is the smallest element of $\mathcal{C}$ containing S, whereas $[S]_{\mathcal{C}}$ is known to contain $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$, so $[S]_{\mathcal{C}} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$. ▪

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KNOWN AND NEW RESULTS

⇝ $G_1 \Box G_2 = (V, E)$, $(V, \mathcal{C}_g)$ g-convexity space.

★ $con(G_1 \Box G_2) = \max\{|V_2| \cdot con(G_1), |V_1| \cdot con(G_2)\}$ [2002].

★ $hn(G_1 \Box K_2) = hn(G_1)$, $gn(G_1 \Box K_2) \geq gn(G_1)$ [2000].

★ $hn(G_1 \Box G_2) = \max\{hn(G_1), hn(G_2)\}$.

★ $\max\{g_1, g_2\} \leq gn(G_1 \Box G_2) \leq g_1 \cdot g_2 - \min\{g_1, g_2\}$ $(g_i = gn(G_i))$.

HULL NUMBER

⊛ $hn(G) = \min\{|S| : S \subseteq V,\ [S]_g = V\}$

$\Rrightarrow$ $hn(G \Box H) = \max\{hn(G), hn(H)\}$

Proof. Let $A = \{a_1, \cdots, a_p\}$ be a minimum g-hull set for G and let $B = \{b_1, \cdots, b_q\}$ be a minimum g-hull set for H, where without loss of generality we can assume that $p \geq q$.

Take $S \subseteq V(G \times H)$. As $[S]_g = [S_1]_g \times [S_2]_g$, then S is a g-hull set for $G \times H$ iff S_1 is a g-hull set for G and S_2 is a g-hull set for H.

$S = \{(a_1, b_1), (a_2, b_2), ..., (a_q, b_q), (a_{q+1}, b_q), (a_{q+2}, b_q), ..., (a_p, b_q)\}$ satisfies this requirement, with $|S| = p$, so $hn(G \Box H) \leq p$.

Also, $hn(G \times H) \geq p$, since S must have at least $hn(G)$ many elements for its projection S_1 to have at least $hn(G)$ elements.

Thus $hn(G \times H) = p$, completing the proof. ∎

GEODETIC NUMBER: TIGHT LOWER BOUND

⊛ $gn(G) = \min\{|S| : S \subseteq V, \text{I[S]=V}\}$

⇛ $gn(G \times H) \geq \max\{gn(G), gn(H)\}$, and the inequality is best possible.

Proof. If S is a minimum geodetic set in $G \times H$ then, $V(G \times H) = I[S] \subseteq I[S_1] \times I[S_2]$. Therefore S_1, S_2 are geodetic sets in G, H respectively, so $gn(G \times H) = |S| \geq \max\{|S_1|, |S_2|\} \geq \max\{gn(G), gn(H)\}$, proving the lower bound.

Consider complete graphs G, H with vertex sets $V(G) = \{u_1, u_2, \cdots u_p\}$ and $V(H) = \{v_1, v_2, \cdots, v_q\}$, where without loss of generality $p \geq q$. Then $gn(G) = p$ and $gn(H) = q$. Let

$$S = \{(u_1, v_1), (u_2, v_2), \cdots, (u_q, v_q), (u_{q+1}, v_q), (u_{q+2}, v_q), \cdots, (u_p, v_q)\}.$$

It is straightforward to verify that S is a geodetic set for $G \times H$. Hence, $gn(G \times H) \leq |S| \leq p = \max\{gn(G), gn(H)\} \leq gn(G \times H)$, so equality holds.

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GEODETIC NUMBER: TIGHT UPPER BOUND

$\Rrightarrow$ $gn(G_1 \times G_2) \leq gn(G) \cdot gn(H) - \min\{gn(G), gn(H)\}$.

$\triangleright$ This upperbound is best possible.

Outline of the**Proof.**

$\Rrightarrow$ Suppose $gn(G) = p$ and $gn(H) = q$ where $p \geq q \geq 2$. Let $A = \{a_1, \cdots, a_p\}$ be a geodetic set of G and $B = \{b_1, \cdots, b_q\}$ a geodetic set of H. Let $S = A \times B - \{(a_1, b_1), (a_2, b_2), \cdots, (a_q, b_q)\}$. Then $|S| = pq - q$. We show that S is a geodetic set of $G \times H$.

$\triangleright$ Let p, t be positive integers such that $t > p$. Let $G = D_p^t$. Then $gn(G) = p$.

Let p, q, t be positive integers such that $t > pq - q$. Let $G = D_p^t$. Let $H = K_q$. Then $gn(G) = p$, $gn(H) = q$, and $gn(G \times H) \geq pq - q$. ∎

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KNOWN AND NEW RESULTS

★ $h(G_1 \Box G_2) = \max\{h(G_1), h(G_2)\}$ [1975].

★ $c(G_1 \Box G_2) = c(G_1) + c(G_2) - k$

where there are $k \in \{0, 1, 2\}$ factors s.t. $e \leq c$ [1975].

★ $e(G_1 \Box G_2) = \max\{e(G_1), c(G_1)\} + \max\{e(G_2), c(G_2)\} - 1$ [1981].

★ $r(G_1) + r(G_2) - 1 \leq r(G_1 \Box G_2) \leq r(G_1) + r(G_2) + 1$

★ $c(G_1 \Box G_2) = e(G_1) + e(G_2) - 2$.

★ $e(G_1 \Box G_2) = e(G_1) + e(G_2) - 1$.

RANK

⊛ $S \subseteq V$ is <u>convexly indep.</u> if for each $x \in S$, $x \notin [S - x]_g$.

► $r(G) = \max\{|\, S \,| : \; S \subseteq V$ is convx. indep$\}$

⊛ $S \subseteq V$ is <u>weakly convexly independent</u> if there exist at most $x \in S$ such that $x \in [S - x]_g$

► $r^1(G) = \max\{|\, S \,| : \; S \subseteq V$ is weakly convx. indep$\}$

↪ $r \leq r^1 \leq r + 1$

⇛ $r(G_1 \times G_2) = r^1(G_1) + r^1(G_2) - 2$

↬ $r(G_1) + r(G_2) - 2 \leq r(G_1 \Box G_2) \leq r(G_1) + r(G_2)$

CARATHEODORY AND EXCHANGE NUMBERS

- $c(G) = \max\{|\,S\,| : \; S \subseteq V \text{ and } \bigcup_{a \in S} [S-a]_g \subsetneq [S]_g\}$
- $e(G) = \max_{S \subseteq V}\{|\,S\,| : \; [S-x]_g \not\subset \bigcup_{a \in S-x} [S-a]_g \text{ for some } x \in S\}$

$\hookrightarrow$ $e - 1 \leq c \leq e$

$\Rightarrow$ $c(G_1 \Box G_2) = c(G_1) + c(G_2) - k$

where there are $k \in \{0, 1, 2\}$ factors s.t. $e \leq c$ [1975].

$\Rrightarrow$ $c(G_1 \Box G_2) = e(G_1) + e(G_2) - 2$

$\Rightarrow$ $e(G_1 \Box G_2) = \max\{e(G_1), c(G_1)\} + \max\{e(G_2), c(G_2)\} - 1$ [1981].

$\Rrightarrow$ $e(G_1 \Box G_2) = e(G_1) + e(G_2) - 1 = c(G_1 \Box G_2) + 1$