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ScienceDirect
Electronic Notes in Discrete Mathematics 29 (2007) 339–343
Electronic Notes in DISCRETE MATHEMATICS
www.elsevier.com/locate/endm

Extremal Graph Theory for Metric Dimension and Diameter [1]

C. Hernando, M. Mora, I. M. Pelayo, C. Seara and D. R. Wood

Departaments de Matemàtica Aplicada I,II, III
Universitat Politècnica de Catalunya
Barcelona, Spain

Abstract

Let $\mathcal{G}_{\beta,D}$ be the set of graphs with metric dimension β and diameter D. The first contribution is to characterize the graphs in $\mathcal{G}_{\beta,D}$ with order $\beta + D$ for all values of β and D. The second contribution is to determine the maximum order of a graph in $\mathcal{G}_{\beta,D}$ for all values of D and β. Only a weak upper bound was previously known.

Keywords: Graph, resolving set, metric dimension, metric basis, diameter, order.

1 Introduction

Let G be a connected graph. A vertex $x \in V(G)$ *resolves* a pair of vertices $v, w \in V(G)$ if $\operatorname{dist}(v, x) \neq \operatorname{dist}(w, x)$. A set of vertices $S \subseteq V(G)$ *resolves* G, and S is a *resolving set* of G, if every pair of distinct vertices of G are resolved

[1] Research supported by projects MCYT-FEDER-BFM2003-00368, MCYT-HU2002-0010, MTM-2004-07891-C02-01, MEC-SB2003-0270, and MCYT-FEDER BFM2003-00368. The research of David R. Wood is supported by a Marie Curie Fellowship of the European Community under contract 023865.

doi:10.1016/j.endm.2007.07.058

by some vertex in S. Informally, S resolves G if every vertex of G is uniquely determined by its vector of distances to the vertices in S. A resolving set S of G with the minimum cardinality is a *metric basis* of G, and $\beta(G) := |S|$ is the *metric dimension* of G. Resolving sets in general graphs were first defined by Slater [7] and Harary and Melter [4]. Resolving sets have since been widely investigated [2,5,9], and arise in diverse areas including coin weighing problems [8], network discovery and verification [1], robot navigation [5], connected joins in graphs [6], and strategies for the Mastermind game [3]. For non-negative integers β and D, let $\mathcal{G}_{\beta,D}$ be the class of connected graphs with metric dimension β and diameter D. Consider the following two extremal questions: (1) *What is the minimum order of a graph in* $\mathcal{G}_{\beta,D}$*?* (2) *What is the maximum order of a graph in* $\mathcal{G}_{\beta,D}$*?*

The first question was independently answered by Yushmanov [9], Khuller et al. [5], and Chartrand et al. [2], who proved that the minimum order of a graph in $\mathcal{G}_{\beta,D}$ is $\beta + D$. Thus it is natural to consider the following problem: *Characterize the graphs in* $\mathcal{G}_{\beta,D}$ *with order* $\beta + D$*.* Such a characterization is simple for $\beta = 1$. In particular, Khuller et al. [5] and Chartrand et al. [2] independently proved that paths P_n (with $n \geq 2$ vertices) are the only graphs with metric dimension 1. Thus $\mathcal{G}_{1,D} = \{P_{D+1}\}$. The characterization is again simple at the other extreme with $D = 1$. In particular, Chartrand et al. [2] proved that the complete graph K_n (with $n \geq 1$ vertices) is the only graph with metric dimension $n-1$ (see Proposition 2.1). Thus $\mathcal{G}_{\beta,1} = \{K_{\beta+1}\}$. Chartrand et al. [2] studied the case $D = 2$, and obtained a non-trivial characterization of graphs in $\mathcal{G}_{\beta,2}$ with order $\beta+2$ (see Proposition 2.2). The first contribution of this paper is to characterize the graphs in $\mathcal{G}_{\beta,D}$ with order $\beta + D$ for all values of $\beta \geq 1$ and $D \geq 3$, thus completing the characterization for all values of D. This result is stated and proved in Section 2. We then study the second question above: What is the maximum order of a graph in $\mathcal{G}_{\beta,D}$? Previously, only a weak upper bound was known. In particular, Khuller et al. [5] and Chartrand et al. [2] independently proved that every graph in $\mathcal{G}_{\beta,D}$ has at most $D^\beta + \beta$ vertices. This bound is tight only for $D \leq 3$ or $\beta = 1$. Our second contribution is to determine the (exact) maximum order of a graph in $\mathcal{G}_{\beta,D}$ for all values of D and β. This result is stated and proved in Section 3.

2 Graphs with minimum order

Twin vertices. Let u be a vertex of a graph G. The *open neighborhood* of u is $\mathrm{N}(u) := \{v \in V(G) : uv \in E(G)\}$, and the *closed neighborhood* of u is $\mathrm{N}[u] := \mathrm{N}(u) \cup \{u\}$. Two distinct vertices u, v are *adjacent twins* if $\mathrm{N}[u] = \mathrm{N}[v]$, and *non-adjacent twins* if $\mathrm{N}(u) = \mathrm{N}(v)$. Observe that if u, v are adjacent twins then $uv \in E(G)$, and if u, v are non-adjacent twins then $uv \notin E(G)$; thus the names are justified. If u, v are adjacent or non-adjacent twins, then u, v are *twins*. A consequence of the definitions is that if u, v are twins in a connected graph G, then $\mathrm{dist}(u, x) = \mathrm{dist}(v, x)$ for every vertex $x \in V(G) \setminus \{u, v\}$. This implies that if u, v are twins in a connected graph G and S resolves G, then u or v is in S. Moreover, if $u \in S$ and $v \notin S$, then $(S \setminus \{u\}) \cup \{v\}$ resolves G.

For a graph G, a set $T \subseteq V(G)$ is a *twin-set* of G if v, w are twins in G for every pair of distinct vertices $v, w \in T$. It it easy to prove that if T is a twin-set of a graph G, then either every pair of vertices in T are adjacent twins, or every pair of vertices in T are non-adjacent twins. If T is a twin-set of a connected graph G with $|T| \geq 3$, it can be proved that $\beta(G) = \beta(G \setminus S) + |S|$ for every subset $S \subset T$ with $|S| \leq |T| - 2$.

The Twin Graph. Let G be a graph. Define a relation $\equiv$ on $V(G)$ by $u \equiv v$ if and only if $u = v$ or u, v are twins. $\equiv$ is an equivalence relation. For each vertex $v \in V(G)$, let v^* be the set of vertices of G that are equivalent to v under $\equiv$. Let $\{v_1^*, \dots, v_k^*\}$ be the partition of $V(G)$ induced by $\equiv$, where each v_i is a representative of the set v_i^*. The *twin graph* of G, denoted by G^*, is the graph with vertex set $V(G^*) := \{v_1^*, \dots, v_k^*\}$, where $v_i^* v_j^* \in E(G^*)$ if and only if $v_i v_j \in E(G)$. Two vertices v^* and w^* of G^* are adjacent if and only if every vertex in v^* is adjacent to every vertex in w^* in G. Each vertex v^* of G^* is a maximal twin-set of G. $G[v^*]$ is a complete graph if the vertices of v^* are adjacent twins, or $G[v^*]$ is a null graph if the vertices of v^* are non-adjacent twins. So it makes sense to consider the following types of vertices in G^*. We say that $v^* \in V(G^*)$ is of *type*: (i) (1) if $|v^*| = 1$; (ii) (K) if $G[v^*] \cong K_r$ and $r \geq 2$; (iii) (N) if $G[v^*] \cong N_r$ and $r \geq 2$; where N_r is the *null* graph with r vertices and no edges. A vertex of G^* is of *type* $(1K)$ if it is of type (1) or (K). A vertex of G^* is of *type* $(1N)$ if it is of type (1) or (N). Observe that the graph G is uniquely determined by G^*, and the type and cardinality of each vertex of G^*. In particular, if v^* is adjacent to w^* in G^*, then every vertex in v^* is adjacent to every vertex in w^* in G. If G is a graph with $\mathrm{diam}(G) \geq 3$ then $\mathrm{diam}(G) = \mathrm{diam}(G^*)$. Theorem 2.3 below characterizes the graphs in $\mathcal{G}_{\beta,D}$ for $D \geq 3$ in terms of the twin graph. Chartrand et al. [2] characterized the graphs in $\mathcal{G}_{\beta,D}$ for $D \leq 2$. For consistency with Theorem 2.3, we describe the characterization by Chartrand et al. [2] in terms of the twin graph.

Proposition 2.1 ([2]) *The following are equivalent for G with n vertices: i) G has metric dimension $\beta(G) = n - 1$; ii) $G \cong K_n$; iii)* $\mathrm{diam}(G) = 1$*; (iv) the twin graph G^* has one vertex, which is of type* $(1K)$.

Proposition 2.2 ([2]) *The following are equivalent for G with $n \geq 3$ vertices: i) G has metric dimension $\beta(G) = n - 2$; ii) G has metric dimension $\beta(G) = n-2$ and diameter $\operatorname{diam}(G) = 2$; iii) the twin graph G^* of G satisfies: a) $G^* \cong P_2$ with at least one vertex of type (N), or b) $G^* \cong P_3$ with one leaf of type (1), the other leaf of type $(1K)$, and the degree-2 vertex of type $(1K)$.*

To describe our characterization we introduce the following notation. Let $P_{D+1} = (u_0, u_1, \dots, u_D)$ be a path of length D. For $k \in [3, D-1]$ let $P_{D+1,k}$ be the graph obtained from P_{D+1} by adding one vertex adjacent to u_{k-1}. For $k \in [2, D-1]$ let $P'_{D+1,k}$ be the graph obtained from P_{D+1} by adding one vertex adjacent to u_{k-1} and u_k.

Theorem 2.3 *Let G be a connected graph of order n and diameter $D \geq 3$. Let G^* be the twin graph of G. Let $\alpha(G^*)$ be the number of vertices of G^* of type (K) or (N). Then $\beta(G) = n - D$ if and only if G^* is one of the following graphs:*

- (i) *$G^* \cong P_{D+1}$ and one of the following cases hold:*
  - (a) *$\alpha(G^*) \leq 1$;*
  - (b) *$\alpha(G^*) = 2$, the two vertices of G^* not of type (1) are adjacent, and if one is a leaf of type (K) then the other is also of type (K);*
  - (c) *$\alpha(G^*) = 2$, the two vertices of G^* not of type (1) are at distance 2 and both are of type (N); or*
  - (d) *$\alpha(G^*) = 3$ and there is a vertex of type (N) or (K) adjacent to two vertices of type (N).*
- (ii) *$G^* \cong P_{D+1,k}$ for some $k \in [3, D-1]$, the degree-3 vertex u^*_{k-1} of G^* is any type, each neighbour of u^*_{k-1} is type $(1N)$, and every other vertex is type (1).*
- (iii) *$G^* \cong P'_{D+1,k}$ for some $k \in [2, D-1]$, the three vertices in the cycle are of type $(1K)$, and every other vertex is of type (1).*

3 Graphs with maximum order

Theorem 3.1 *For all integers $D \geq 2$ and $\beta \geq 0$, the maximum order of a connected graph with diameter D and metric dimension β is*

$$m(D,\beta) = \left(\left\lfloor \frac{2D}{3} \right\rfloor + 1 \right)^{\beta} + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i-1)^{\beta-1}. \tag{1}$$

Lemma 3.2 *For every graph $G \in \mathcal{G}_{\beta,D}$, $|V(G)| \leq m(D,\beta)$.*

To prove the lower bound in Theorem 3.1 we construct a graph $G \in \mathcal{G}_{\beta,D}$

with as many vertices as in Equation (1). Let $A = \lceil D/3 \rceil$, $B = \lceil D/3 \rceil + \lfloor D/3 \rfloor$, and $Q = \{(x_1, \dots, x_\beta) : A \leq x_i \leq D, i \in [1, \beta]\}$. For each $i \in [1, \beta]$ and $r \in [0, A-1]$, let $P_{i,r} = \{(x_1, \dots, x_{i-1}, r, x_{i+1}, \dots, x_\beta) : x_j \in [B-r, B+r], j \neq i\}$. Let $P_i = \bigcup\{P_{i,r} : r \in [0, A-1]\}$ and $P = \bigcup\{P_i : i \in [1, \beta]\}$. Let G be the graph with $V(G) = Q \cup P$, where $(x_1, \dots, x_\beta)$ and $(y_1, \dots, y_\beta)$ in $V(G)$ are adjacent if and only if $|y_i - x_i| \leq 1$ for each $i \in [1, \beta]$. Let $S = \{v_1, \dots, v_\beta\}$, where $v_i = (t, \dots, t, 0, t, \dots, t) \in P_i$. For all $D, \beta > 0$, $|V(G)| = m(D, \beta)$.

Lemma 3.3 *For all vertices $x = (x_1, \dots, x_\beta)$ and $y = (y_1, \dots, y_\beta)$ of G, $\text{dist}(x, y) = \max\{|y_i - x_i| : i \in [1, \beta]\} \leq D$.*

Lemma 3.4 *For every $x = (x_1, \dots, x_\beta)$ and for each $v_i \in S$ $\text{dist}(x, v_i) = x_i$.*

By Lemma 3.3 $\text{diam}(G) = D$. By Lemma 3.4 S resolves G. If the metric dimension of G is $< |S| = \beta$ then, by Lemma 3.2, we get a contradiction.

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