LECTURE 20

Geodesic convexity in graphs

Geodetic sets formed by boundary vertices

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- **Geodesic convexity**
- Boundary-type vertices
- Geodetic sets: General results
- Geodetic sets on the perfect family

GEODESIC CONVEXITY

⋆ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_g \subseteq 2^V$.

- A $u - v$ geodesic is a $u - v$ path of minimum length.
- Closed interval: $I[u, v] = \{V(\rho) : \ \rho$ is a $u - v$ geodesic$\}$
- Geodetic closure: $I[S] = \bigcup_{u,v \in S} I[u, v]$
- g-convex set: $S \in \mathcal{C}_g \Leftrightarrow S = I[S]$.
- g-convex hull:

$$S \subseteq I[S] \subseteq I^2[S] \subseteq \ldots \subseteq I^r[S] = [S]_g \subseteq V$$

DOMINATION PARAMETERS (G-CONVEXITY)

⋆ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_g)$ g-convexity space.

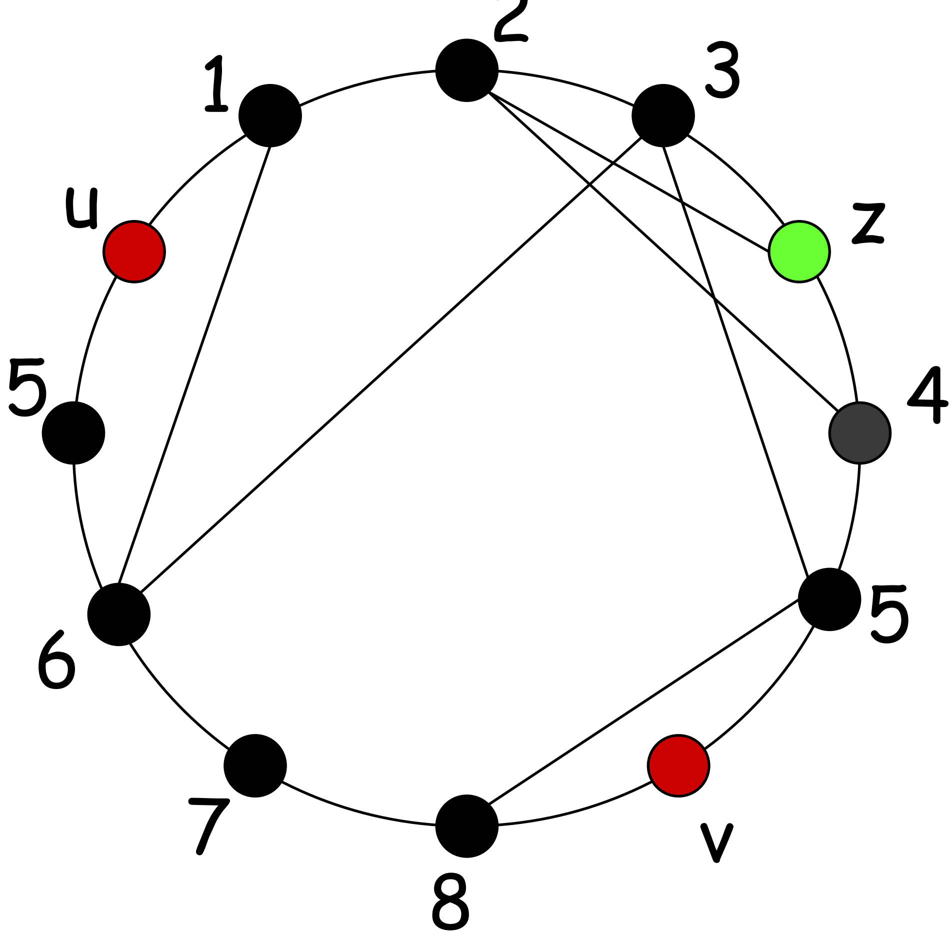
- ▶ Geodetic set: $I[S] = V$

- ⊗ Geodetic number: $gn(G) = \min\{|S| : S$ is a geodetic set of $G\}$

- ▶ Hull set: $[S]_g = V$.

- ⊗ Hull number: $hn(G) = \min\{|S| : S$ is a hull set of $G\}$

$\hookrightarrow hn(G) \leq gn(G)$



d(u,v)=5

I[u,v]=V-z

I[V-z]=V

{u,v} is a hull set

hn(G)=2

S={u,v,z}: I[S]=V ↔ S is a geodetic set

gn(G)=3

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BOUNDARY VERTICES

[•] **Boundary vertices**:

$$\partial(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \forall w \in N(v): \ d(u,w) \le d(u,v)\}$$

[•] **Eccentric vertices**:

$$Ecc(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \forall w \in V : d(u,w) \le d(u,v)\} \subseteq \partial(G)$$

[•] **Contour vertices**:

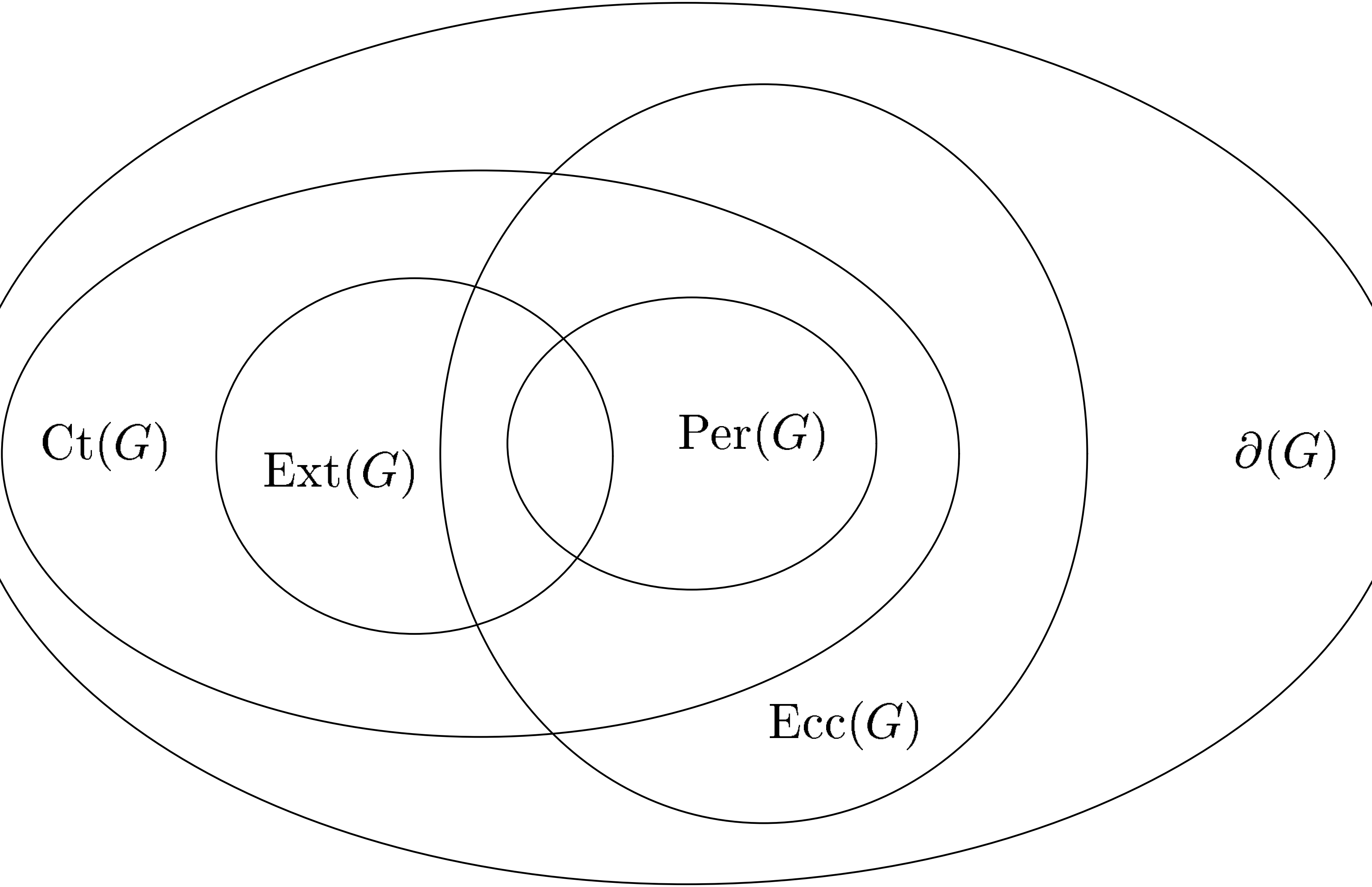
$$Ct(G) = \{v \in V \mid ecc(w) \le ecc(v), \ \forall w \in N(v)\} \subseteq \partial(G)$$

[•] **Peripheral vertices**:

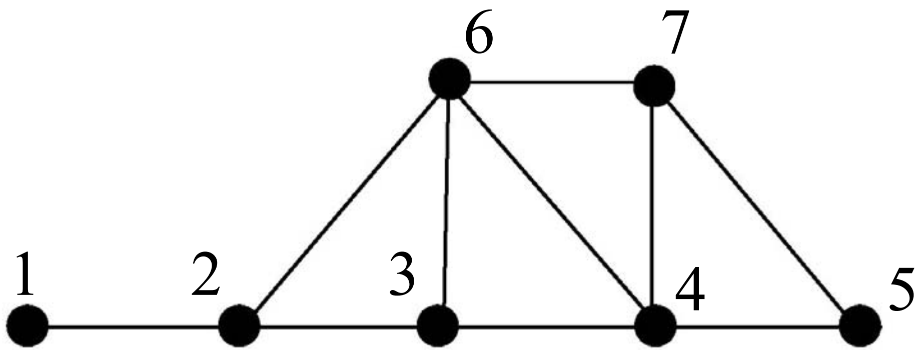
$$Per(G) = \{v \in V \mid ecc(w) \le ecc(v), \ \forall w \in V\} \subseteq Ct(G) \cap Ecc(G)$$

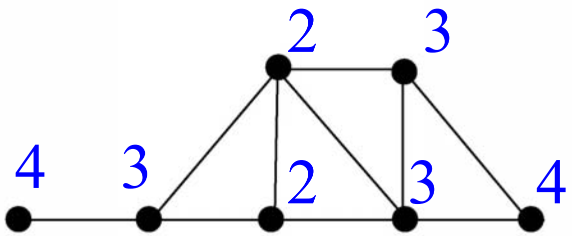
[•] **Extreme vertices**:

$$Ext(G) = \{v \in V \mid G[N(v)] \text{ is a clique}\} \subseteq Ct(G)$$



Ct(G)
Ext(G)
Per(G)
∂(G)
Ecc(G)





Ext(G)={1,5}

Per(G)={1,5}

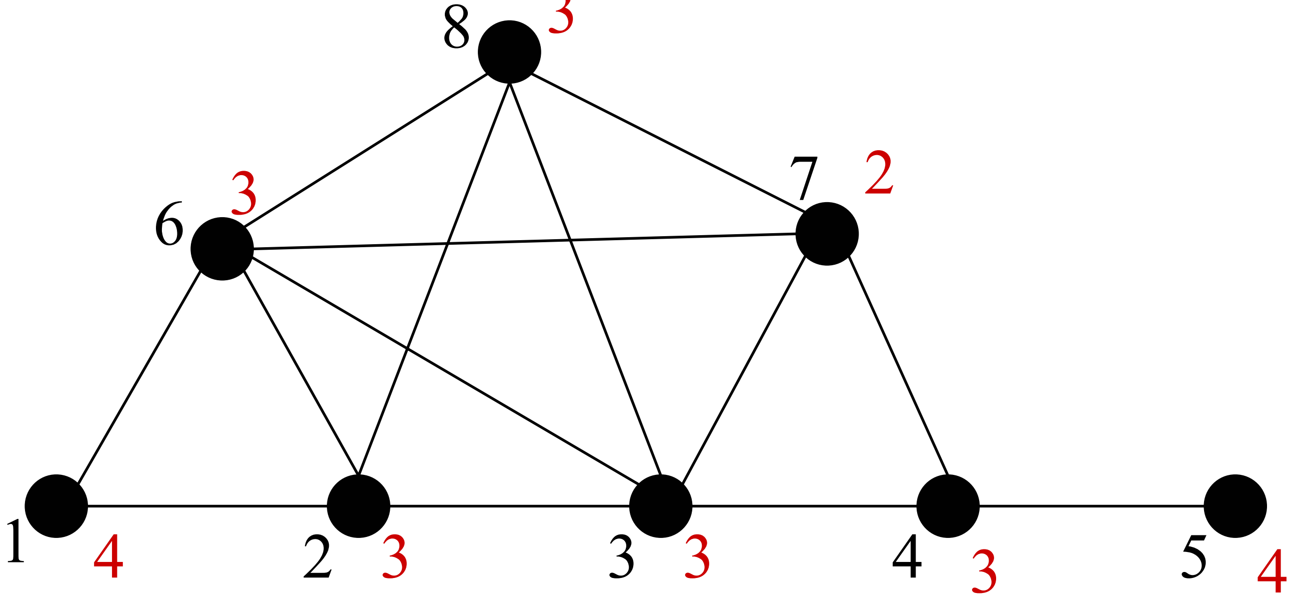
Ct(G)={1,5}

Ecc(G)={1,5,7}

d(7,N(3)) =d(7,3)

→

∂(G)={1,3,5,7}



$\partial(1)=\{5,8\}$ $\partial(2)=\{1,5,7\}$ $\partial(3)=\{1,5,7,8\}$ $\partial(4)=\{1,5,8\}$

$\partial(5)=\{1,8\}$ $\partial(6)=\{1,5\}$ $\partial(7)=\{1,2,5\}$ $\partial(8)=\{1,5\}$

$\mathrm{Ext}(G)=\{1,5\}$ $\mathrm{Per}(G)=\{1,5\}$ $\mathrm{Ct}(G)=\{1,5,8\}$

$\mathrm{Ecc}(G)=\{1,2,5\}$ $\partial(G)=\{1,2,5,7,8\}$

- (Chartrand et alt, 2003) For each triple a, c, d of integers with $2 \leq a \leq c \leq d$, there is a connected graph G such that $Per(G)$ has order a, $Ecc(G)$ has order c, and $\partial(G)$ has order d.

★ If $|Ecc(G)| > |Per(G)|$, then $|\partial(G)| \geq |Per(G)| + 2$.

★ If $|Per(G)| = |Ct(G)| = 2$, then either $|\partial(G)| = 2$ or $|\partial(G)| \geq 4$.

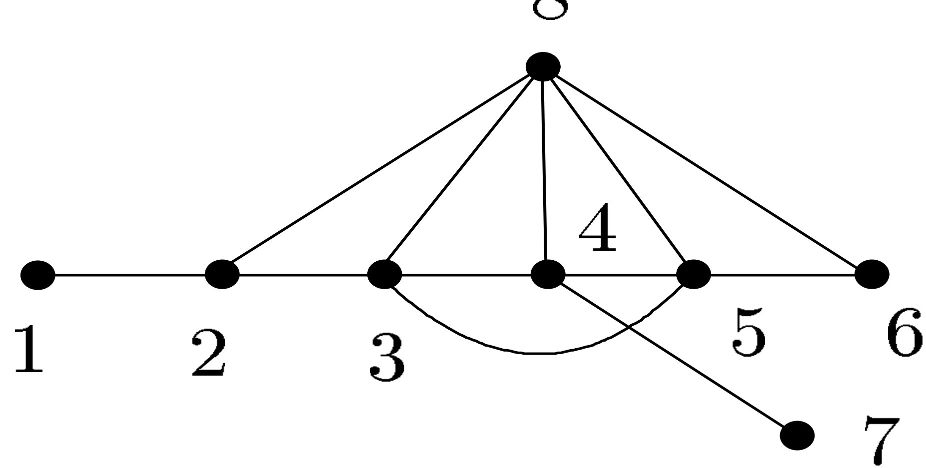
⇛⇛ Let $(a, b, c, d) \in \mathbb{Z}^4$ be integers satisfying the constraints

$$\begin{cases} 2 \leq a \leq b \leq d, \\ 2 \leq a \leq c \leq d, \\ (a, b, c, d) \neq (2, 2, 2, 3), \ [*] \\ (a, b, c, d) \neq (a, b, a+1, a+1). \ [**] \end{cases}.$$

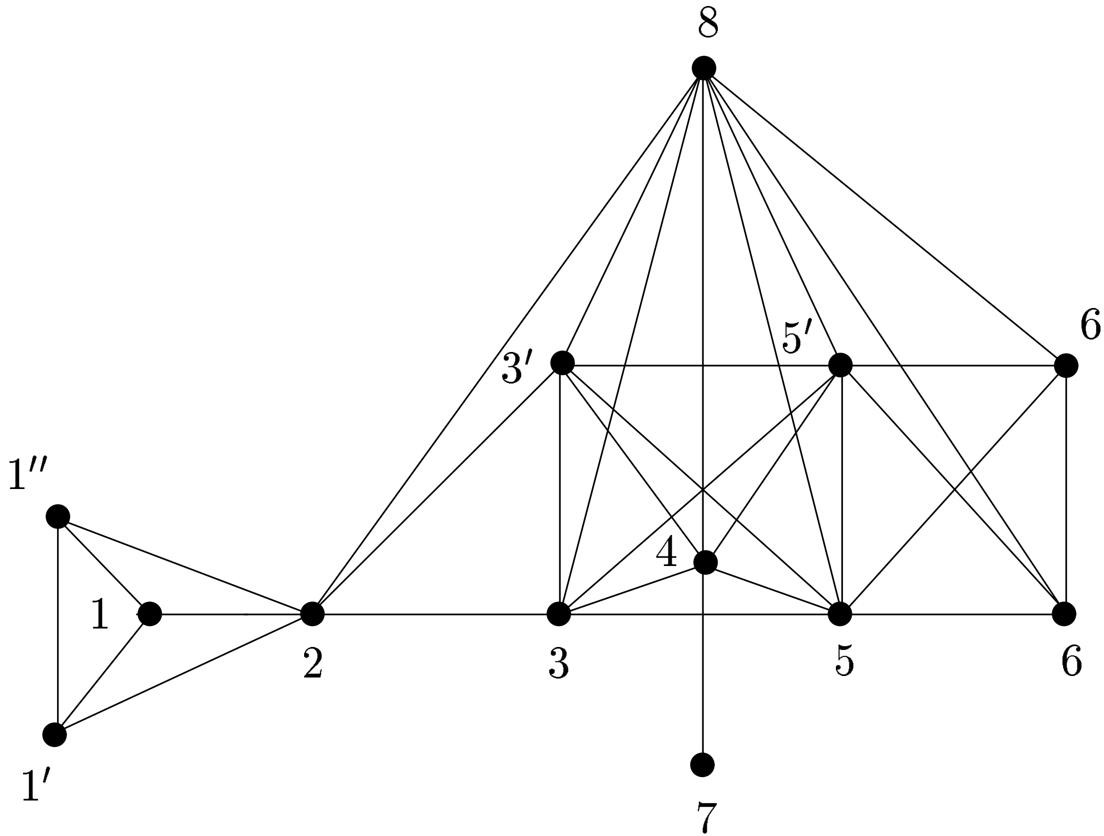
Then, there exists a connected graph $G = (V, E)$ satisfying:

$$|Per(G)| = a, \quad |Ct(G)| = b, \quad |Ecc(G)| = c, \quad |\partial(G)| = d.$$

17-1567-167-13567: [2,4,3,5]



[4,8,6,10]



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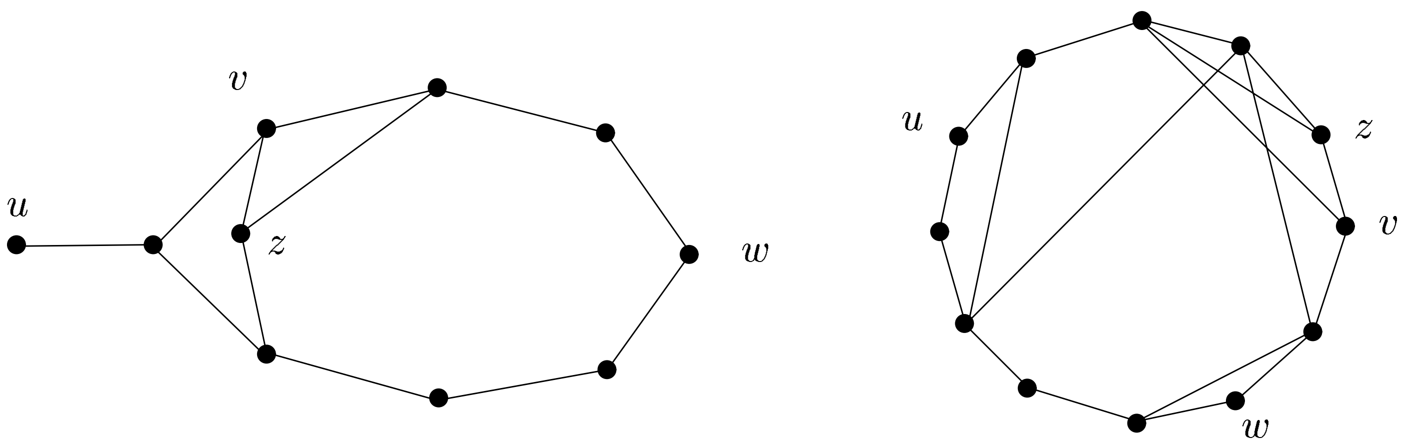
The contour needs not to be geodetic

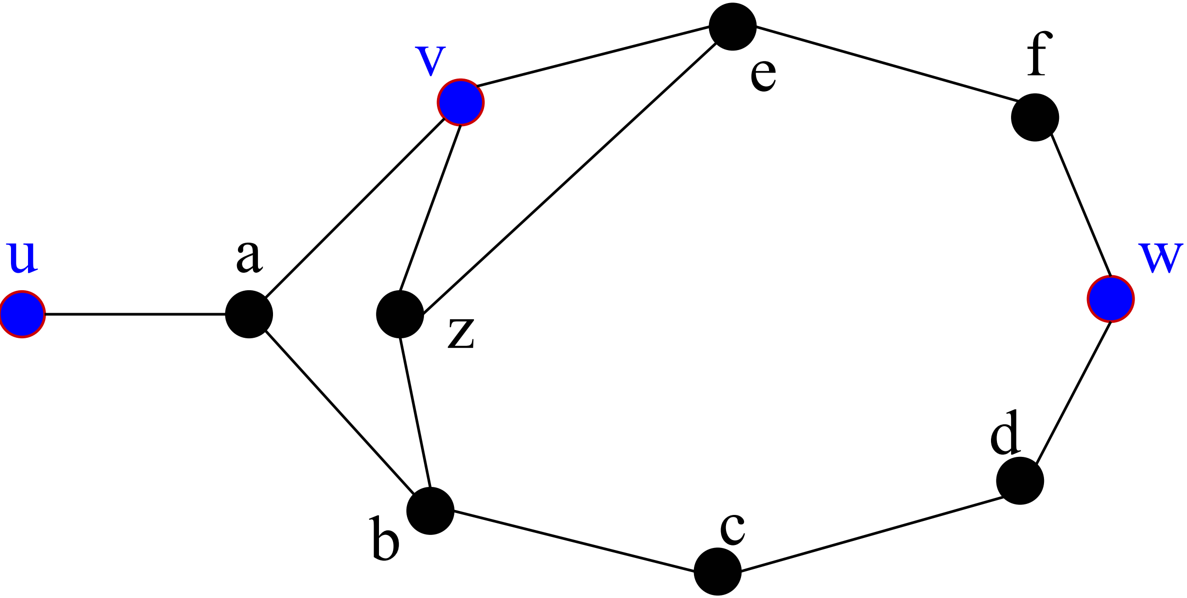
★ The contour of every graph is a **hull set**. (J. C., 2004).

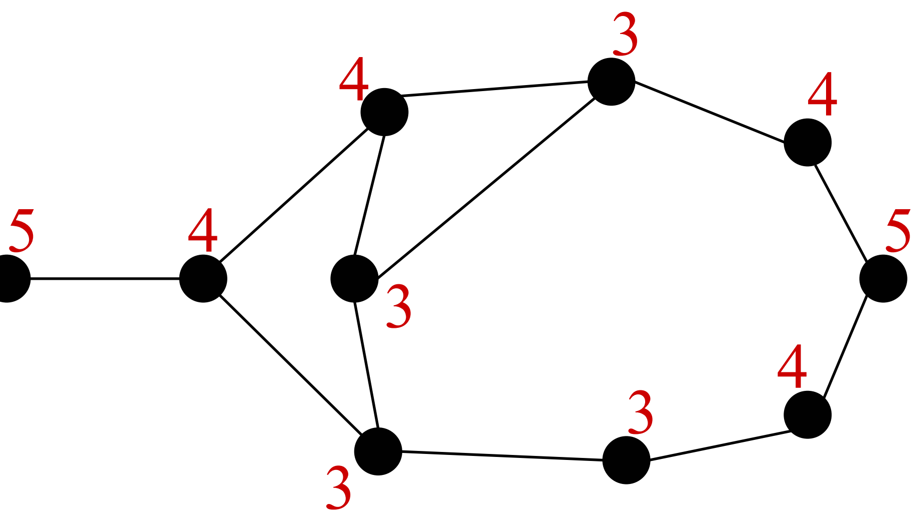
★ The contour of every graph is **monophonic**. (I.M.P., 2004).



a) b)

Two graphs with a non geodetic contour: Ct(G)={u,v,w} and I[Ct(G)]=V(G)-z.





Ct(G)={u,v,w}

I[u,v] = {u,a,v}

I[u,w] = V-z

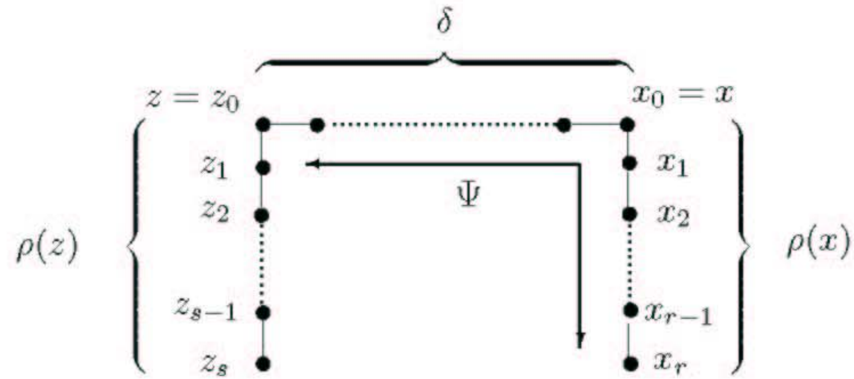
I[v,w] = {v,e,f,w}

I[Ct(G)]=V-z

The contour of any connected graph G is a monophonic set.

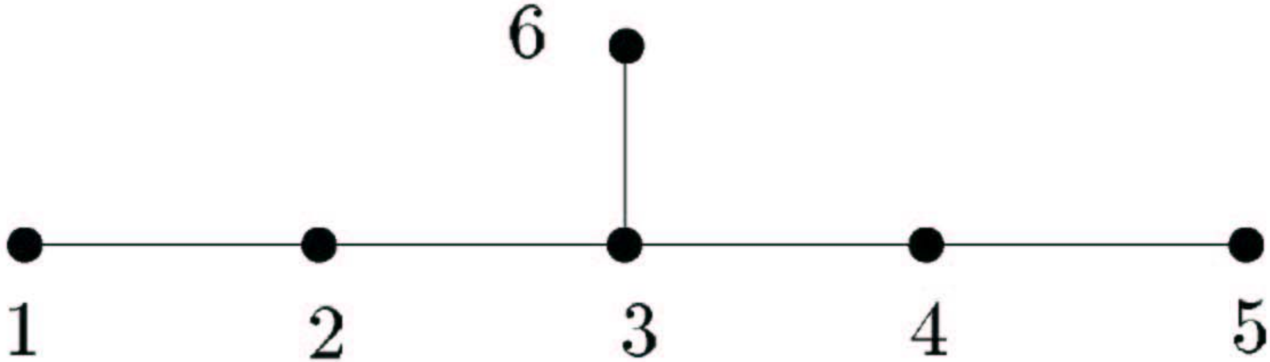
Outline of the **Proof.** Let x be a vertex of $V(G) \setminus Ct(G)$. Hence, there exists a vertex $x_r \in Ct(G)$ and an $x - x_r$ geodesic $\rho(x)$ of length r such that $ecc(x_r) = l + r$ where $l = ecc(x)$.

There exits a vertex z at a distance exactly l from x and $l + r$ from x_r, and x lies on a shortest path $\Psi = z \cdots x x_1 \cdots x_r$ between z and x_r.



Suppose that z is not a contour vertex. Let us construct a path $\rho(z) = z_0 z_1 \cdots z_s$ such that $z = z_0$, $z_i \notin Ct(G)$ for $i \in \{0, \ldots, s-1\}$, $z_s \in Ct(G)$ and $ecc(z_i) = ecc(z_{i-1}) + 1 = ecc(z) + i$ for $i \in \{1, \ldots, s\}$. ■ ■ ■

The eccentric set needs not to be geodetic



Tree T for which $Ecc(T) = \{1, 5\}$ and $[Ecc(T)] = V(T) \setminus 6$.

If the contour of a graph is equal to its periphery, then it is geodetic.

Proof. Let x be a vertex of $V(G) \setminus Ct(G)$.

Since $x \notin Ct(G)$, there exist a vertex $x_r \in Ct(G)$ and an $x - x_r$ geodesic $\rho(x)$ of length r such that $ecc(x_r) = l + r$, where $l = ecc(x)$.

But $x_r \in Ct(G) = Per(G)$ implies that $ecc(x_r) = D$ and $D = l + r$.

Thus, there exists a vertex $z \in Per(G)$ such that $D = d(z, x_r) \leq d(z, x) + d(x, x_r) \leq ecc(x) + r = l + r = D$, that is, $d(z, x_r) = d(z, x) + d(x, x_r)$.

Hence, x is on a shortest path between the vertices $z, x_r \in Per(G) = Ct(G)$. ∎

The expanded contour $\Omega(G) = Ct(G) \cup Ecc_G(Ct(G))$ of every graph $G = (V, E)$ is geodetic.

Proof. Let x be a vertex of $V(G) \setminus \Omega(G)$.

Since $x \notin Ct(G)$, there exists a vertex $x_r \in Ct(G)$ and an $x - x_r$ geodesic $\rho(x)$ of length r such that $ecc(x_r) = ecc(x) + r$.

Let y_r be an eccentric vertex of x_r, i.e., such that $d(y_r, x_r) = ecc(x_r)$. Then,

$$ecc(x) + r = ecc(x_r) = d(y_r, x_r) \leq \underbrace{d(y_r, x)}_{\leq ecc(x)} + \underbrace{d(x, x_r)}_{=r} \leq ecc(x) + r$$

and hence we conclude that the inequalities in the formula above are all equalities, which means that the vertex x lies in a shortest path joining $x_r \in Ct(G) \subset \Omega(G)$ and $y_r \in Ecc_G(Ct(G)) \subset \Omega(G)$. ■

The boundary of every graph is edge geodetic.

Proof.

Take an edge e=xy of G.

Consider the set Υ_e of all shortest paths containing e.

Let P=(a......xy.......b) be an element of Υ_e of maximum length.

Clearly, $\{a,b\} \subset \partial(G)$, since vertex b (resp. a) belongs to the boundary of a (resp. b). ∎

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The contour of every distance hereditary graph G is geodetic.

Proof. The contour of every graph G is monophonic.

If G is distance hereditary, then a path is monophonic if and only if it is a shortest path.

Hence, being monophonic is equivalent to being geodetic. ∎

The contour of every chordal graph is geodetic.

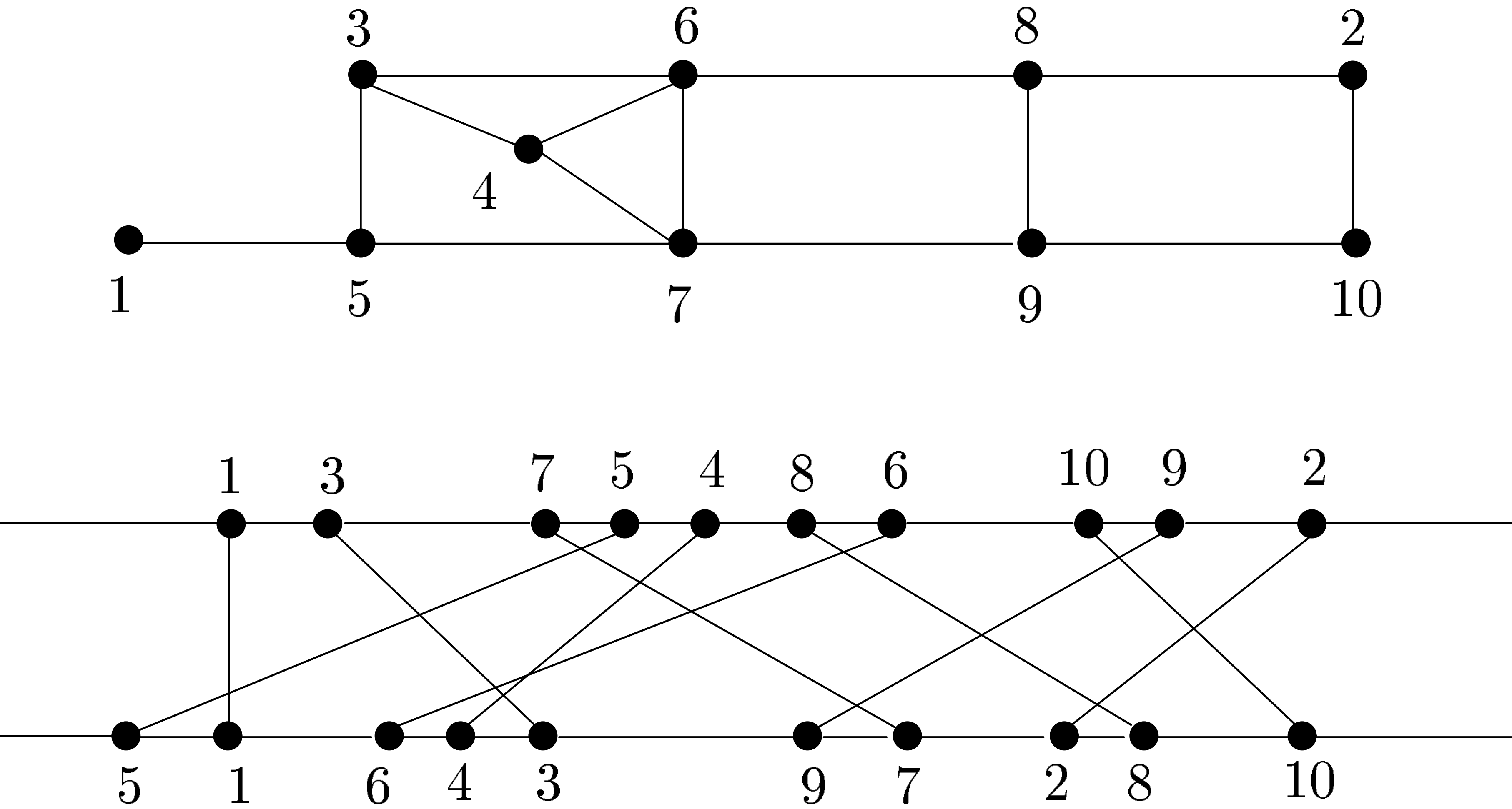
Outline of the Proof.

Let G be a chordal graph and let $x_0 \in V(G)$.

If $x_0 \notin Ct(G)$, then it has a neighbor x_1 s.t. $ecc(x_1) = ecc(x_0) + 1$. We repeat this argument to obtain a path $(x_0, x_1, \ldots, x_k)$ where $ecc(x_{i+1}) = ecc(x_i) + 1$, $\forall i \in \{0, 1, \ldots, k-1\}$, and x_k is a contour vertex.

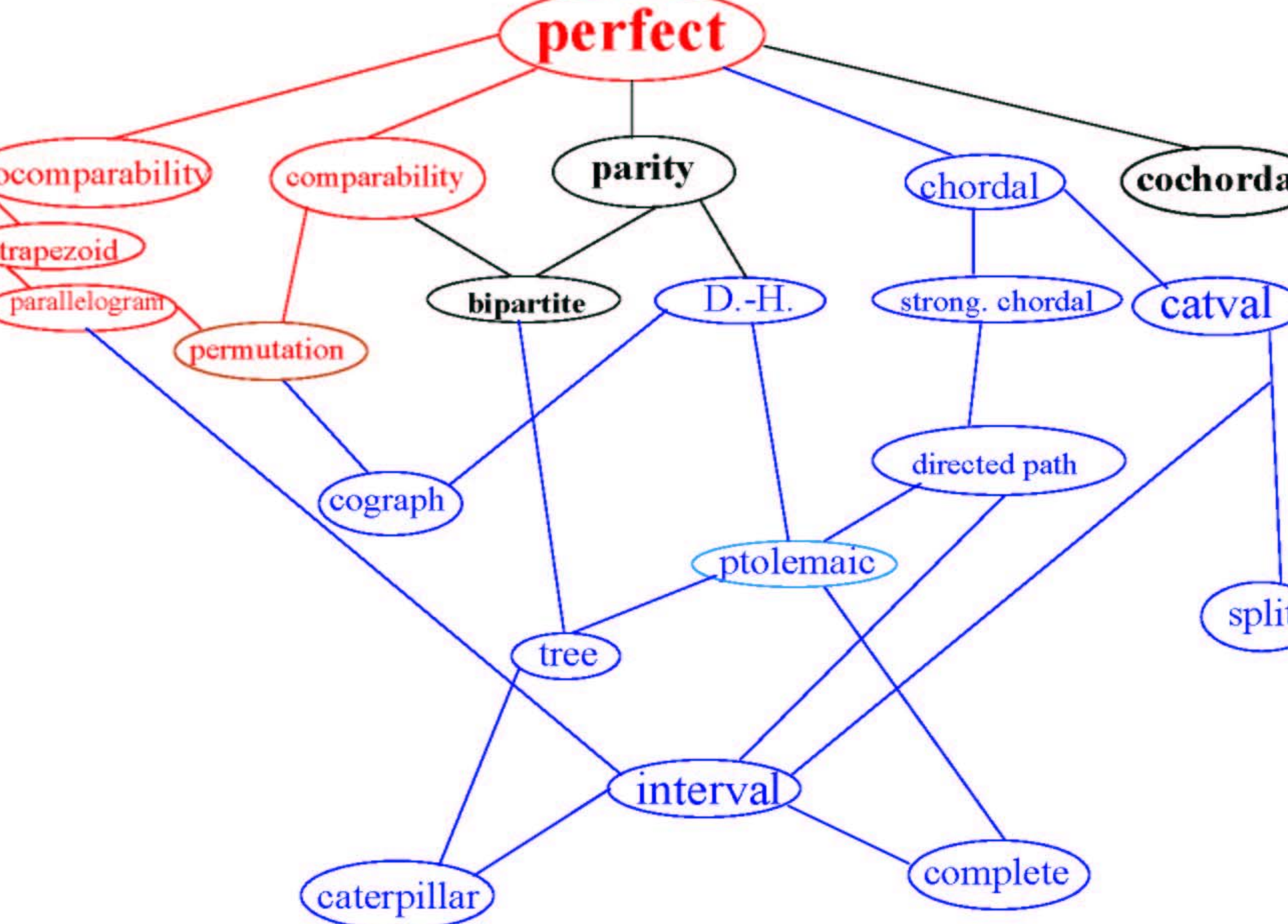
$\leadsto$ Every chordal graph G admits a tree intersection representation $(T, \mathcal{F})$ s. t. every leaf u of T belongs to a trivial subtree of $\mathcal{F}$. There exists an eccentric vertex of x_k, say z, which is represented by a leaf, and such a vertex is a contour vertex.

$\leadsto$ There exists a shortest path between x_k and z containing x_0, as desired. ∎



Permutation graph P and its matching diagram.

Notice that Ct(P)={1,2,3} and [Ct(P)]=V(P)-4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ocomparability
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cochorda
trapezoid
parallelogram
bipartite
D.-H.
strong. chordal
catval
permutation
directed path
cograph
ptolemaic
spli
tree
interval
complete
caterpillar