LECTURE 10

Geodesic and monophonic convexity

Searching for geodetic and/or monophonic sets

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- **Geodesic and monophonic convexities**
- Generalizing the Krein-Milman property
- Steiner set problem
- Boundary-type sets

GRAPH CONVEXITY SPACE

⋆ $G = (V, E)$ is a connected graph and $\mathcal{C} \subseteq 2^V$.

- $(V, \mathcal{C})$ is a GRAPH CONVEXITY SPACE if:

(C1) $\emptyset \in \mathcal{C}$, $V \in \mathcal{C}$.

(C2) $\{W_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} W_i \in \mathcal{C}$.

(C3) $\{W_i\}_{i \in I} \subseteq \mathcal{C}$ s.t. $W_i \subseteq W_{i+1} \Rightarrow \bigcup_{i \in I} W_i \in \mathcal{C}$.

(C4) For every $U \in \mathcal{C}$, $\langle U \rangle_G$ is connected.

GEODESIC CONVEXITY

⋆ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_g \subseteq 2^V$.

- A $u - v$ geodesic is a $u - v$ path of minimum length.
- Closed interval: $I[u, v] = \{V(\rho) : \rho$ is a $u - v$ geodesic$\}$
- Geodetic closure: $I[S] = \bigcup_{u,v \in S} I[u, v]$
- g-convex set: $S \in \mathcal{C}_g \Leftrightarrow S = I[S]$.
- g-convex hull: $S \subseteq I[S] \subseteq I^2[S] \subseteq \ldots \subseteq I^r[S] = [S]_g \subseteq V$

⋆ $(V, \mathcal{C}_g)$ is an interval, path and metric convexity space.

MONOPHONIC CONVEXITY

⋆ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_m \subset 2^V$.

- A $u - v$ monophonic path is a $u - v$ chordless path.
- Closed interval: $J[u, v] = \{V(\rho) : \rho$ is a $u{-}v$ monophonic path$\}$
- Monophonic closure: $J[S] = \bigcup_{u,v \in S} J[u, v]$
- m-convex set: $S \in \mathcal{C}_m \Leftrightarrow S = J[S]$.
- m-convex hull: $S \subseteq J[S] \subseteq J^2[S] \subseteq \ldots \subseteq J^r[S] = [S]_m \subseteq V$

⋆ $(V, \mathcal{C}_m)$ is an interval and path convexity space.

DOMINATION PARAMETERS (G-CONVEXITY)

⋆ $G=(V,E)$ conn. graph, $S\subseteq V$, $(V,\mathcal{C}_g)$ g-convexity space.

▶ Geodetic set: $I[S]=V$

⊗ Geodetic number: $gn(G)=\min\{|S| : S \text{ is a geodetic set of } G\}$

▶ Hull set: $[S]_g=V$.

⊗ Hull number: $hn(G)=\min\{|S| : S \text{ is a hull set of } G\}$

↪ $hn(G)\leq gn(G)$

DOMINATION PARAMETERS (M-CONVEXITY)

⋆ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_m)$ m-convexity space.

- Monophonic set: $J[S] = V$

⊗ Monophonic number: $mn(G) = \min\{|S| : S \subseteq V$ is monophonic$\}$

- m-Hull set: $[S]_m = V$.

⊗ m-Hull number: $hn(G) = \min\{|S| : S$ is an m − hull set of $G\}$

↪ $mhn(G) \leq hn(G) \leq gn(G)$

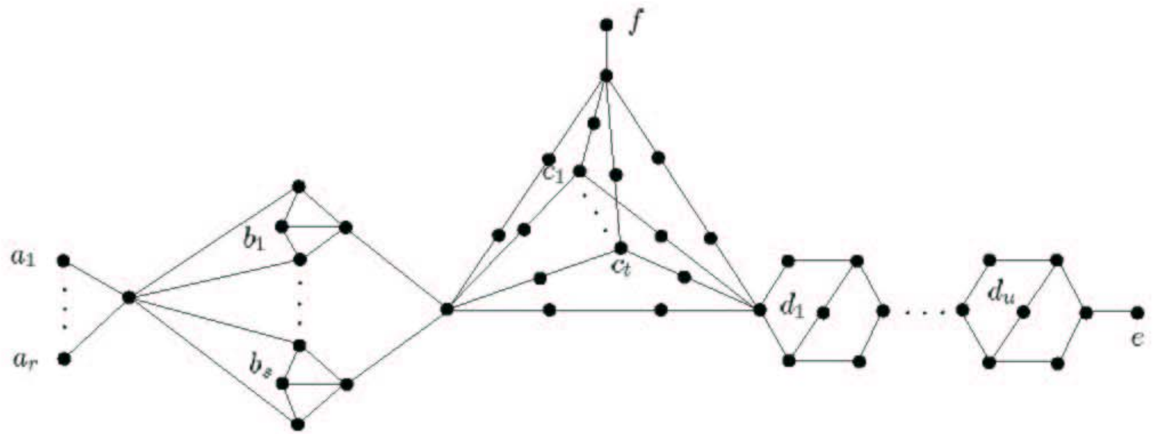
↪ $mhn(G) \leq mn(G) \leq gn(G)$

⇛ What about $mn(G)$ and $hn(G)$?.

- (I.M.P. et alt., 2004) For any integers a, b, c, d such that $3 \leq a \leq b \leq c \leq d$, there exists a connected graph G s. t.:

  1. $a = mhn(G)$, $b = mn(G)$, $c = hn(G)$, and $d = gn(G)$,

  2. $a = mhn(G)$, $b = hn(G)$, $c = mn(G)$, and $d = gn(G)$.



$W_1 = Ext(G) = \{a_1, \ldots, a_r, e, f\}$ is a minimum m-hull set,

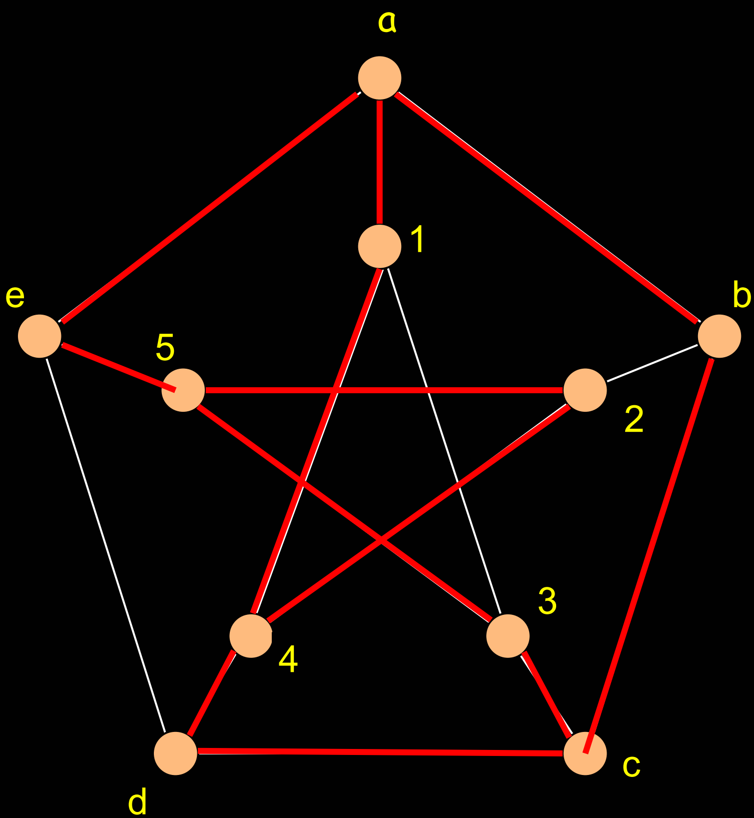
$W_2 = W_1 \bigcup \{b_1, \ldots, b_s\}$ is a minimum monophonic set,

$W_3 = W_1 \bigcup \{c_1, \ldots, c_t\}$ is a minimum g-hull set, and

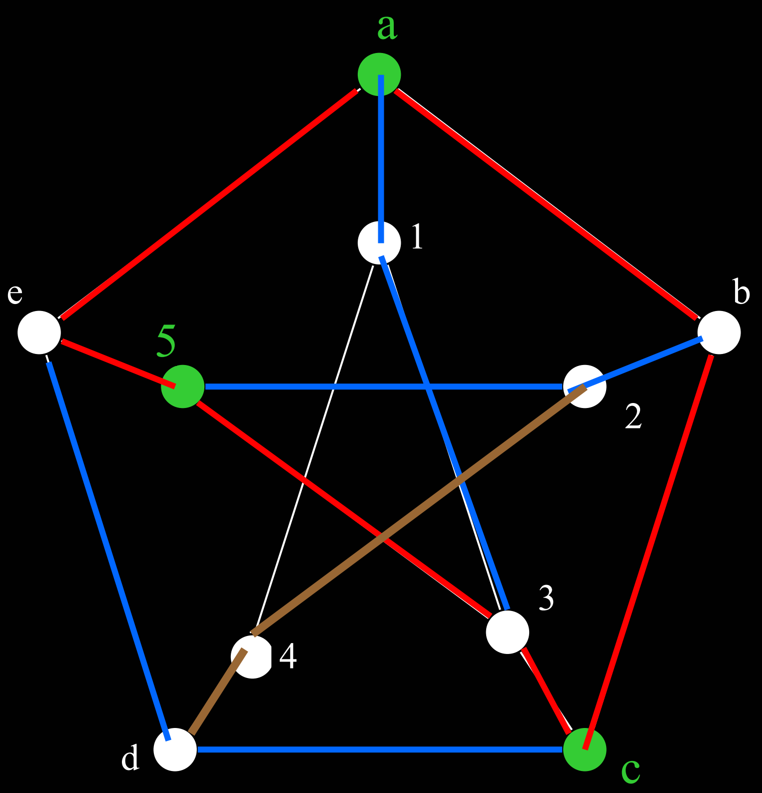
$W_4 = W_2 \bigcup \{c_1, \ldots, c_t\} \bigcup \{d_1, \ldots, d_u\}$ is a minimum geodetic set.

PETERSEN GRAPH

con	gn	hn	gin	mn
5	4	3	3	3



S={a,c,4,5} is geodetic



S={a,c,5}

I[S]={a,b,c,3,5,e}

$I^2[S]=V-4$

$I^3[S]=V$

gin(S)=3

con	gn	hn	gin	mn
5	4	3	3	3

S={a,c,5} is a hull set

S={a,c,5} is monophonic: J[S]=V

- Geodesic and monophonic convexities
- <u>Generalizing the Krein-Milman property</u>
- Steiner set problem
- Boundary-type sets

EXTREME VERTICES

⋆ Let $G = (V, E)$ be a graph and $(V, \mathcal{C})$ a graph convexity space.

► A vertex $x \in W \in \mathcal{C}$ is an *extreme vertex* iff $W - x \in \mathcal{C}$.

► A vertex $x \in V$ is *simplicial* iff $N_G(v)$ is a clique.

♦ A vertex $x \in W \in \mathcal{C}_g$ is an extreme vertex iff it is *simplicial* in $\langle W \rangle_G$.

♦ A vertex $x \in W \in \mathcal{C}_m$ is an extreme vertex iff it is *simplicial* in $\langle W \rangle_G$.

CONVEX GEOMETRY: KREIN-MILMAN PROPERTY

► A convexity space $(V, \mathcal{C})$ is called a *convex geometry* if every convex set is the convex hull of its extreme vertices.

♦ (Farber and Jamison, 1986) Let $G = (V, E)$ be connected graph. Then,

- $(V, \mathcal{C}_g)$ **is a convex geometry iff** G **is Ptolemaic.**
- $(V, \mathcal{C}_m)$ **is a convex geometry iff** G **is chordal.**

► *Chordal graphs* are those without induced cycles of length greater than 3. A chordal graph G is called *Ptolemaic* if it distance-hereditary, i.e., if every induced path is a shortest path.

GENERALIZING THE KREIN-MILMAN PROPERTY

[•] $(V, \mathcal{C}_g)$ is a convex geometry iff G is Ptolemaic

OPERATOR	DOMINATING SET	FAMILY
$[\ ] : 2^V \mapsto \mathcal{C}_g$	EXTREME VERTICES	PTOLEMAIC

[•] $(V, \mathcal{C}_m)$ is a convex geometry iff G is chordal

OPERATOR	DOMINATING SET	FAMILY
$[\ ] : 2^V \mapsto \mathcal{C}_m$	EXTREME VERTICES	CHORDAL

GENERALIZING THE KREIN-MILMAN PROPERTY

[•] $G = (V, E)$, $(V, \mathcal{C}_x)$:

OPERATORS	DOMINATING SETS	FAMILIES
$[\ ] : 2^V \mapsto \mathcal{C}_g$	EXTREME SET	PTOLEMAIC
$I : 2^V \mapsto 2^V$	BOUNDARY SUBSETS	PERFECT
$J : 2^V \mapsto 2^V$	STEINER SETS	ALL GRAPHS

- Every g-convex set is the convex hull of its **contour** [Caceres et alt., 2003].
- Every g-convex set is the geodetic closure of its **boundary** [Pelayo et alt., 2004].
- Every g-convex set of a **chordal graph** is the geodetic closure of its **contour** [Pelayo et alt., 2004].
- Every m-convex set is the monophonic closure of all of its **Steiner sets** [Pelayo et alt., 2004].

- Geodesic and monophonic convexities
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Steiner set problem

$G = (V, E)$, $W \subseteq V$

- *Steiner W-tree T*: tree of minimum order s.t. $W \subseteq V(T)$.

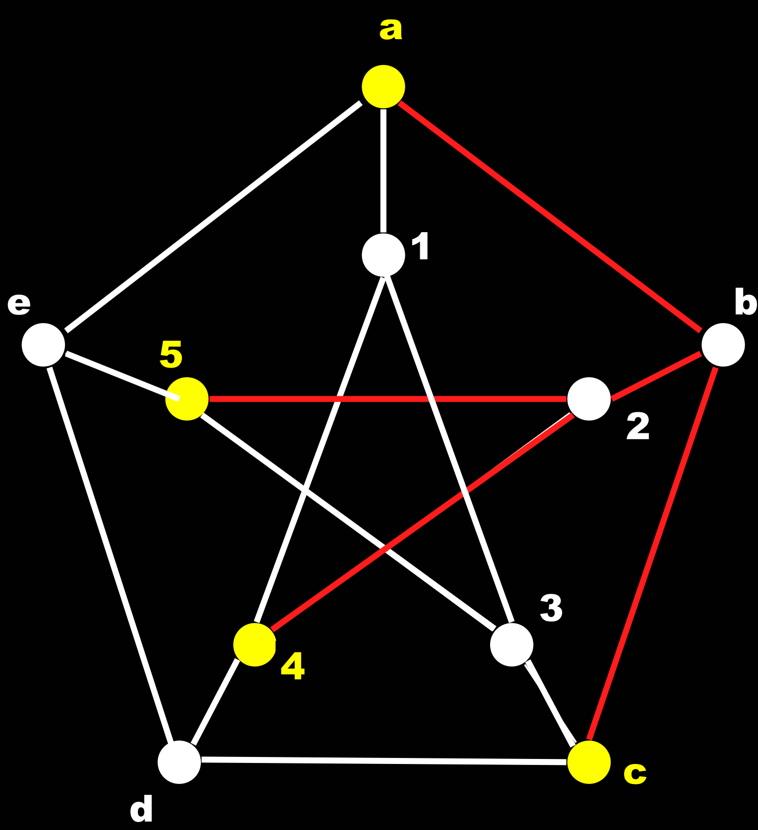
- *Steiner interval $S(W)$*: Consists of all vertices in G that lie on some Steiner W-tree.

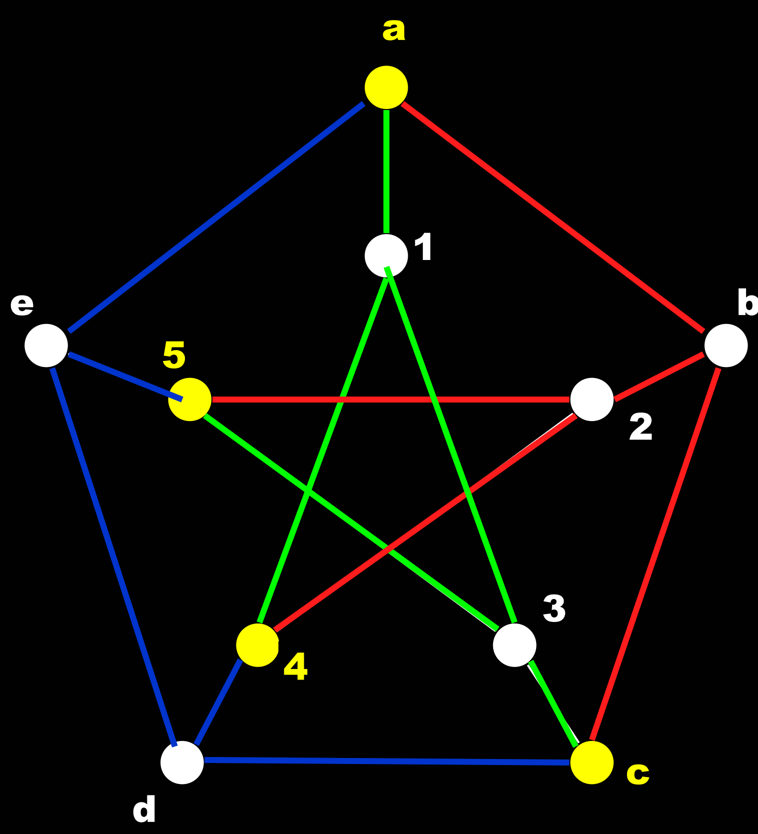
- *Steiner set*: $S(W) = V(G)$.

- *Steiner number*: $st(G) = \min\{|W| : S$ is a Steiner set of $G\}$.

S={a,c,4,5,2,b} is a Steiner tree of W={a,c,4,5}

W={a,c,4,5} is a Steiner set



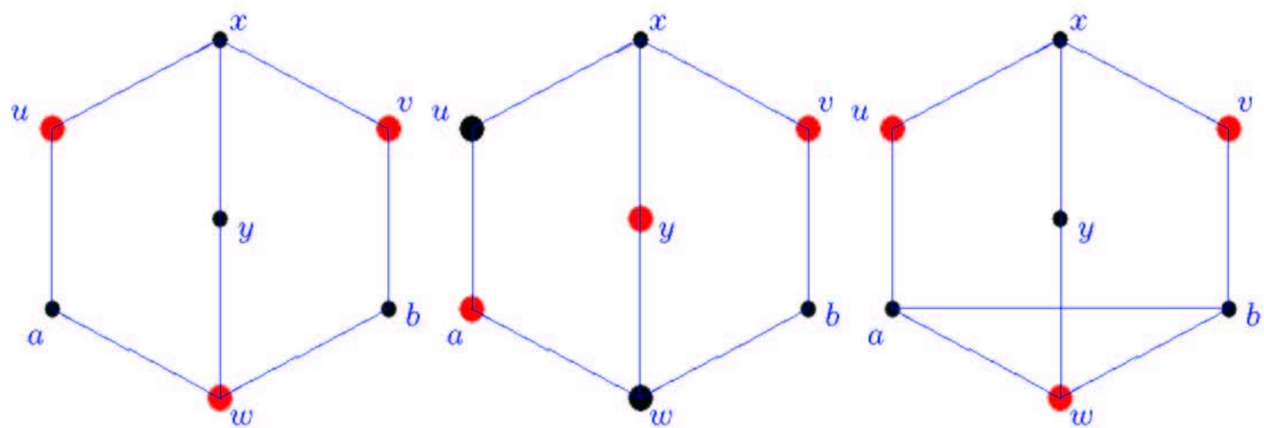


Every Steiner set is geodetic?

- G. Ch. and P. Zh., The Steiner number of a graph, *Disc. Math.* **242**.

Theorem 3.2: **Every Steiner set is geodetic**. Hence, $gn(G) \leq st(G)$.

G	C_n	T_n	K_n	$K_{p,q}$	$W_{1,p}$	Q_n	$K_m \times K_n$	P
$gn(G)$	2,3	|♠|	n	4	$\lceil \frac{p}{2} \rceil$	2	n	4
$st(G)$	2,3	|♠|	n	p	p-2	2	n	4



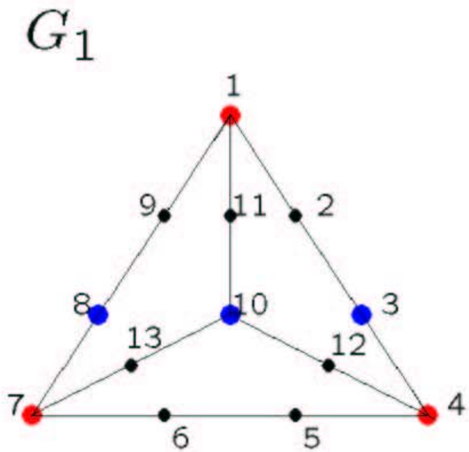
{u,v,w}is a Steiner set and not geodetic

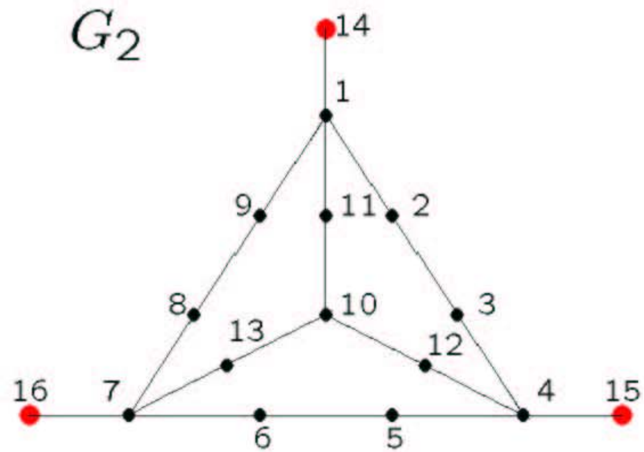
{a,y,v}is both a Steiner set and geodetic

{u,v,w}is a Steiner set and not geodetic

Every geodetic set has at least four vertices

Every Steiner set is a hull set?

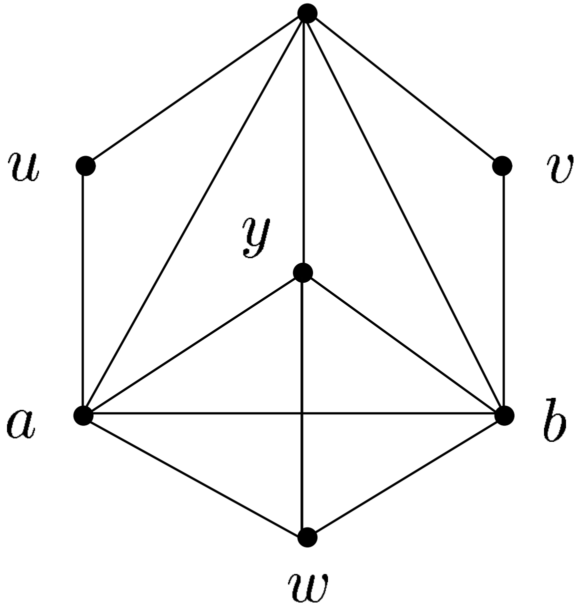




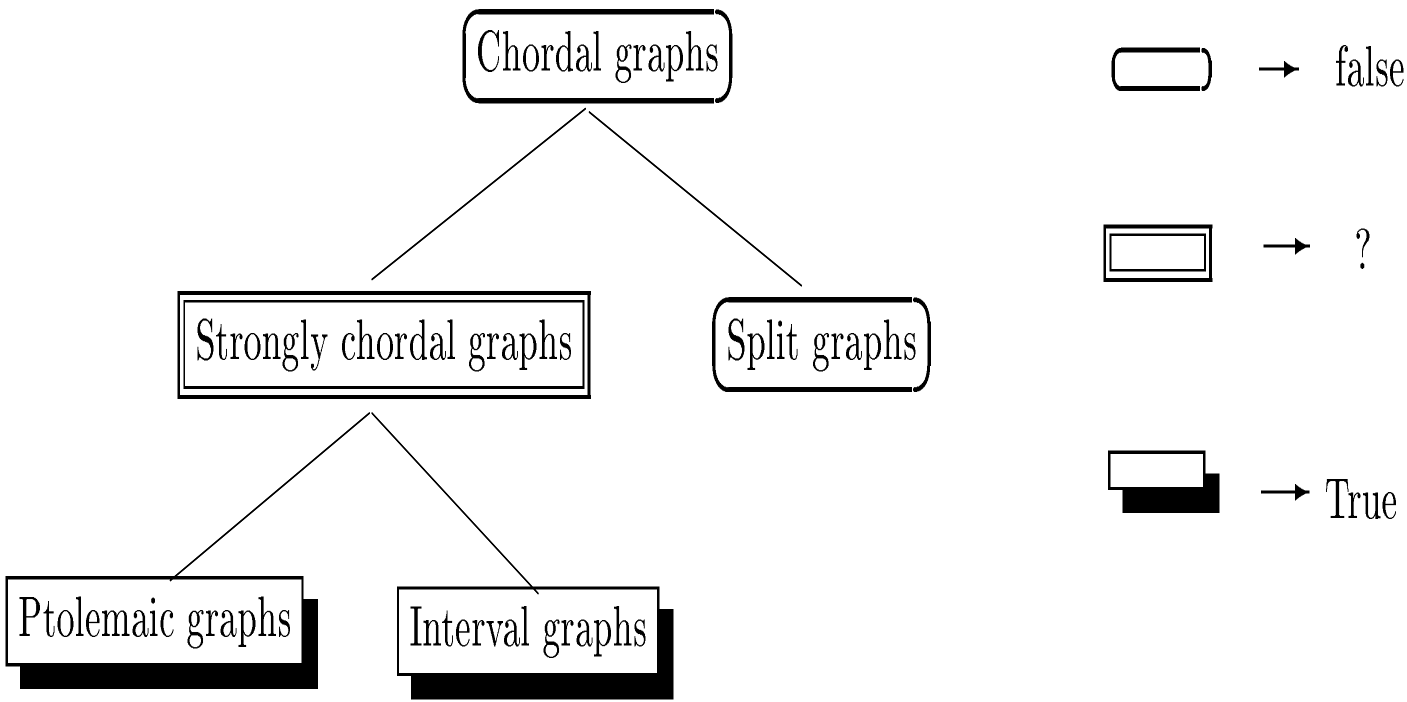
- $W_1 = \{1, 4, 7\}$ is a Steiner set in G_1.
- $I[W_1] = V - \{10, 11, 12, 13\} = [W]_g$ in G_1.
- $hn(G_1) = 3$, as $\Omega = \{3, 8, 10\}$ is a hull set.
- $W_2 = \{14, 15, 16\}$ is a Steiner set in G_2.
- $hn(G_2) = 4$.

Every Steiner set is monophonic

- Every Steiner set is geodetic if G is distance hereditary.
- Every Steiner set is geodetic if G is Ptolemaic.
- Every Steiner set is geodetic if G is an interval graph.
- ◎ Is every Steiner set geodetic if G is an strongly chordal graph ?.
- **Every Steiner set is monophonic**.
- Every edge Steiner set is edge monophonic.



Split graph G satisfying that its vertex subset S={u,v,w} is a Steiner set, but it is not geodetic since I[S]=V(G)-y.



- Geodesic and monophonic convexities
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- **Boundary-type sets**

Boundary-type sets [1]

- v is a locally radial vertex of G if it is a **local maximum of $d(u, *)$ for some u**; i.e., if there exists another vertex $u \in V(G)$ such that no neighbor of v is further away from u than v.

- **Boundary** $\partial(G)$: set of all locally radial vertices.

- v is a radial vertex of G if it is a **global maximum of $d(u, *)$ for some u**; i.e., if there exists another vertex $u \in V(G)$ such that $d(u, v) = ecc(u) = max\{d(u, w) : w \in V\}$.

- **Eccentric set** $Ecc(G)$: set of all radial vertices.

Boundary-type sets [2]

- v is a <u>locally diametral vertex</u> of G if it is a **local maximum of** $ecc(*)$; i.e., if $ecc(v) \geq ecc(u)$, for all $u \in N(v)$.

- **Contour** $Ct(G)$: set of all locally diametral vertices.

- v is a <u>diametral vertex</u> of G if it is a **global maximum of** $ecc(*)$; i.e., if $ecc(v) \geq ecc(u)$, for all $u \in V(G)$.

- **Periphery** $Per(G)$: set of all diametral vertices.

BOUNDARY VERTICES

[•] **Boundary vertices**:

$$\partial(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \forall w \in N(v): \; d(u,w) \leq d(u,v)\}$$

[•] **Eccentric vertices**:

$$Ecc(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \forall w \in V : d(u,w) \leq d(u,v)\} \subseteq \partial(G)$$

[•] **Contour vertices**:

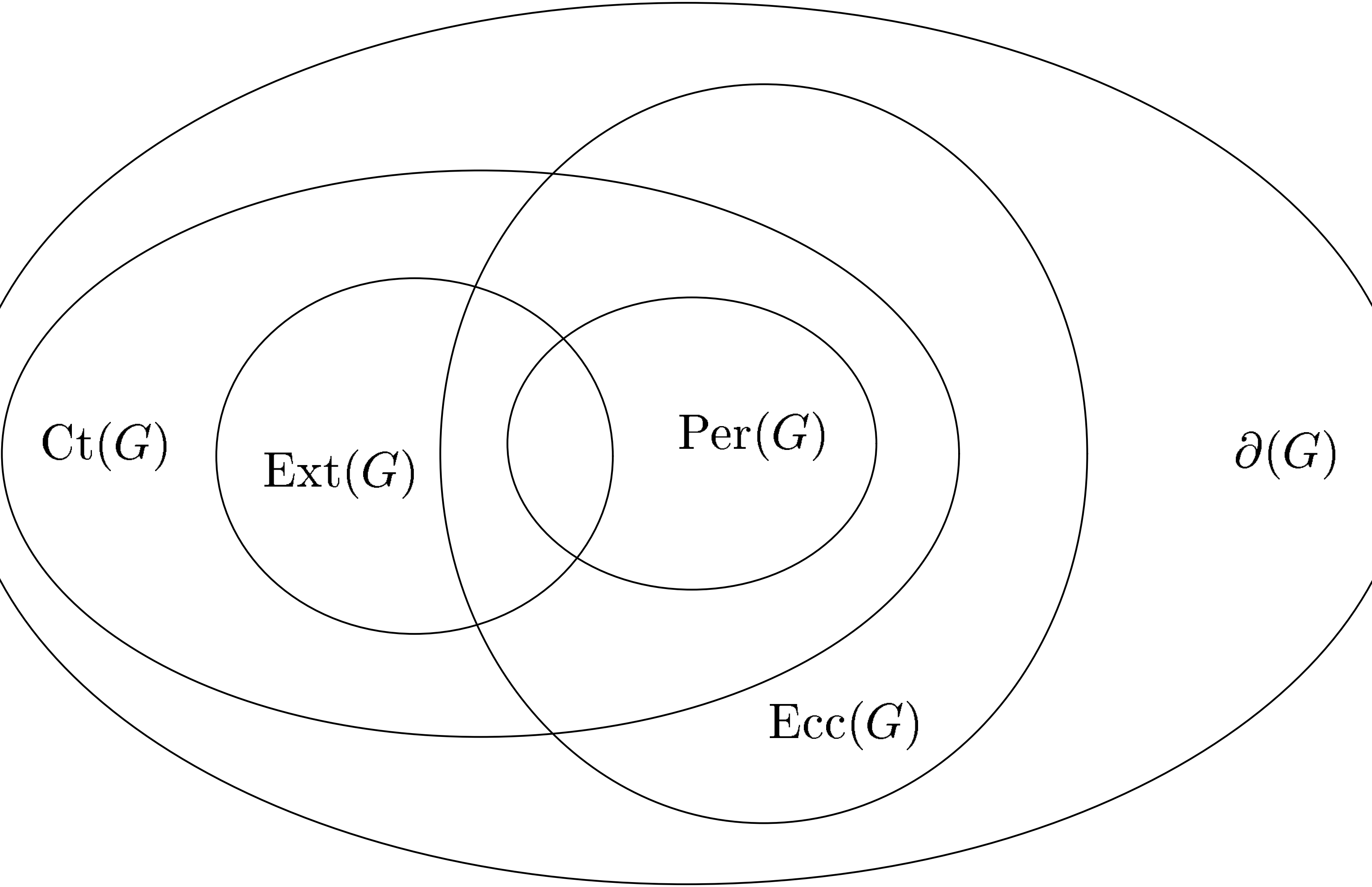
$$Ct(G) = \{v \in V \mid ecc(w) \leq ecc(v), \; \forall w \in N(v)\} \subseteq \partial(G)$$

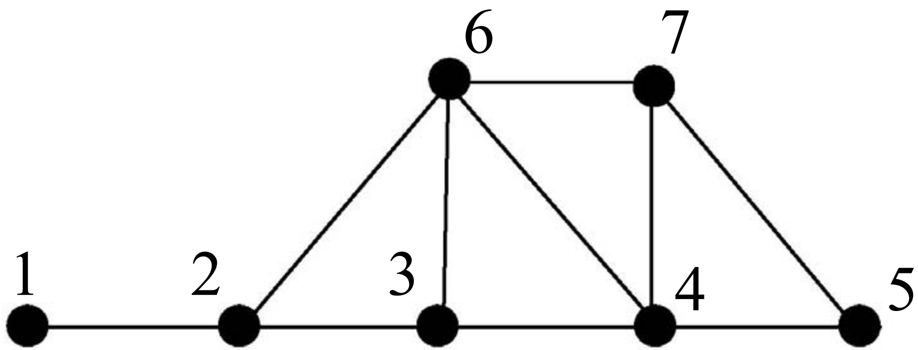
[•] **Peripheral vertices**:

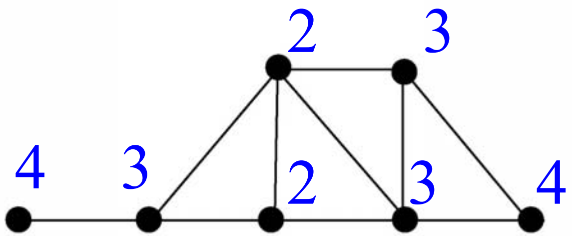
$$Per(G) = \{v \in V \mid ecc(w) \leq ecc(v), \; \forall w \in V\} \subseteq Ct(G) \cap Ecc(G)$$

[•] **Extreme vertices**:

$$Ext(G) = \{v \in V \mid G[N(v)] \text{ is a clique}\} \subseteq Ct(G)$$


Ct(G)
Ext(G)
Per(G)
∂(G)
Ecc(G)





Ext(G)={1,5}

Per(G)={1,5}

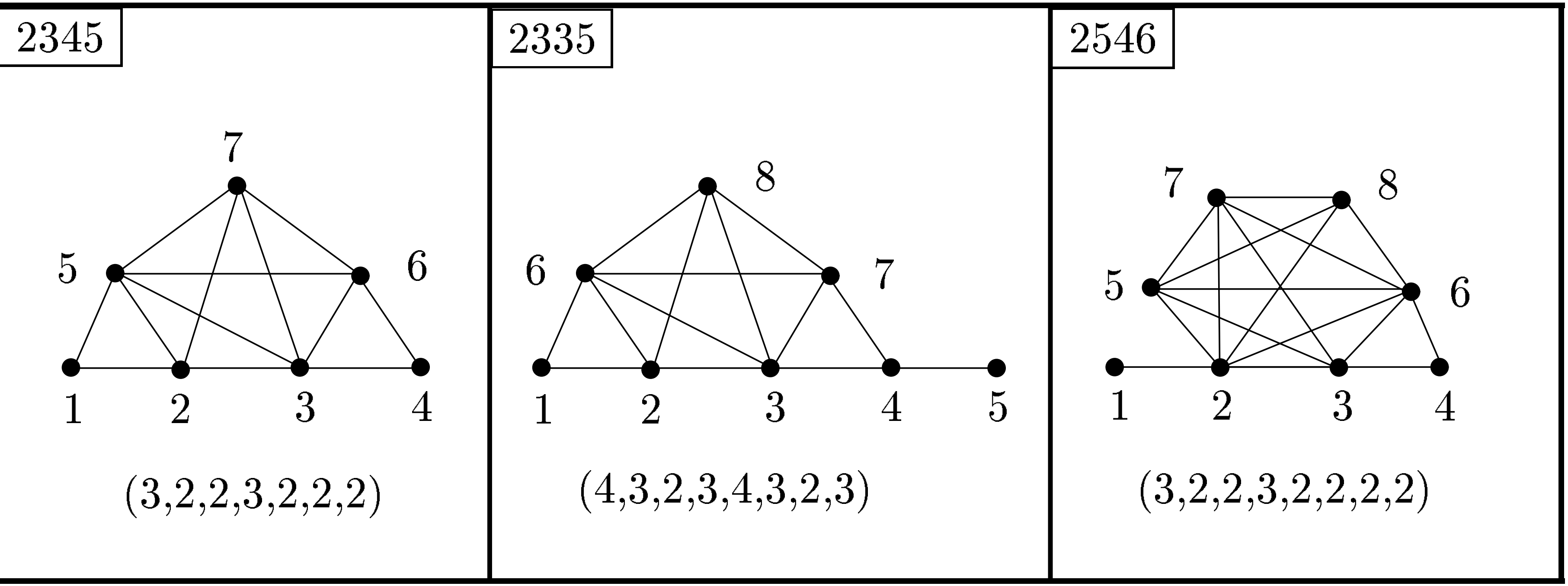
Ct(G)={1,5}

Ecc(G)={1,5,7}

d(7,N(3)) =d(7,3)

→

∂(G)={1,3,5,7}

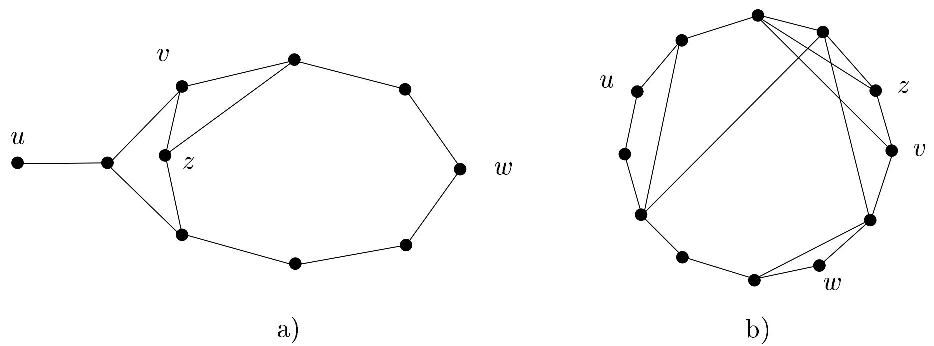


G	Per(G)	Ct(G)	Ecc(G)	∂ (G)
[2345]	{1,4}	{1,4,7}	{1,2,4,6}	{1,2,4,6,7}
[2335]	{1,5}	{1,5,8}	{1,2,5}	{1,2,5,7,8}
[2546]	{1,4}	{1,4,5,7,8}	{1,3,4,8}	{1,3,4,5,7,8}

Krein-Milman-type results [1]

★ The extreme set of every Ptolemaic graph G is a **hull set**. (M. Farber and R. E. Jamison, 1986).

★ The contour of every graph is a **hull set**. (J. Cáceres et alt., 2004).



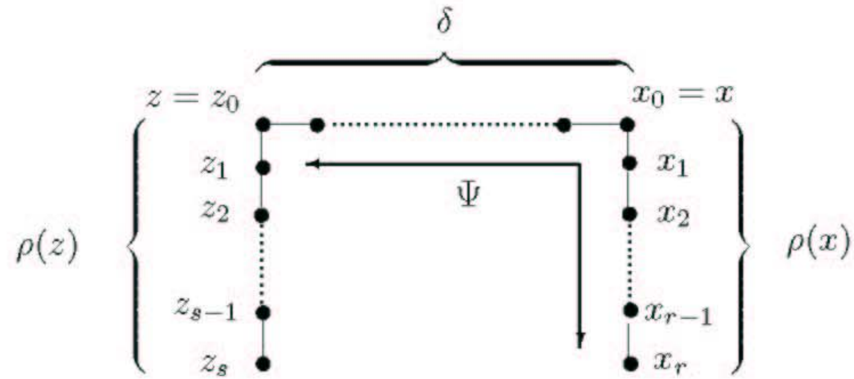
Two graphs with a non geodetic contour: Ct(G)={u,v,w} and I[Ct(G)]=V(G)-z.

★ The contour of every graph is **monophonic**. (I.M.P. et alt., 2004).

The contour of any connected graph G is a monophonic set.

Outline of the **Proof.** Let x be a vertex of $V(G) \setminus Ct(G)$. Hence, there exists a vertex $x_r \in Ct(G)$ and an $x - x_r$ geodesic $\rho(x)$ of length r such that $ecc(x_r) = l + r$ where $l = ecc(x)$.

There exits a vertex z at a distance exactly l from x and $l + r$ from x_r, and x lies on a shortest path $\Psi = z \cdots x x_1 \cdots x_r$ between z and x_r.



Suppose that z is not a contour vertex. Let us construct a path $\rho(z) = z_0 z_1 \cdots z_s$ such that $z = z_0$, $z_i \notin Ct(G)$ for $i \in \{0, \ldots, s-1\}$, $z_s \in Ct(G)$ and $ecc(z_i) = ecc(z_{i-1}) + 1 = ecc(z) + i$ for $i \in \{1, \ldots, s\}$. ■ ■ ■