LECTURE 6

Graphs seen as convexity spaces

Graph-theoretical and classical parameters

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❖ **<u>Graph Convexity Spaces</u>**

❖ Graph-theoretical Parameters (def., ex.)

❖ Independency Parameters (def., ex.)

❖ Independency parameters (prop.)

CONVEXITY SPACE

⋆ V is a non-empty set, $\mathcal{C}$ is collection of V-sets.

• $(V, \mathcal{C})$ is a convexity space if:

(C1) $\emptyset \in \mathcal{C}$, $V \in \mathcal{C}$.

(C2) $\{W_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} W_i \in \mathcal{C}$.

(C3) $\{W_i\}_{i \in I} \subseteq \mathcal{C}$ s.t. $W_i \subseteq W_{i+1} \Rightarrow \bigcup_{i \in I} W_i \in \mathcal{C}$.

CONVEX HULL

$\star$ <u>CONVEX HULL</u> of $S \subset V$:

$$[S]_{\mathcal{C}} = \bigcap\{W : \ S \subseteq W, W \in \mathcal{C}\},$$

$\star$ These conditions are equivalent (J. Schmidt,1952):

(C3) $\{W_i\}_{i \in I} \subseteq \mathcal{C}$ s.t. $W_i \subseteq W_{i+1} \Rightarrow \bigcup\limits_{i \in I} W_i \subseteq \mathcal{C}$.

(C3') $A \subset V$, $x \in [A]_{\mathcal{C}} \Rightarrow x \in [F]_{\mathcal{C}}$ for some finite A-subset F.

(J. Schmidt, 1952) Let $(V, \mathcal{C})$ be a convexity space. Then,

The union of every convex chain is convex $\Leftrightarrow \forall\, A \subseteq V,\ [A]_{\mathcal{C}} = \bigcup_{\substack{F \subseteq A \\ F\,finite}} [F]_{\mathcal{C}}.$

Proof.

$\Leftarrow)$: Take $\{A_i\}_{i \in I} \in \mathcal{P}(\mathcal{C})$ s. t., for every $i \in I$, $A_i \subseteq A_{i+1}$.

If $D = \bigcup_{i \in I} A_i$, then $[D]_{\mathcal{C}} = \bigcup_{\substack{F \subseteq D \\ F\,finite}} [F]_{\mathcal{C}}.$

If F is a finite D-subset, then there exists an $i \in I$ s.t. $F \subseteq [F]_{\mathcal{C}} \subseteq A_i$.

Hence, $[D]_{\mathcal{C}} = \bigcup_{\substack{F \subseteq D \\ F\,finite}} [F]_{\mathcal{C}} \subseteq \bigcup_{i \in I} A_i = D$

$\Rightarrow$): Take $A \subseteq V$. To prove that $[A]_\mathcal{C} = \bigcup_{\substack{F \subseteq A \\ F \, finite}} [F]_\mathcal{C}$ we proceed by transfinite induction on the cardinality $|A|$ of A.

Certainly this equality is true for finite sets. Suppose that the formula holds for sets of smaller cardinality than A.

Well-order A such that, for every $a \in A$, the set $P(a) = \{x \in A : x < a\}$ has a cardinality strictly less than S. Notice that $A = \bigcup_{a \in A} P(a)$.

Take the chain $\{[P(a)]_\mathcal{C}\}_{a \in A}$. By hypothesis, its union $D = \bigcup_{a \in A}[P(a)]_\mathcal{C}$ is a convex set, and thus we have that $D = [A]_\mathcal{C}$.

By the the inductive assumption, $[P(a)]_\mathcal{C} = \bigcup_{\substack{F \subseteq P(a) \\ F \, finite}} [F]_\mathcal{C}$.

As each finite set in A is inside some $P(a)$, the assertion holds for A. $\blacksquare$

SOME EXAMPLES

- $V \neq \emptyset$, $\mathcal{C} = \{\emptyset, V\}$
- $V \neq \emptyset$, $\mathcal{C} = 2^V$

- $V = \mathbb{N}$, $k \geq 1$, $\mathcal{C} = \{A \subset V : |A| \leq k\} \cup \{V\}$

- Standard convexity in a K-vector space V (K totally ordered):

  $C \subseteq V$ is convex $\Leftrightarrow \forall x, y \in C$, $\forall t \in [0, 1]$: $t \cdot x + (1 - t) \cdot y \in C$.

- Order convexity in a poset $(V, \leqslant)$:

  $C \subseteq V$ is order convex $\Leftrightarrow \forall x, y \in C$: $x \leqslant z \leqslant y \Rightarrow z \in C$.

- Metric convexity in a metric space (V, d):

  $A \in \mathcal{C} \Leftrightarrow \forall x, y \in A$, $\{z \in V : d(x, z) + d(z, y) = d(x, y)\} \subseteq A$

- $V = \mathbb{R}^n$, euclidean convexity (resp. taxicab convexity).

GRAPH CONVEXITY SPACE

⋆ $G = (V, E)$ is a connected graph and $\mathcal{C} \subseteq 2^V$.

• $(V, \mathcal{C})$ is a <u>GRAPH CONVEXITY SPACE</u> if:

(C1) $\emptyset \in \mathcal{C}$, $V \in \mathcal{C}$.

(C2) $\{W_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} W_i \in \mathcal{C}$.

(C3) $\{W_i\}_{i \in I} \subseteq \mathcal{C}$ s.t. $W_i \subseteq W_{i+1} \Rightarrow \bigcup_{i \in I} W_i \in \mathcal{C}$.

(C4) For every $U \in \mathcal{C}$, $\langle U \rangle_G$ is connected.

PATH CONVEXITIES

- A path convexity $(V, \mathcal{C})$ of a graph $G = (V, E)$ is a type of interval convexity defined by a set $\mathcal{P}$ of paths in G containing all shortest paths:

Given $u, v \in V$, the $\mathcal{P}$-interval $\mathcal{P}(u, v)$ is the set of all vertices w belonging to paths from $\mathcal{P}$ joining u and v.

▶ *Coarsest convexity*: $\mathcal{P}$ is the set of all paths.

▶ *Triangle-path convexity*: $\mathcal{P}$ is the set of all triangle paths.

▶ *Total convexity*: $\mathcal{P}$ is the set of all triangle free paths.

▶ *Monophonic convexity*: $\mathcal{P}$ is the set of all induced paths.

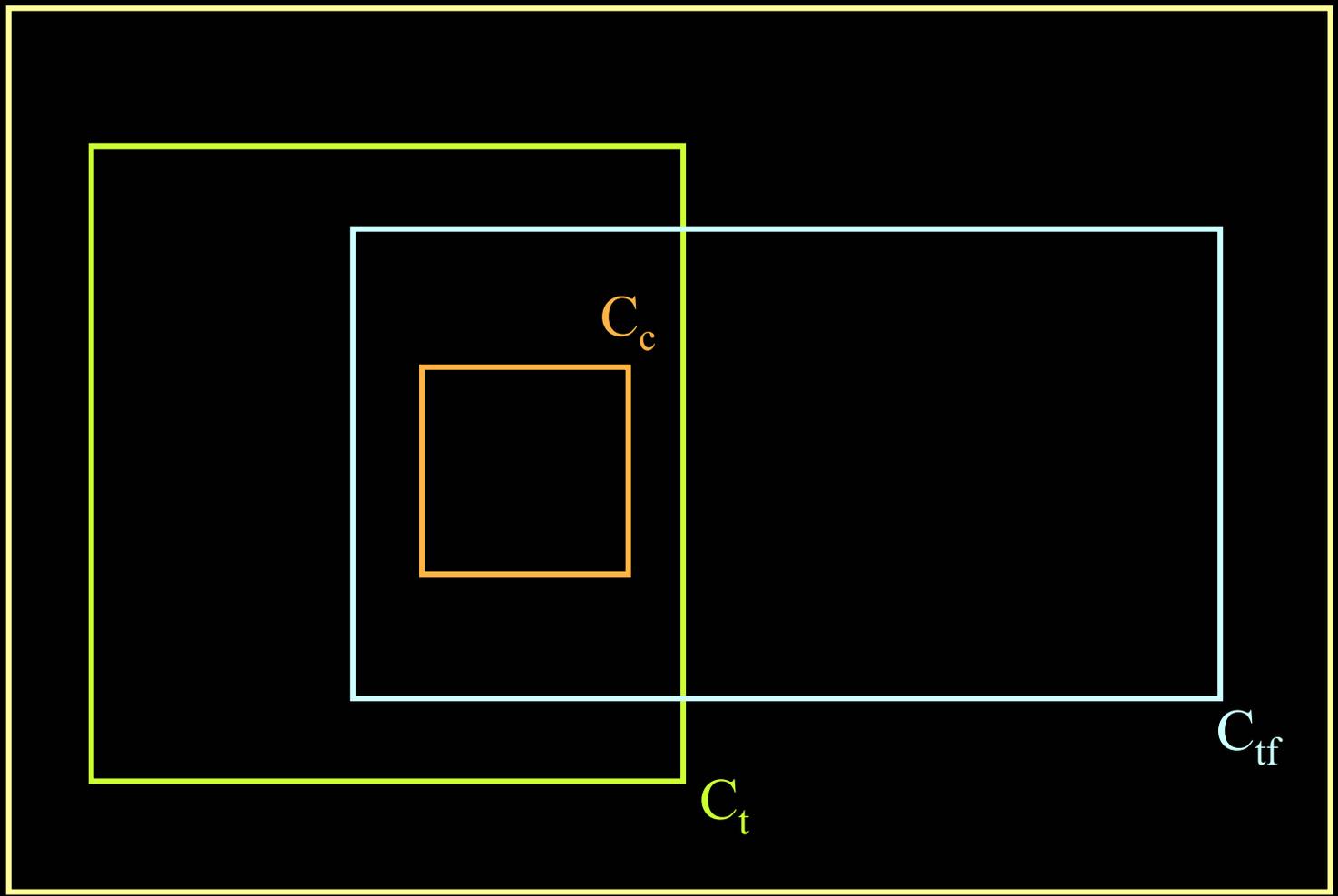▶ *Geodesic convexity*: $\mathcal{P}$ is the set of all shortest paths.
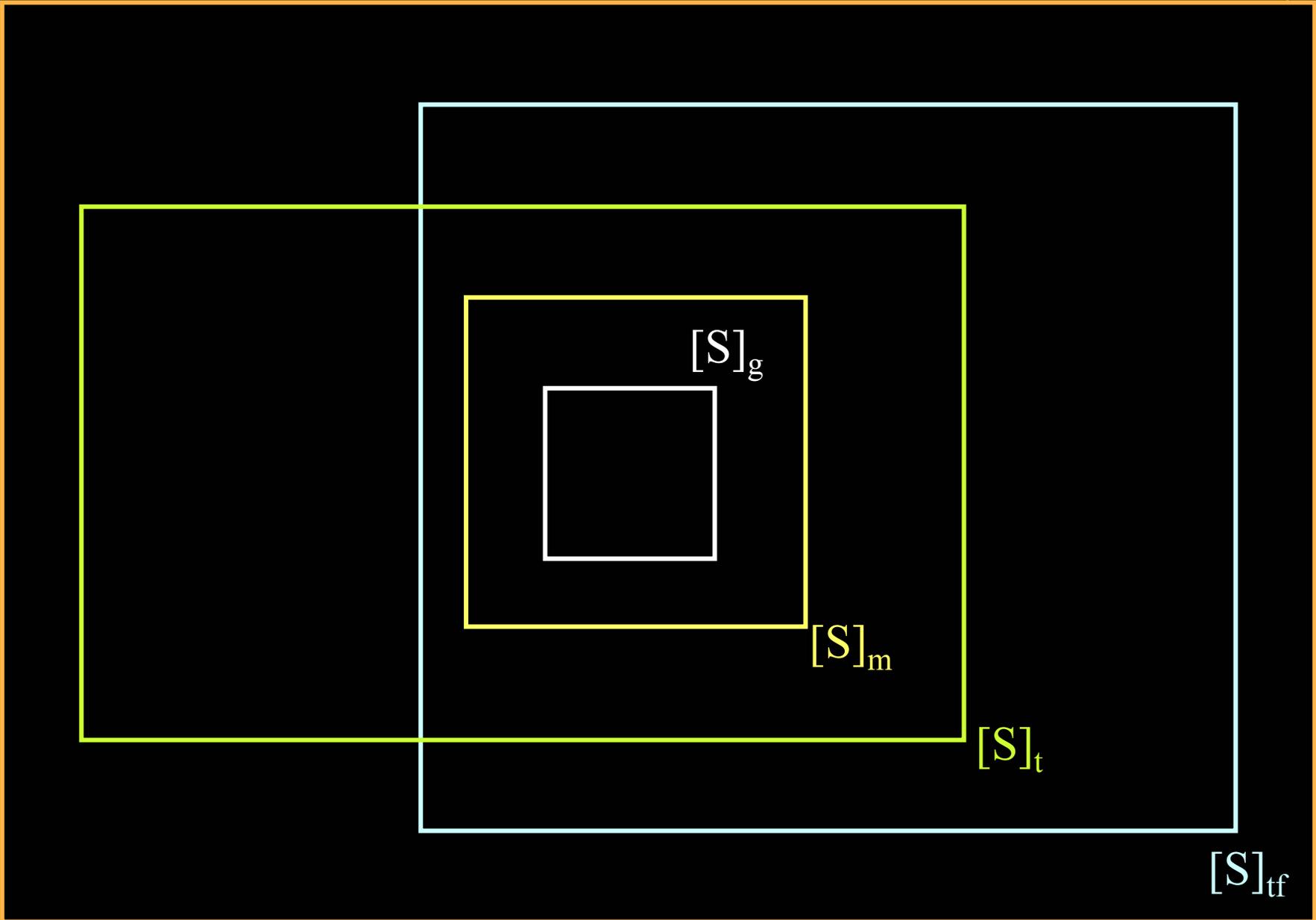
$[S]_c$

$[S]_{tf}$

$[S]_t$
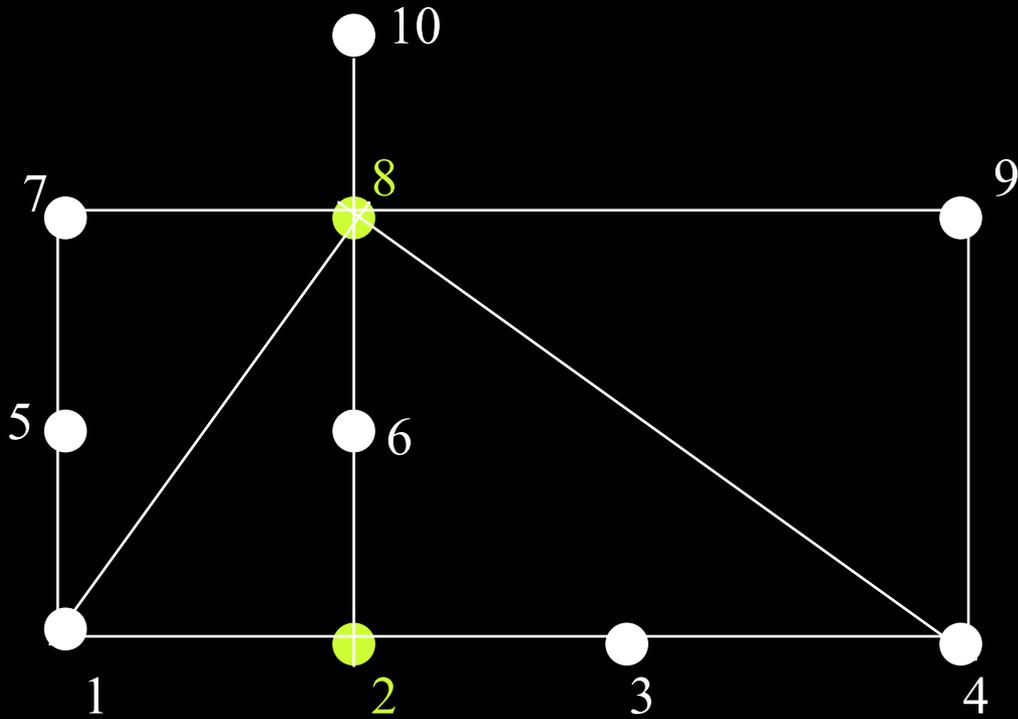
$[S]_m$

$[S]_g$

S={2,8}

[S]_g ={1,2,6,8}

[S]_m = [S]_g ∪ {3,4}

[S]_t = [S]_m ∪ {9}

[S]_tf = [S]_m ∪ {5,7}

[S]_c = V-10

GEODESIC CONVEXITY

$\star$ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_g \subseteq 2^V$.

▶ A $u - v$ geodesic is a $u - v$ path of minimum length.

▶ Closed interval: $I[u, v] = \{V(\rho) : \rho$ is a $u - v$ geodesic$\}$

▶ Geodetic closure: $I[S] = \bigcup_{u,v \in S} I[u, v]$

▶ g-convex set: $S \in \mathcal{C}_g \Leftrightarrow S = I[S]$.

▶ g-convex hull: $S \subseteq I[S] \subseteq I^2[S] \subseteq \ldots \subseteq I^r[S] = [S]_g \subseteq V$

$\star$ $(V, \mathcal{C}_g)$ is an interval, path and metric convexity space.

MONOPHONIC CONVEXITY

$\star$ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_m \subset 2^V$.

▶ A $u - v$ monophonic path is a $u - v$ chordless path.

▶ Closed interval: $J[u, v] = \{V(\rho) : \rho$ is a $u-v$ monophonic path$\}$

▶ Monophonic closure: $J[S] = \bigcup_{u,v \in S} J[u, v]$

▶ m-convex set: $S \in \mathcal{C}_m \Leftrightarrow S = J[S]$.

▶ m-convex hull: $S \subseteq J[S] \subseteq J^2[S] \subseteq \ldots \subseteq J^r[S] = [S]_m \subseteq V$

$\star$ $(V, \mathcal{C}_m)$ is an interval and path convexity space.

❖ Graph Convexity Spaces

❖ **Graph-theoretical Parameters (def.,ex.**

❖ Independency Parameters (def., ex.)

❖ Independency parameters (prop.)

DOMINATION PARAMETERS (G-CONVEXITY)

$\star$ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_g)$ g-convexity space.

▶ Geodetic set: $I[S] = V$

⊗ Geodetic number: $gn(G) = \min\{|S| : S$ is a geodetic set of $G\}$

▶ Hull set: $[S]_g = V$.

⊗ Hull number: $hn(G) = \min\{|S| : S$ is a hull set of $G\}$

↪ $hn(G) \leq gn(G)$

OTHER PARAMETERS (G-CONVEXITY)

$\star$ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_g)$ g-convexity space.

▶ g-convex set: $I[S] = S$

⊗ Convexity number: $con(G) = \max\{|S| : S \subsetneq V,\ S \text{ is convex}\}$

▶ $S \subsetneq I[S] \subsetneq \ldots \subsetneq I^r[S] = I^{r+1}[S] = [S]_g \subseteq V \ \Rightarrow \ gin(S) = r$

⊗ Geodetic iteration number: $gin(G) = \max\{gin(S) : S \subseteq V\}$

DOMINATION PARAMETERS (M-CONVEXITY)

$\star$ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_m)$ m-convexity space.

▶ Monophonic set: $J[S] = V$

⊗ Monophonic number: $mn(G) = \min\{|S| : S \subseteq V \text{ is monophonic}\}$

▶ m-Hull set: $[S]_m = V$.

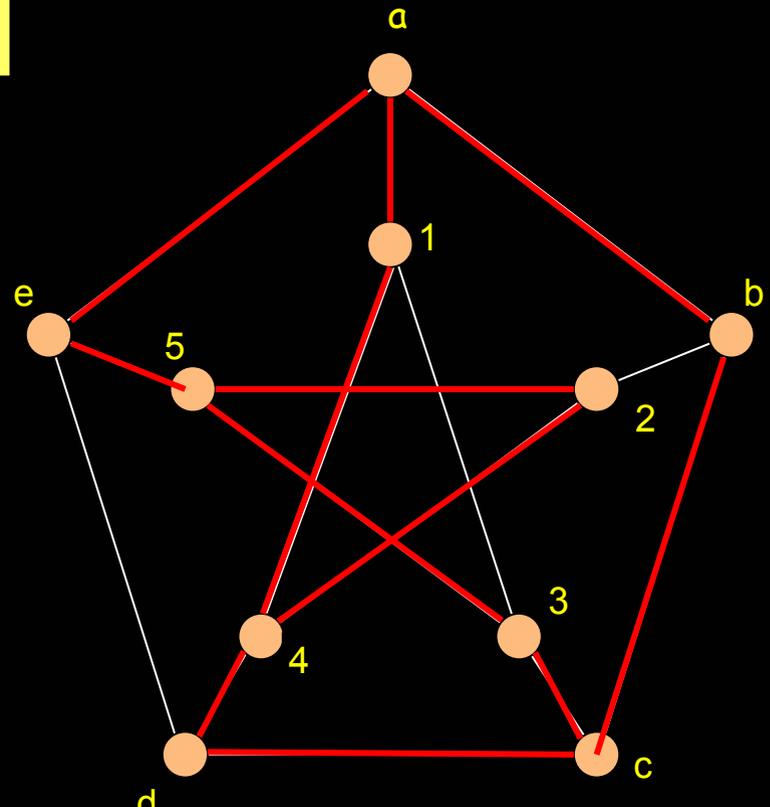⊗ m-Hull number: $hn(G) = \min\{|S| : S \text{ is an } m - \text{hull set of } G\}$

↪ $mhn(G) \leq mn(G)$

↪ $mn(G) \leq gn(G)$

PETERSEN GRAPH

n	m	Δ	D	g
10	15	3	2	5

con	gn	hn	gin	mn
5	4	3	3	3

S={a,c,4,5} is geodetic

❖ Graph Convexity Spaces

❖ Graph-theoretical Parameters (def., ex.)

❖ **Independency Parameters (def., ex.)**

❖ Independency parameters (prop.)

SOME TYPES OF DEPENDENCE

$\star$ $(V, \mathcal{C})$ convexity space, $S \subseteq V$ is:

▶ <u>convexly dependent</u> if there exist $x \in S$ s.t. $[S]_\mathcal{C} = [S-x]_\mathcal{C}$.

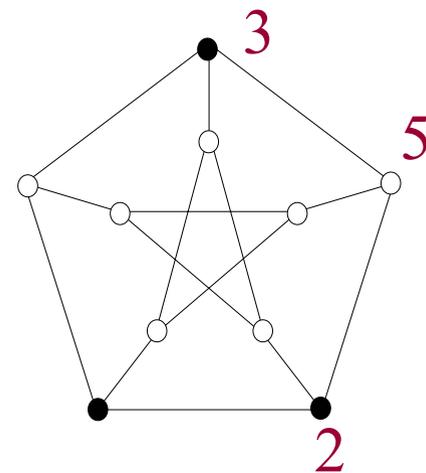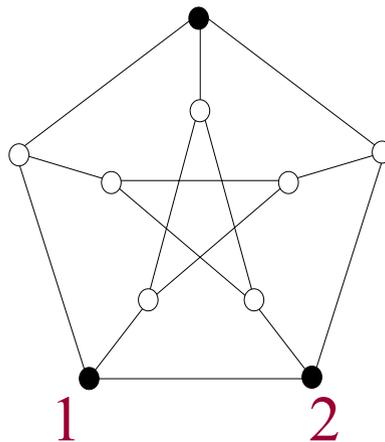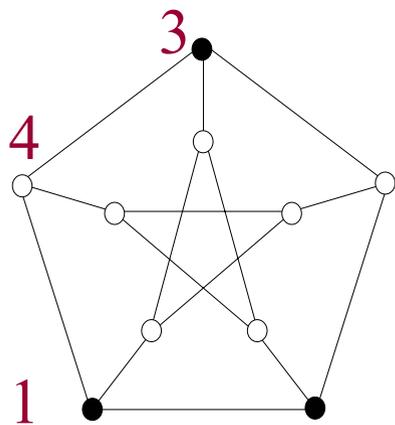▶ <u>redundant</u> if $[S]_\mathcal{C} = \bigcup_{a \in S} [S - a]_\mathcal{C}$.

▶ <u>Helly dependent</u> if $\bigcap_{a \in A} [S - a]_\mathcal{C} \neq \emptyset$.

▶ <u>exchange dependent</u> if for all $x \in S$, $[S-x]_\mathcal{C} \subseteq \bigcup_{a \in S-x} [S-a]_\mathcal{C}$.

[1,3]={1,4,3}   [1,2]={1,2}   [2,3]={2,5,3}

[1,2,3]={1,2,3,4,5}
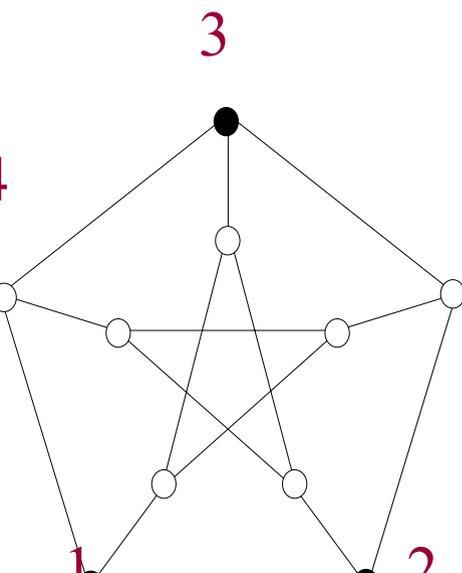
$[1,2,3] = [1,3]$ NO

$[1,2,3] = [1,2]$ NO

5   $[1,2,3] = [2,3]$ NO

} convex indep.

$[1,3] \bigcup [1,2] \bigcup [2,3] = [1,2,3]$ SI   redundant

$[1,3] \bigcap [1,2] \bigcap [2,3] = \emptyset$ SI   Helly indep.

CLASSICAL PARAMETERS

IN EVERY CASE, $x(G)$ is the maximum cardinality of ...

$r(G)$: ...of a convexly independent set [Rank].

$c(G)$: ...of an irredundant set [Caratheodory number].

$h(G)$: ...of a Helly independent set [Helly number].

$e(G)$: ...of a exchange independent set [Exchange number].

Convexity parameters of some convexity spaces

X	cx	c	e	h	r	con	gn	hn	gin	mn	Obs.
R^2	d_2	3	3	3	∞	∞	∞	∞	2	ns	
R^3	d_2	4	4	4	∞	∞	∞	∞	3	ns	
R^n	d_2	n+1	n+1	n+1	∞	∞	∞	∞	n	ns	
R^2	d_1	2	3	2	4	∞	∞	∞	1	ns	
R^3	d_1	3	4	2	6	∞	∞	∞	1	ns	
R^n	d_1	n	n+1	2	2n	∞	∞	∞	1	ns	
$N(k)$	$k\geq3$	k+1	2	k+1	k+1	k	ns	k+1	ns	ns	
K_1	g	1	0	1	1	0	1	1	0	1	dh
K_2	g	1	2	2	2	1	2	2	0	2	dh
K_n	g	1	2	n	n	n-1	n	n	0	n	dh
C_4	g	2	3	2	2	2	2	2	1	2	dh
C_{2h}	g	2	3	3	3	h	2	2	1	2	$h\geq3$
C_3	g	1	2	3	3	2	3	3	0	3	dh
C_{2h+1}	g	2	3	3	3	h+1	3	3	1	3	$h\geq2$
T_n	g	2	2	2	♠	n-1	♠	♠	1	♠	dh, $n\geq3$
$K_{p,q}$	g	2	3	2	2	2	min{4,p}	2	2	min{4,p}	dh, $2\leq p\leq q$, $q\geq3$
$W_{1,p}$	g					p-1	ceil(p/2)	ceil(p/2)	1	3	$p\geq4$
Q_n	g	n	n+1	2	n	2^{n-1}	2	2	1	2	$n\geq2$
P	g					5	4	3	3	3	

notation: cx: class of convexity, dh: distance hereditary, $N(k)$: cx:=subsets in N of cardinality at most k, ns:nonsense

❖ Graph Convexity Spaces

❖ Graph-theoretical Parameters (def., ex.)

❖ Independency Parameters (def., ex.)

❖ **Independency parameters (prop.)**

SOME PROPERTIES

$\star$ $(V, \mathcal{C})$ convexity space.

▶ Every convexly dependent set is redundant. Hence, $c \leq r$.

▶ Every convexly dep. set is Helly dependent. Hence, $h \leq r$.

▶ Convex independency is an hereditary property.

▶ Helly independency is an hereditary property.

▶ $e - 1 \leq c \leq \max\{h, e - 1\}$ (Sierksma,1977).

▶ Interval convexities: $e - 1 \leq c \leq e$ (Duchet, I.M.P., 2005)

Every convexly dependent set is both redundant and Helly dependent.

Proof.

Let $S \subseteq V$ be a convex dependent set, i.e., $[S]_\mathcal{C} = [S - x]_\mathcal{C}$ for some $x \in S$.

▶ In particular, $[S]_\mathcal{C} = \bigcup_{s \in S}[S - s]_\mathcal{C}$, i.e., S is redundant.

▶ Observe that $x \in S - s$, for every $s \in S - x$ and thus: $x \in \bigcap_{s \in S - x}[S - s]_\mathcal{C}$.

Notice also that $x \in S \subseteq [S]_\mathcal{C} = [S - x]_\mathcal{C}$.

In consequence, $x \in \bigcap_{s \in S}[S - s]_\mathcal{C} \neq \emptyset$, i.e., S is Helly dependent. ∎

Both convex and Helly independence are hereditary properties.

Proof.

Take $B \subset A$.

▶ If B is convexly dependent, then $x \in [B - x]_\mathcal{C}$ for some $x \in B$.

Hence, $x \in [A - x]_\mathcal{C}$, since $[B - x]_\mathcal{C} \subseteq [A - x]_\mathcal{C}$.

▶ If B is Helly dependent, then $\bigcap_{b \in B}[B - b]_\mathcal{C} \neq \emptyset$.

Observe that $\bigcap_{b \in B}[B - b]_\mathcal{C} \subseteq \bigcap_{b \in B}[A - b]_\mathcal{C}$.

Note also that, for every $a \in A - B$, $\bigcap_{b \in B}[B - b]_\mathcal{C} \subseteq [B]_\mathcal{C} \subseteq [A - a]_\mathcal{C}$. Hence:

$$\emptyset \neq \bigcap_{b \in B}[B - b]_\mathcal{C} \subseteq \left(\bigcap_{b \in B}[A - b]_\mathcal{C}\right) \cap \left(\bigcap_{a \in A - B}[A - a]_\mathcal{C}\right) = \bigcap_{a \in A}[A - a]_\mathcal{C}. \quad \blacksquare$$

(G. Sierksma, 1977) $c \leq \max\{h, e - 1\}$.

Proof.

It suffices to prove that every finite set $S \subseteq V$ s.t. $\max\{h + 1, e\} \leq |S|$ is redundant.

As $h + 1 \leq |S|$, then $\bigcap_{s \in S}[S - s]_C \neq \emptyset$. Take $p \in \bigcap_{s \in S}[S - s]_C$.

If $p \in S$, then $[S]_C = [S - p]_C$ and we are done.

Suppose thus that $p \notin S$ and consider the set $T = S + p$, which satisfies $|T| = |S| + 1 \geq e + 1$.

Hence,

$$[S]_C \subseteq \bigcup_{s \in S}[S + p - s]_C = \bigcup_{s \in S}[S - s]_C,$$

since $p \in [S - s]_C \Leftrightarrow [S + p - s]_C = [S - s]_C$. ∎

(G. Sierksma, 1977) $e - 1 \leq c$.

Proof. Corollary of Lemma below. ∎

Lemma Let $(V, \mathcal{C})$ be a convexity space and $k \geq 2$. Then, $(i) \Rightarrow (ii)$, where:

(i) Every k-subset of V is redundant.

(ii) Every $(k+1)$-subset of V is exchange dependent.

Proof.

Take $S \subseteq V$ s.t. $|S| = k + 1$. Take $x \in S$.

Consider the set $T = S - x$. Observe that T is redundant , since $|T| = k$.

Hence, $[T]_{\mathcal{C}} = \bigcup_{t \in T}[T - t]_{\mathcal{C}}$.

In consequence, $[S - x]_{\mathcal{C}} = \bigcup_{s \in S - x}[S - x - s]_{\mathcal{C}} \subseteq \bigcup_{s \in S - x}[S - s]_{\mathcal{C}}$. ∎

(P. Duchet, 2006) In every interval convexity: $c \leq e$.

Proof. Corollary of Lemma below. ∎

Lemma Let $(V, \mathcal{C})$ be a convexity space and $k \geq 2$.

Then, $(i) \Rightarrow (ii) \Rightarrow (iii) \Rightarrow (iv)$, where:

(i) Every k-subset of V is redundant.

(ii) **Every $(k+1)$-subset of V is exchange dependent.**

(iii) **Every $(k+1)$-subset of V is redundant.**

(iv) Every $(k+2)$-subset of V is exchange dependent.