Quasiperfect Dominating Codes in Graphs [1]

Ignacio M. Pelayo[2]

[2]Department de Matemàtica Aplicada III
Universitat Politècnica de Catalunya
Barcelona, Catalunya, Spain

9th International colloquium on graph theory and combinatorics

University Joseph Fourier, Grenoble, France

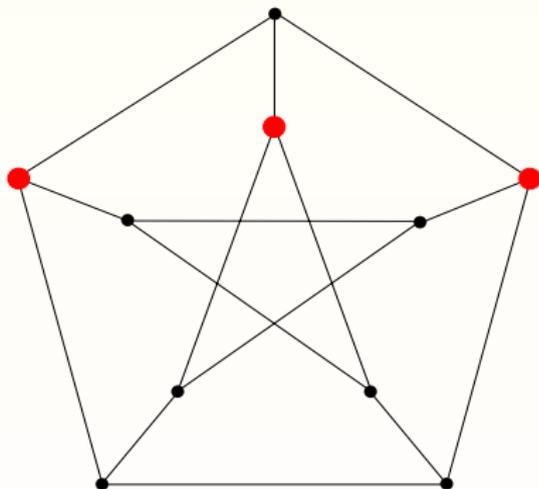[1]Joint work with José Cáceres, Carmen Hernando, Mercè Mora and Maria Luz Puertas.

1. QP-dominating sets, codes and parameters.

2. $\Delta \notin \{4, \ldots, n-4\}$

3. Trees

4. Cographs

5. Claw-free graphs

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in D, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in D, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.
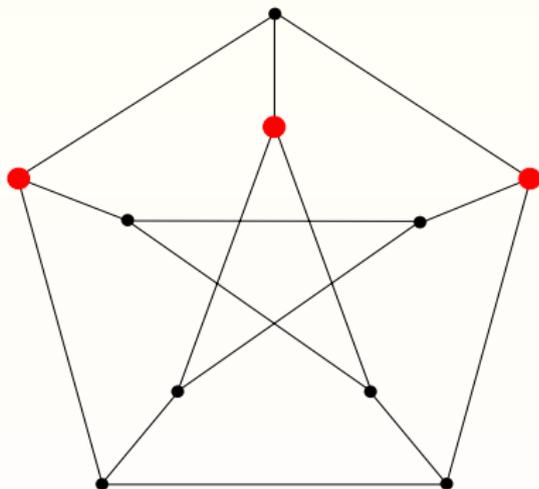
▷ The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in D, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.

▷ The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

▷ A dominating set of cardinality $\gamma(G)$ is called a γ-*code*.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in D, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.

▷ The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

▷ A dominating set of cardinality $\gamma(G)$ is called a γ-*code*.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex of $V \setminus D$ has a neighbour in D, i.e., for each $u \in V \setminus D$, $N(u) \cap D \neq \emptyset$.

▷ The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

▷ A dominating set of cardinality $\gamma(G)$ is called a γ-*code*.
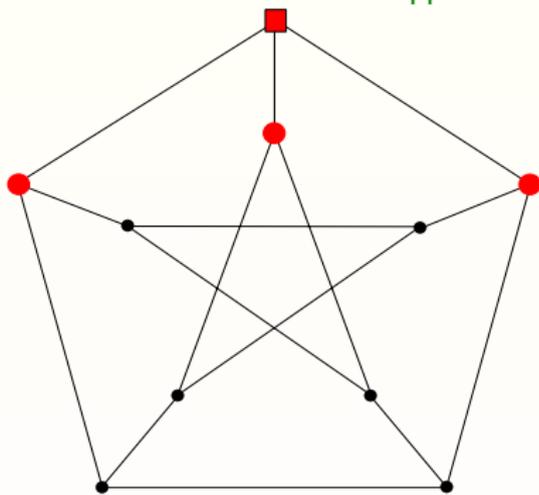

● $\gamma(P) = 3$, since red vertices form a γ-code.

▷ A dominating set $D \subset V(G)$ of a graph G is a *perfect dominating set* if every vertex of $V \setminus D$ has exactly a neighbour in D, i.e., for each $u \in V \setminus D$, $|N(u) \cap D| = 1$.

▷ A dominating set $D \subset V(G)$ of a graph G is a *perfect dominating set* if every vertex of $V \setminus D$ has exactly a neighbour in D, i.e., for each $u \in V \setminus D$, $|N(u) \cap D| = 1$.

▷ The *perfect domination number*, denoted by $\gamma_{11}(G)$, is the minimum cardinality of a perfect dominating set of G.

▷ A dominating set $D \subset V(G)$ of a graph G is a *perfect dominating set* if every vertex of $V \setminus D$ has exactly a neighbour in D, i.e., for each $u \in V \setminus D$, $|N(u) \cap D| = 1$.
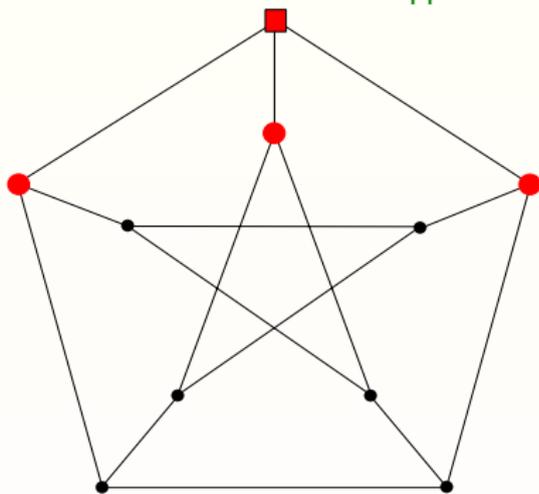
▷ The *perfect domination number*, denoted by $\gamma_{11}(G)$, is the minimum cardinality of a perfect dominating set of G.

▷ A perfect dominating set of cardinality $\gamma_{11}(G)$ is called a γ_{11}-*code*.

▷ A dominating set $D \subset V(G)$ of a graph G is a *perfect dominating set* if every vertex of $V \setminus D$ has exactly a neighbour in D, i.e., for each $u \in V \setminus D$, $|N(u) \cap D| = 1$.

▷ The *perfect domination number*, denoted by $\gamma_{11}(G)$, is the minimum cardinality of a perfect dominating set of G.

▷ A perfect dominating set of cardinality $\gamma_{11}(G)$ is called a γ_{11}-*code*.

▷ A dominating set $D \subset V(G)$ of a graph G is a *perfect dominating set* if every vertex of $V \setminus D$ has exactly a neighbour in D, i.e., for each $u \in V \setminus D$, $|N(u) \cap D| = 1$.

▷ The *perfect domination number*, denoted by $\gamma_{11}(G)$, is the minimum cardinality of a perfect dominating set of G.

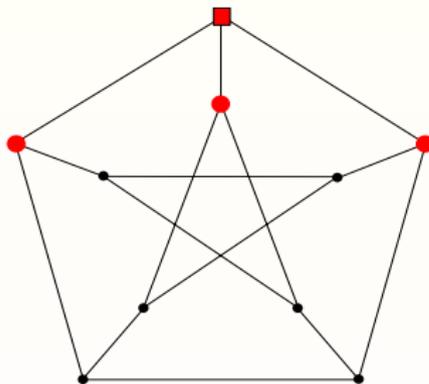▷ A perfect dominating set of cardinality $\gamma_{11}(G)$ is called a γ_{11}-*code*.

● $\gamma_{11}(P) = 4$, since red vertices form a γ_{11}-code.

▷ A dominating set $D \subset V(G)$ of a graph G is a *k-quasiperfect dominating set* if every vertex of $V \setminus D$ has at most k neighbours in D, i.e., for each $u \in V \setminus D$, $1 \leq |N(u) \cap D| = k$.

▷ A dominating set $D \subset V(G)$ of a graph G is a *k-quasiperfect dominating set* if every vertex of $V \setminus D$ has at most k neighbours in D, i.e., for each $u \in V \setminus D$, $1 \leq |N(u) \cap D| = k$.

▷ The *k-quasiperfect perfect domination number*, denoted $\gamma_{1k}(G)$, is the minimum cardinality of a *k*-quasiperfect dominating set of G.
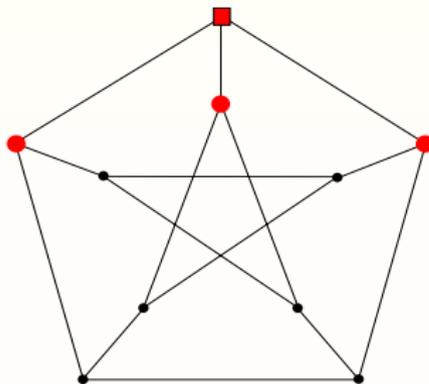
▷ A dominating set $D \subset V(G)$ of a graph G is a *k-quasiperfect dominating set* if every vertex of $V \setminus D$ has at most k neighbours in D, i.e., for each $u \in V \setminus D$, $1 \leq |N(u) \cap D| = k$.

▷ The *k-quasiperfect perfect domination number*, denoted $\gamma_{1k}(G)$, is the minimum cardinality of a *k*-quasiperfect dominating set of G.

▷ A γ_{1k}-*code* is a *k*-QP dominating set of cardinality $\gamma_{1k}(G)$.

▷ A dominating set $D \subset V(G)$ of a graph G is a *k-quasiperfect dominating set* if every vertex of $V \setminus D$ has at most k neighbours in D, i.e., for each $u \in V \setminus D$, $1 \leq |N(u) \cap D| = k$.

▷ The *k-quasiperfect perfect domination number*, denoted $\gamma_{1k}(G)$, is the minimum cardinality of a *k*-quasiperfect dominating set of G.

▷ A γ_{1k}-*code* is a *k*-QP dominating set of cardinality $\gamma_{1k}(G)$.

▷ A dominating set $D \subset V(G)$ of a graph G is a *k-quasiperfect dominating set* if every vertex of $V \setminus D$ has at most k neighbours in D, i.e., for each $u \in V \setminus D$, $1 \leq |N(u) \cap D| = k$.

▷ The *k-quasiperfect perfect domination number*, denoted $\gamma_{1k}(G)$, is the minimum cardinality of a *k*-quasiperfect dominating set of G.

▷ A γ_{1k}-*code* is a *k*-QP dominating set of cardinality $\gamma_{1k}(G)$.

● Notice that $n = 10$, $\gamma_{11}(P) = \gamma_{12}(P) = 4$, $\gamma_{13}(P) = \gamma(P) = 3$.

Let G be a graph of order n and maximum degree Δ. Then,

$$n \geq \gamma_{11}(G) \geq \gamma_{12}(G) \geq \ldots \geq \gamma_{1\Delta}(G) = \gamma(G)$$

Table: QP domination parameters of some basic graphs ($n > 6$).

G	P_n	C_n	K_n	$K_{1,n-1}$	$K_{p,q}$	W_n
$\Delta(G)$	2	2	$n-1$	$n-1$	$\max\{p,q\}$	$n-1$
$\gamma_{11}(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{2n}{3} \rceil - \lfloor \frac{n}{3} \rfloor$	1	1	2	1
$\gamma_{12}(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2	1
$\gamma(G)$	$\lceil \frac{n}{3} \rceil$	$\lceil \frac{n}{3} \rceil$	1	1	2	1

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2}$ $\Longleftrightarrow$ G is either a path or a cycle.

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2} \Longleftrightarrow G$ is either a path or a cycle.

▷ $\boxed{\Delta = n - 1} \Longleftrightarrow \gamma_{11}(G) = \gamma(G) = 1$.

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2} \Longleftrightarrow G$ is either a path or a cycle.

▷ $\boxed{\Delta = n-1} \Longleftrightarrow \gamma_{11}(G) = \gamma(G) = 1$.

⇒ $\boxed{\Delta = 3} \Longrightarrow \gamma_{11}(G) \leq n-3$ (unless G is the bull graph).

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2} \Longleftrightarrow G$ is either a path or a cycle.

▷ $\boxed{\Delta = n-1} \Longleftrightarrow \gamma_{11}(G) = \gamma(G) = 1$.

$\Rightarrow$ $\boxed{\Delta = 3} \Longrightarrow \gamma_{11}(G) \leq n-3$ (unless G is the bull graph).

$\Rightarrow$ $\boxed{\Delta = n-2} \Longrightarrow \gamma_{12}(G) = \gamma(G) = 2$.

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2} \Longleftrightarrow G$ is either a path or a cycle.

▷ $\boxed{\Delta = n - 1} \Longleftrightarrow \gamma_{11}(G) = \gamma(G) = 1$.

⇒ $\boxed{\Delta = 3} \Longrightarrow \gamma_{11}(G) \leq n - 3$ (unless G is the bull graph).

⇒ $\boxed{\Delta = n - 2} \Longrightarrow \gamma_{12}(G) = \gamma(G) = 2$.

⇒ $n \geq 6$ and $2 \leq k \leq n \Longrightarrow$ there exists a graph G of order n s.t. $\Delta(G) = n - 2$, $\gamma_{11}(G) = k$.

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2} \Longleftrightarrow G$ is either a path or a cycle.

▷ $\boxed{\Delta = n-1} \Longleftrightarrow \gamma_{11}(G) = \gamma(G) = 1$.

$\Rightarrow$ $\boxed{\Delta = 3} \Longrightarrow \gamma_{11}(G) \leq n-3$ (unless G is the bull graph).

$\Rightarrow$ $\boxed{\Delta = n-2} \Longrightarrow \gamma_{12}(G) = \gamma(G) = 2$.

$\Rightarrow$ $n \geq 6$ and $2 \leq k \leq n \Longrightarrow$ there exists a graph G of order n s.t. $\Delta(G) = n-2$, $\gamma_{11}(G) = k$.

$\Rightarrow$ $\boxed{\Delta = n-3} \Longrightarrow \gamma_{12}(G) = \gamma(G) \in \{2, 3\}$.

G is a graph of order $n \geq 3$ and maximum degree $\Delta \notin \{4, \ldots, n-4\}$.

▷ $\boxed{\Delta = 2} \Longleftrightarrow G$ is either a path or a cycle.

▷ $\boxed{\Delta = n - 1} \Longleftrightarrow \gamma_{11}(G) = \gamma(G) = 1$.

⇒ $\boxed{\Delta = 3} \Longrightarrow \gamma_{11}(G) \leq n - 3$ (unless G is the bull graph).

⇒ $\boxed{\Delta = n - 2} \Longrightarrow \gamma_{12}(G) = \gamma(G) = 2$.

⇒ $n \geq 6$ and $2 \leq k \leq n \Longrightarrow$ there exists a graph G of order n s.t. $\Delta(G) = n - 2$, $\gamma_{11}(G) = k$.

⇒ $\boxed{\Delta = n - 3} \Longrightarrow \gamma_{12}(G) = \gamma(G) \in \{2, 3\}$.

⇒ $n \geq 9$ and $2 \leq k \leq n \Longrightarrow$ there exists a graph G of order n and $\gamma(G) = 2$ (resp. $\gamma(G) = 3$) s.t. $\Delta(G) = n - 3$, $\gamma_{11}(G) = k$.

G has order $n \geq 6$ and maximum degree $\Delta = 3 \Rightarrow \gamma_{11}(G) \leq n - 3$.

Sketch of proof:

G has order $n \geq 6$ and maximum degree $\Delta = 3 \Rightarrow \gamma_{11}(G) \leq n - 3$.

Sketch of proof:

▷ If G is a tree, then $V(G) \setminus \{l, m, k\}$ is a γ_{11}-set, where $\{l, m, k\}$ are three leaves. Assume thus that G contains an induced cycle C.

G has order $n \geq 6$ and maximum degree $\Delta = 3 \Rightarrow \gamma_{11}(G) \leq n - 3$.

Sketch of proof:

$\triangleright$ If G is a tree, then $V(G) \setminus \{l, m, k\}$ is a γ_{11}-set, where $\{l, m, k\}$ are three leaves. Assume thus that G contains an induced cycle C.

Two conditions:

(A) All vertices in C have degree 3.
(B) There are two non-adjacent vertices $u, v \in V(G)$ of degree 2 and an induced path P joining them s.t. all internal vertices of P have degree 3.

G has order $n \geq 6$ and maximum degree $\Delta = 3 \Rightarrow \gamma_{11}(G) \leq n - 3$.

Sketch of proof:

▷ If G is a tree, then $V(G) \setminus \{l, m, k\}$ is a γ_{11}-set, where $\{l, m, k\}$ are three leaves. Assume thus that G contains an induced cycle C.

Two conditions:

(A) All vertices in C have degree 3.
(B) There are two non-adjacent vertices $u, v \in V(G)$ of degree 2 and an induced path P joining them s.t. all internal vertices of P have degree 3.

▷ Lemma: (A) or (B) $\Rightarrow \gamma_{11}(G) \leq n - 3$.

G has order $n \geq 6$ and maximum degree $\Delta = 3 \Rightarrow \gamma_{11}(G) \leq n - 3$.

Sketch of proof:

▷ If G is a tree, then $V(G) \setminus \{l, m, k\}$ is a γ_{11}-set, where $\{l, m, k\}$ are three leaves. Assume thus that G contains an induced cycle C.

Two conditions:

(A) All vertices in C have degree 3.
(B) There are two non-adjacent vertices $u, v \in V(G)$ of degree 2 and an induced path P joining them s.t. all internal vertices of P have degree 3.

▷ Lemma: (A) or (B) $\Rightarrow \gamma_{11}(G) \leq n - 3$.

Four cases:

(1) All vertices in C have degree 3.
(2) C contains exactly one vertex a with $\delta(a) = 2$.
(3) C contains exactly two adjacent vertices a, b with $\delta(a) = \delta(b) = 2$.
(4) C contains two non-adjacent vertices a, b with $\delta(a) = \delta(b) = 2$.

$$n \geq 6 \implies \exists \ G \text{ s.t. } [\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, n, n]$$

$$n \geq 6 \implies \exists\ G \text{ s.t. } [\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, n, n]$$
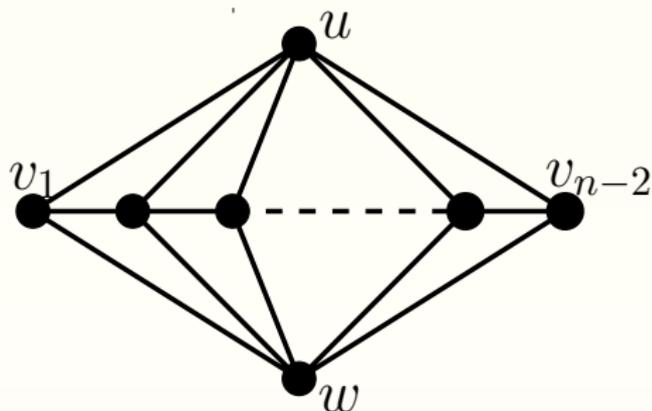
▷ <u>Sketch of proof</u>:

Figure: $G = P_{n-2} \vee \bar{K}_2$

$n \geq 6 \implies \exists\ G$ s.t. $[\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, n, n]$
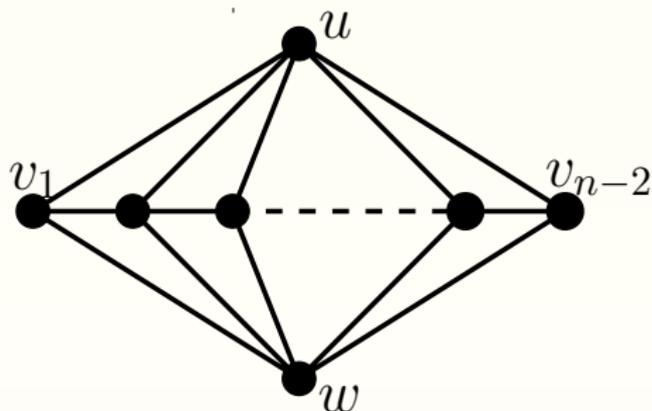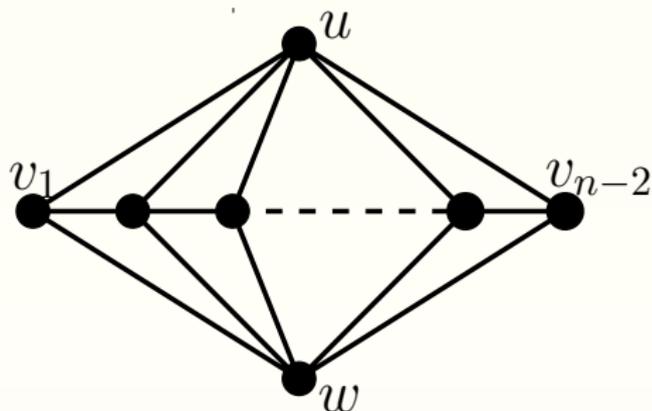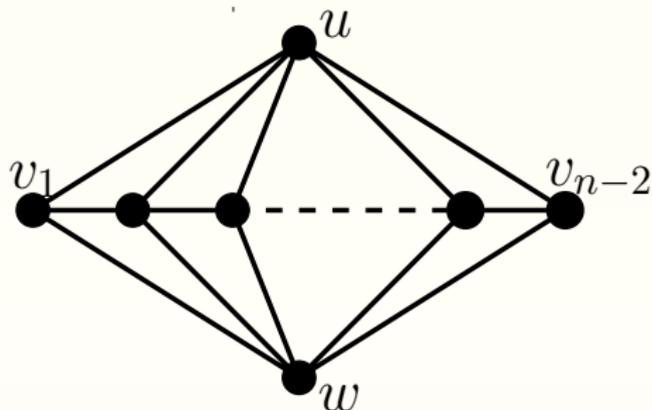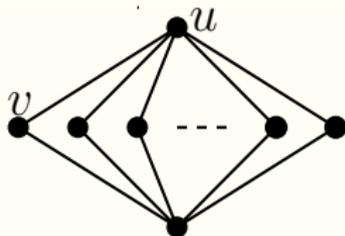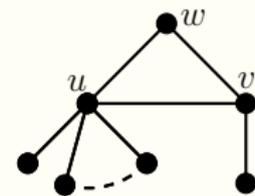
▷ <u>Sketch of proof</u>:

Figure: $G = P_{n-2} \vee \bar{K}_2$

▷ $deg(u) = n - 2$

$$n \geq 6 \Longrightarrow \exists \, G \text{ s.t. } [\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, n, n]$$

▷ <u>Sketch of proof</u>:

Figure: $G = P_{n-2} \vee \bar{K}_2$

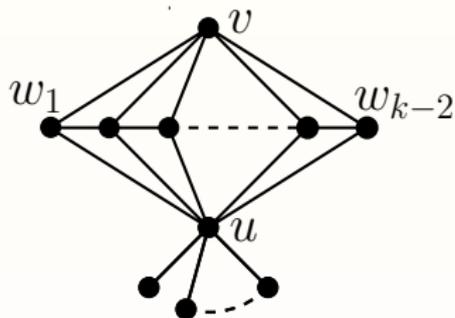▷ $deg(u) = n - 2$

▷ $\{u, w\}$ is a γ-code.

$n \geq 6 \Longrightarrow \exists\, G$ s.t. $[\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, n, n]$
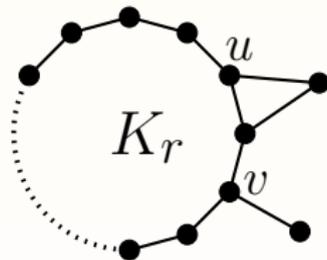
▷ Sketch of proof:

Figure: $G = P_{n-2} \vee \bar{K}_2$

▷ $deg(u) = n - 2$

▷ $\{u, w\}$ is a γ-code.

▷ The unique γ_{11}-set is the whole graph.

$$n \geq 6, 2 \leq k < n \Longrightarrow \exists\ G \text{ s.t. } [\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, k, n]$$

$n \geq 6, 2 \leq k < n \Longrightarrow \exists\, G$ s.t. $[\Delta(G), \gamma(G), \gamma_{11}(G), n] = [n-2, 2, k, n]$

▷ <u>Sketch of proof</u>:

$[2, 2, n]$

$[2, 3, n]$

$[2, k, n], \; {\scriptstyle 4 \leq k \leq n-2}$

$[2, n-1, n]$

T is a tree of order $n \geq 4$.

T is a tree of order $n \geq 4$.

  ▷ Every $\gamma_{11} -$ *code* of *T* contains all its strong support vertices.

  [Caro, Hansberg, Henning,DM,2012]

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - $ *code* of T contains all its strong support vertices.

[Caro, Hansberg, Henning,DM,2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.

[Caro, Hansberg, Henning,DM,2012]

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - $ **code** of *T* contains all its strong support vertices.
[Caro, Hansberg, Henning,DM,2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.
[Caro, Hansberg, Henning,DM,2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - $ ***code*** of T contains all its strong support vertices.

[Caro, Hansberg, Henning,DM,2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.

[Caro, Hansberg, Henning,DM,2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

Sketch of proof:

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - $ *code* of *T* contains all its strong support vertices.

[Caro, Hansberg, Henning,DM,2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.

[Caro, Hansberg, Henning,DM,2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

Sketch of proof:

● *S*: dominating set of *T*.

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - $ **code** of *T* contains all its strong support vertices.

[Caro, Hansberg, Henning, DM, 2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.

[Caro, Hansberg, Henning, DM, 2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

Sketch of proof:

- *S*: dominating set of *T*.
- $T_1, \ldots, T_r$: the connected components of $T - S$ such that at least one of its vertices has two neighbors in *S*.

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - code$ of T contains all its strong support vertices.
[Caro, Hansberg, Henning,DM,2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.
[Caro, Hansberg, Henning,DM,2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

Sketch of proof:

- S: dominating set of T.
- $T_1, \ldots, T_r$: the connected components of $T - S$ such that at least one of its vertices has two neighbors in S.
- $S^* = S \cup (\cup_{i=1}^{r} V(T_i))$ is a 1-QP dominating set of T.

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} - $ **code** of T contains all its strong support vertices.
[Caro, Hansberg, Henning, DM, 2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.
[Caro, Hansberg, Henning, DM, 2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

Sketch of proof:

- S: dominating set of T.
- $T_1, \ldots, T_r$: the connected components of $T - S$ such that at least one of its vertices has two neighbors in S.
- $S^* = S \cup (\cup_{i=1}^{r} V(T_i))$ is a 1-QP dominating set of T.
- S^* is the smallest 1-QP dominating set of T containing S.

T is a tree of order $n \geq 4$.

▷ Every $\gamma_{11} -$ *code* of *T* contains all its strong support vertices.
  [Caro, Hansberg, Henning,DM,2012]

▷ $\gamma(T) = n/2 \Leftrightarrow \gamma_{11}(T) = n/2 \Leftrightarrow T = T' \odot K_1$, for some tree T'.
  [Caro, Hansberg, Henning,DM,2012]

▷▷ $\gamma_{11}(T) \leq 2\gamma(T) - 1$.

  Sketch of proof:

- S: dominating set of *T*.
- $T_1, \ldots, T_r$: the connected components of $T - S$ such that at least one of its vertices has two neighbors in S.
- $S^* = S \cup (\cup_{i=1}^{r} V(T_i))$ is a 1-QP dominating set of *T*.
- S^* is the smallest 1-QP dominating set of *T* containing S.
- $|S^*| \leq 2|S| - 1$.

A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.
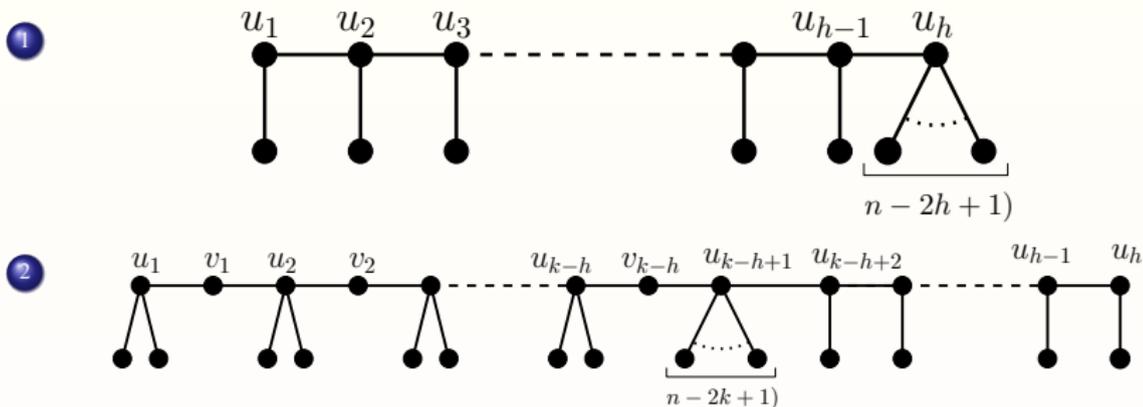
A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.

▷ 1. Let h, n be integers such that $1 \leq h$ and $2h \leq n$.
   Then, there is a caterpillar T s.t. $[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, h, n]$.

   2. Let h, k, n be integers such that $2 \leq h < k \leq 2h - 1$ and $2k < n$.
   Then, there is a caterpillar T s.t. $[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$.

A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.

▷ ① Let h, n be integers such that $1 \leq h$ and $2h \leq n$.
Then, there is a caterpillar T s.t. $[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, h, n]$.

② Let h, k, n be integers such that $2 \leq h < k \leq 2h - 1$ and $2k < n$.
Then, there is a caterpillar T s.t. $[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$.

Sketch of proof:

$\star$ A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.
$\star$ A *k-ary tree* is a rooted tree such that each vertex has at most *k* children.
$\star$ A *full k-ary tree* is a *k*-ary tree such that all vertices that are not leaves have exactly *k* children.

⋆ A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.
⋆ A *k-ary tree* is a rooted tree such that each vertex has at most *k* children.
⋆ A *full k-ary tree* is a *k*-ary tree such that all vertices that are not leaves have exactly *k* children.

▷ If T is a caterpillar, then:

$$\boxed{2\gamma(T) > \gamma_{11}(T) \geq \gamma_{12}(T) = \gamma(T)}$$

$\star$ A *caterpillar* is a tree s.t. the removal of all its leaves gives rise to a path.
$\star$ A *k-ary tree* is a rooted tree such that each vertex has at most k children.
$\star$ A *full k-ary tree* is a *k*-ary tree such that all vertices that are not leaves have exactly k children.

$\triangleright$ If T is a caterpillar, then:

$$2\gamma(T) > \gamma_{11}(T) \geq \gamma_{12}(T) = \gamma(T)$$

$\triangleright$ If T is full *k*-ary tree, then:

$$n - \ell(T) = \gamma_{11}(T) = \cdots = \gamma_{1,k-1}(T) > \gamma_{1,k}(T) = \gamma_{1,k+1}(T) = \gamma(T)$$

G is a cograph, i.e., a P_4-free graph.

G is a cograph, i.e., a P_4-free graph.

▷ *diam*$(G) \leq 2$.

G is a cograph, i.e., a P_4-free graph.

   ▷ $diam(G) \leq 2$.

   ▷ $\gamma_{12}(G) = \gamma(G)$ [Chellali,Haynes,Hedetniemi,DAM,2013]

G is a cograph, i.e., a P_4-free graph.

$\triangleright$ $diam(G) \leq 2$.

$\triangleright$ $\gamma_{12}(G) = \gamma(G)$ [Chellali,Haynes,Hedetniemi,DAM,2013]

$\triangleright$ $\gamma_{11}(G) \in \{1, 2, n\}$

G is a cograph, i.e., a P_4-free graph.

- ▷ $diam(G) \leq 2$.
- ▷ $\gamma_{12}(G) = \gamma(G)$ [Chellali,Haynes,Hedetniemi,DAM,2013]
- ▷ $\gamma_{11}(G) \in \{1, 2, n\}$
- ▷ $\gamma_{11}(G) = 2 \Longleftrightarrow G$ is as in Figure.
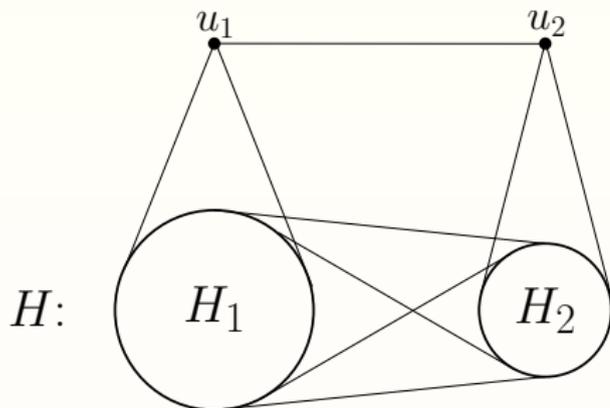
Figure: $H = H_1 \vee H_2$, $N_G(u_1) = V(H_1) + u_2$, $N_G(u_2) = V(H_2) + u_1$.

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

$\triangleright$ $\gamma_{12}(G) = i(G) = \gamma(G)$ [Allan,Laskar,DM,1978]

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

▷ $\gamma_{12}(G) = i(G) = \gamma(G)$ [Allan,Laskar,DM,1978]

▷ Let h, k, n be integers such that $2 \leq h \leq k$ and $h \leq \frac{n}{2}$.
   Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

   if at least one of the following conditions holds:

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

$\triangleright$ $\gamma_{12}(G) = i(G) = \gamma(G)$ [Allan,Laskar,DM,1978]

$\triangleright$ Let h, k, n be integers such that $2 \leq h \leq k$ and $h \leq \frac{n}{2}$.
Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

if at least one of the following conditions holds:

(1) $4 \leq n \leq 7$: $3h + k < 2n$ or $[h, k, n] \in \{[2, 6, 6], [3, 3, 6]\}$ (converse true).

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

▷ $\gamma_{12}(G) = i(G) = \gamma(G)$ [Allan,Laskar,DM,1978]

▷ Let h, k, n be integers such that $2 \le h \le k$ and $h \le \frac{n}{2}$.
Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

if at least one of the following conditions holds:

(1) $4 \le n \le 7$: $3h + k < 2n$ or $[h, k, n] \in \{[2, 6, 6], [3, 3, 6]\}$ (converse true).

(2) $h + k \le n$.

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

▷ $\gamma_{12}(G) = i(G) = \gamma(G)$ [Allan,Laskar,DM,1978]

▷ Let h, k, n be integers such that $2 \leq h \leq k$ and $h \leq \frac{n}{2}$.
Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

if at least one of the following conditions holds:

(1) $4 \leq n \leq 7$: $3h + k < 2n$ or $[h, k, n] \in \{[2, 6, 6], [3, 3, 6]\}$ (converse true).
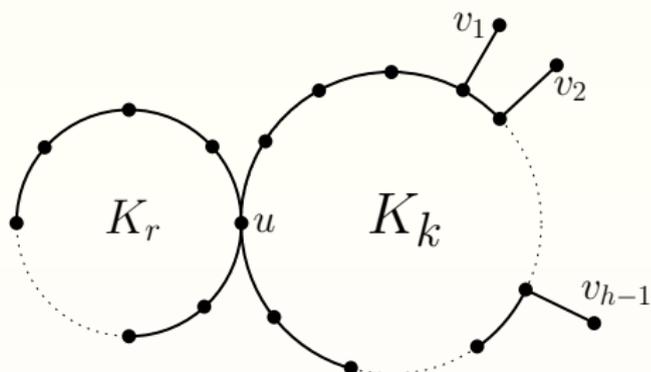
(2) $h + k \leq n$.

(3) $3h + k + 1 \leq 2n$.

G is a claw-free graph, i.e., a $K_{1,3}$-free graph.

▷ $\gamma_{12}(G) = i(G) = \gamma(G)$ [Allan,Laskar,DM,1978]

▷ Let h, k, n be integers such that $2 \leq h \leq k$ and $h \leq \frac{n}{2}$. Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

if at least one of the following conditions holds:

(1) $4 \leq n \leq 7$: $3h + k < 2n$ or $[h, k, n] \in \{[2, 6, 6], [3, 3, 6]\}$ (converse true).

(2) $h + k \leq n$.

(3) $3h + k + 1 \leq 2n$.

▷ Open Problem: $3 \leq h < k$, $2n \leq 3h + k$.

(2) Let h, k, n be integers such that $2 \le h \le k < n$ and $h + k \le n$.
Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

(2) Let h, k, n be integers such that $2 \le h \le k < n$ and $h + k \le n$. Then, there exists a claw-free graph G s.t.
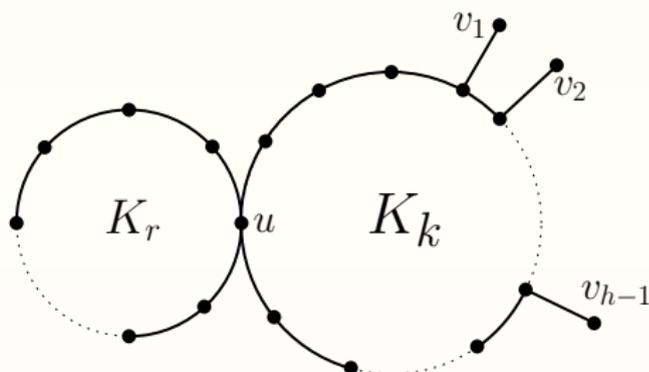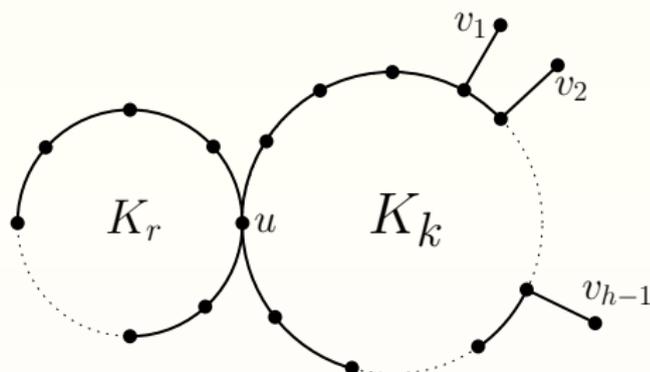
$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$$2 \le h \le k, \, 2 \le r$$

(2) Let h, k, n be integers such that $2 \le h \le k < n$ and $h + k \le n$. Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$
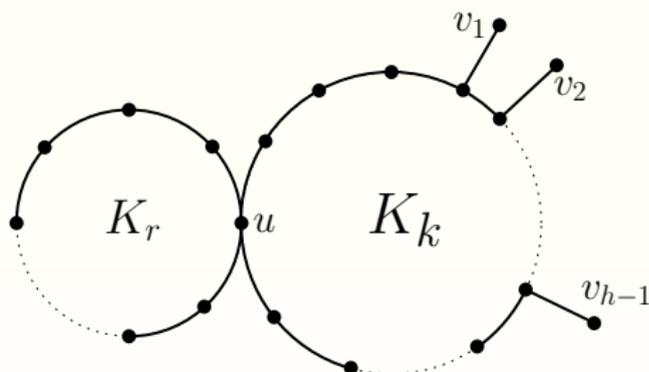

$$2 \le h \le k, \, 2 \le r$$

- $\{v_1, \ldots, v_{h-1}, u\}$ is a γ-code of G.

(2) Let h, k, n be integers such that $2 \le h \le k < n$ and $h + k \le n$.
Then, there exists a claw-free graph G s.t.

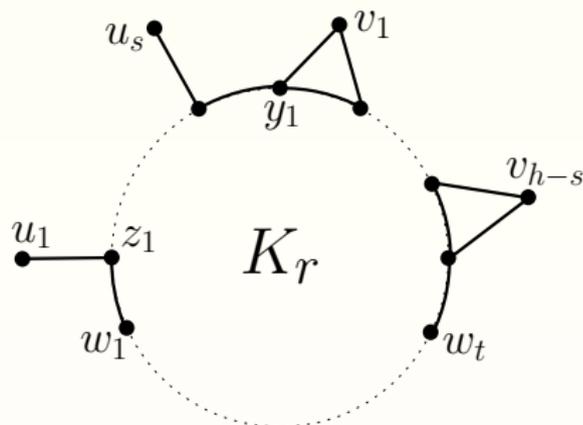$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$$2 \le h \le k, 2 \le r$$

- $\{v_1, \ldots, v_{h-1}, u\}$ is a γ-code of G.
- K_k is a γ_{11}-code of G.

(2) Let h, k, n be integers such that $2 \le h \le k < n$ and $h + k \le n$.
Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$$2 \le h \le k,\, 2 \le r$$

- $\{v_1, \ldots, v_{h-1}, u\}$ is a γ-code of G.
- K_k is a γ_{11}-code of G.
- $n = h + k + r - 2 \ge h + k$.

(3) Let h, k, n be integers such that $2 \leq h \leq k$ and $3h + k + 1 \leq 2n$.
Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

(3) Let h, k, n be integers such that $2 \leq h \leq k$ and $3h + k + 1 \leq 2n$. Then, there exists a claw-free graph G s.t.
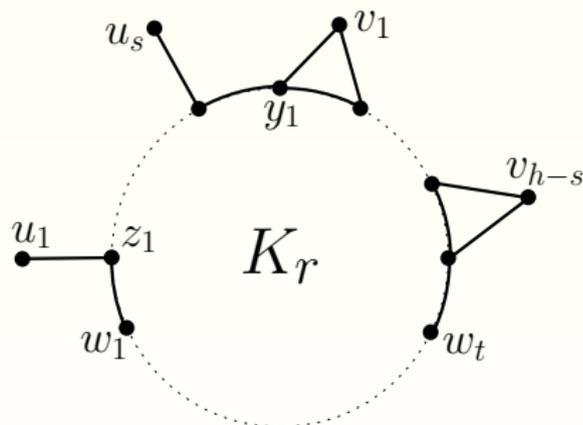
$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$1 \leq t,\ 0 \leq s,\ \max\{2, s\} \leq h,\ r = 2h - s + t$

(3) Let h, k, n be integers such that $2 \le h \le k$ and $3h + k + 1 \le 2n$. Then, there exists a claw-free graph G s.t.
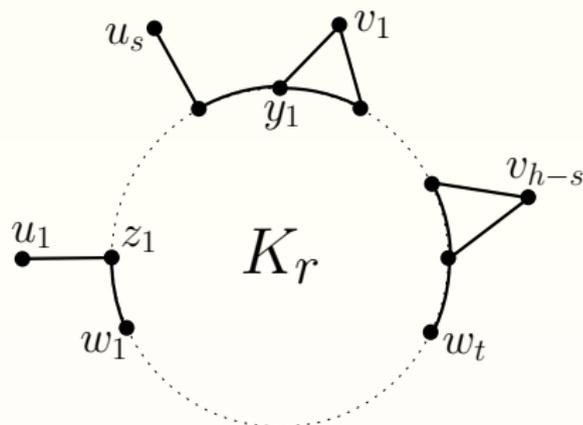
$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$1 \le t,\, 0 \le s,\, \max\{2, s\} \le h,\, r = 2h - s + t$

- $\{z_1, u_2, \ldots, u_s, v_1, \ldots, v_{h-s}\}$ or $\{u_1, \ldots, u_s, y_1, v_2, \ldots, v_{h-s}\}$ is a γ-code of G.

(3) Let h, k, n be integers such that $2 \le h \le k$ and $3h + k + 1 \le 2n$. Then, there exists a claw-free graph G s.t.
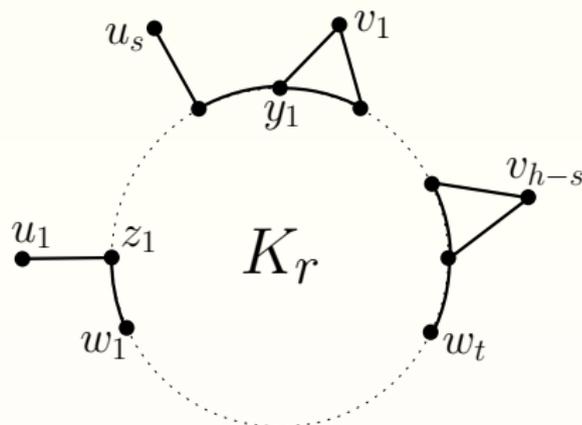
$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$1 \le t,\, 0 \le s,\, \max\{2, s\} \le h,\, r = 2h - s + t$

- $\{z_1, u_2, \ldots, u_s, v_1, \ldots, v_{h-s}\}$ or $\{u_1, \ldots, u_s, y_1, v_2, \ldots, v_{h-s}\}$ is a γ-code of G.
- $V(G) \setminus \{u_1, \ldots, u_s\}$ is a γ_{11}-code of G.

(3) Let h, k, n be integers such that $2 \leq h \leq k$ and $3h + k + 1 \leq 2n$. Then, there exists a claw-free graph G s.t.

$$[\gamma(G), \gamma_{11}(G), |V(G)|] = [h, k, n]$$

$1 \leq t,\ 0 \leq s,\ \max\{2, s\} \leq h,\ r = 2h - s + t$

- $\{z_1, u_2, \ldots, u_s, v_1, \ldots, v_{h-s}\}$ or $\{u_1, \ldots, u_s, y_1, v_2, \ldots, v_{h-s}\}$ is a γ-code of G.
- $V(G) \setminus \{u_1, \ldots, u_s\}$ is a γ_{11}-code of G.
- $n = r + h = 3h - s + t = 3\gamma(G) - n + \gamma_{11}(G) + t \Rightarrow 2n \geq 3h + k + 1$.