Geodesic Convexity and Cartesian Products in Graphs

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Abstract

In this work we investigate the behavior of various geodesic convexity parameters with respect to the Cartesian product operation for graphs. First, we show that the convex sets arising from geodesic convexity in a Cartesian product of graphs are exactly the same as the convex sets arising from the usual binary operation $\oplus$ for making a convexity space out of the Cartesian product of any two convexity spaces. To be more precise, we prove that if $(G_1, \mathcal{C}_1)$, $(G_2, \mathcal{C}_2)$ are two graph (geodesic) convexity spaces and if $(G_1 \times G_2, \mathcal{C})$ is the graph (geodesic) convexity space determined by the graph $G_1 \times G_2$, then $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1,\ B \in \mathcal{C}_2\}$. Second, we study results involving a number of classical and graph-theoretic convexity parameters as applied to Cartesian products of graphs. For example, concerning geodetic numbers of graphs, we prove that for every two graphs G, H such that $gn(G) = p \geq gn(H) = q \geq 1$,

$$p \leq gn(G \times H) \leq pq - q,$$

and that both bounds are tight.

1 Geodesic Convexity and Cartesian Product

We consider only finite, simple, connected graphs. For undefined basic concepts we refer the reader to introductory graph theoretical literature, e.g., [60]. Given vertices u, v in a graph G we let $d_G(u, v)$ denote the distance between u and v in G. When there is no confusion, subscripts will be omitted. A $u - v$ path ρ is called a $u - v$ *geodesic* if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$. The *geodetically closed interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics. For $S \subseteq V$, the *geodetic closure* $I[S]$ of S is the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in S$, i.e. $I[S] = \bigcup_{u,v \in S} I[u, v]$. A *convexity* on a finite set X is a family $\mathcal{C}$ of subsets of X (each such set called a *convex set*), which is closed under intersection and which contains both X and the empty set. The pair $(X, \mathcal{C})$ is called a convexity space. A (finite) *graph convexity* space is a pair $(G, \mathcal{C})$, formed by a finite connected graph $G = (V, E)$ and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying that every member of $\mathcal{C}$ induces a connected subgraph of G (see [23, 30, 56]). In this paper we consider only the so-called *geodesic convexity*, $\mathcal{C}_g$ defined as follows. A vertex set $W \subseteq V$ is called *geodesically convex* (or simply *g-convex*) if $I[W] = W$. In a convexity space $(X, \mathcal{C})$, the smallest convex set containing a set $A \subseteq V$ is denoted $[A]_{\mathcal{C}}$ and is called the *convex hull* of A. In the case where the space is a graph convexity space and the convexity used is geodesic convexity, we let $[A]_g$ denote the convex hull. A non-empty set $A \subseteq V$ is called a *hull set* if $[A]_g = V$.

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In graph theory, the definition of $G_1 \times G_2$, the *Cartesian product* of graphs $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$, is that it is the graph on vertex set $V_1 \times V_2$ in which vertices (x_1, x_2) and (y_1, y_2) are adjacent if and only if either $x_1 = x_2$ and $y_1 y_2 \in E_2$ or $y_1 = y_2$ and $x_1 x_2 \in E_1$. By contrast, in the theory of convexity spaces, a standard binary operation $\oplus$ exists for producing what is also naturally called a Cartesian product, as follows. Given convexity spaces $(X_1, \mathcal{C}_1)$ and $(X_2, \mathcal{C}_2)$, the so-called *convex product space* $(X_1 \times X_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$ on the set of points $X_1 \times X_2$ is defined by letting $\mathcal{C}_1 \oplus \mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1,\ B \in \mathcal{C}_2\}$. In this section we reconcile whether the $\oplus$ operator as applied to the Cartesian product of two geodesic graph convexity spaces yields the same space as the geodesic graph convexity space induced by the usual Cartesian product graph formed from those graphs. We begin with the following well known proposition. We omit its proof.

Proposition 1.1 *Let G, H be connected graphs. Given any two vertices (u, x), (v, y) in $G \times H$, we have $d_{G \times H}((u, x), (v, y)) = d_G(u, x) + d_H(v, y)$. Furthermore, if P is a (u, x)-(v, y) geodesic in $G \times H$ and P_1 and P_2 are the projections of $V(P)$ onto G and H, respectively, then P_1 induces a $u - v$ geodesic in G and P_2 induces a $x - y$ geodesic in H.*

For S a subset of a Cartesian product $X_1 \times X_2$ of any two sets X_1 and X_2, we make frequent use of the *projection* $\pi_i(S)$ of S onto coordinate i $(i = 1, 2)$, given as usual by $\pi_1(S) = \{u \in X_1 : \exists v \in X_2, (u, v) \in S\}$ and $\pi_2(S) = \{v \in X_2 : \exists u \in X_1, (u, v) \in S\}$. Often, when the context is clear, we write S_1 and S_2 as shorthand for $\pi_1(S)$ and $\pi_2(S)$ respectively.

The following proposition tells us, when we create a convexity space by use of the operator $\oplus$, how to relate the closure of a set S of points in the Cartesian product to the projections of S on each coordinate. The proposition after that tells us the same information, but for when we create a graph convexity space on the graph Cartesian product of two graphs, and instead consider the geodesic convexity space associated with that resulting graph.

Proposition 1.2 *Let $(X, \mathcal{C}) = (X_1 \times X_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$. If $S \subseteq X$, then*

$$[S]_{\mathcal{C}} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}.$$

Proof. Clearly the set $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$ specified is an element of $\mathcal{C}$. A convex set containing S must be of the form $A \times B$ with $A \in \mathcal{C}_1,\ B \in \mathcal{C}_2$, where $S_1 \subseteq A$ and $S_2 \subseteq B$ in order for $S \subseteq A \times B$. Thus $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$ is the smallest element of $\mathcal{C}$ containing S, so it equals $[S]_{\mathcal{C}}$. ∎

Proposition 1.3 *Let $G_1 = ((V_1, E_1), \mathcal{C}_1)$ and $G_2 = ((V_2, E_2), \mathcal{C}_2)$, where the elements of $\mathcal{C}_i$ are the g-convex sets in G_i. Let $G = (V, E)$ be the graph Cartesian product $G_1 \times G_2$ and let $\mathcal{C}$ be the set of g-convex sets in G.*

1. *If $S \subseteq V$ then, $[S]_{\mathcal{C}} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$.*
2. *If $a = (a_1, a_2), b = (b_1, b_2) \in V(G)$ then, $I[a, b] = I[a_1, b_1] \times I[a_2, b_2]$.*
3. *If $S \subseteq V$ then, $I[S] \subseteq I[S_1] \times I[S_2]$.*

Proof. Consider any $S \in \mathcal{C}$. We show that $\pi_1([S]_{\mathcal{C}}) \in \mathcal{C}_1$, as follows. Suppose for contradiction that for some $x, x' \in \pi_1([S]_{\mathcal{C}})$ that there exists an $x - x'$ geodesic in G_1 containing a vertex $z \notin \pi_1([S]_{\mathcal{C}})$. Then there exist $(x, y), (x', y') \in [S]_{\mathcal{C}}$. The distance from (x, y) to (x', y') is $d_{G_1}(x, x') + d_{G_2}(y, y')$, so there exists an $(x, y) - (x', y')$ geodesic that goes through (z, y) along the way to (x', y) before continuing on to (x', y'). Since $[S]_\mathcal{C} \in \mathcal{C}$ we have that $(z, y) \in [S]_\mathcal{C}$, so $z \in \pi_1([S]_\mathcal{C})$, a contradiction. By symmetric argument, $\pi_2([S]_\mathcal{C}) \in \mathcal{C}_2$.

Therefore $[S_i]_{\mathcal{C}_i} \subseteq \pi_i([S]_\mathcal{C})$ for $i = 1, 2$, since $\pi_i([S]_\mathcal{C}) \in \mathcal{C}_i$ and $S_i \subseteq \pi_i([S]_\mathcal{C})$.

Next we show that $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \subseteq [S]_\mathcal{C}$. Consider any $(x, y) \in [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$. There exist x', y' for which $(x, y') \in [S]_\mathcal{C}$ and $(x', y) \in [S]_\mathcal{C}$, since $[S_i]_{\mathcal{C}_i} \subseteq \pi_i([S]_\mathcal{C})$. There is an $(x, y') - (x', y)$ geodesic that passes through (x, y), so $(x, y) \in [S]_\mathcal{C}$. Therefore $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \subseteq [S]_\mathcal{C}$.

It is easy to see that $S \subseteq [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \subseteq [S]_\mathcal{C}$ where $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2} \in \mathcal{C}$. Yet $[S]_\mathcal{C}$ is the smallest element of $\mathcal{C}$ containing S, whereas $[S]_\mathcal{C}$ is known to contain $[S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$, so $[S]_\mathcal{C} = [S_1]_{\mathcal{C}_1} \times [S_2]_{\mathcal{C}_2}$, proving item 1.

Concerning item 2, consider any $x = (x_1, x_2) \in I[a, b]$, along an $a - b$ geodesic. Then since $d(a, b) = d(a_1, b_1) + d(a_2, b_2)$, it must be for $i = 1, 2$ that x_i is on an $a_i - b_i$ geodesic, so $x \in I[a_1, b_1] \times I[a_2, b_2]$. Conversely, consider any $x = (x_1, x_2) \in I[a_1, b_1] \times I[a_2, b_2]$. Then there exists an $a_1 - b_1$ geodesic which follows a path P from a_1 to x_1 and from there on to b_1 via a path Q, and likewise an $a_2 - b_2$ geodesic which follows a path R through x_2. Then the path from $a = (a_1, a_2)$ to (x_1, a_2) (formed as in P while holding the second entry a_2 fixed) followed by the path from there to (x_1, x_2) and then on to x_1, b_2 (formed as in R while holding the first entry x_1 fixed) followed by the path from there to (b_1, b_2) (formed as in Q while holding the second entry b_1 fixed) yields an $a - b$ geodesic containing x. Therefore item 2 holds.

Concerning item 3, $I[S] = \bigcup_{a,b \in S} I[a, b] = \bigcup_{a,b \in S} I[a_1, b_1] \times I[a_2, b_2] \subseteq (\bigcup_{a_1,b_1 \in S} I[a_1, b_1]) \times (\bigcup_{a_2,b_2 \in S} I[a_2, b_2]) = I[S_1] \times I[S_2]$. ■

Now we can show, as promised, that the g-convex sets of the graphically defined $G_1 \times G_2$ turn out to be exactly those convex sets that we get by instead using the convexity operator $\oplus$ in combining the g-convex sets for G_1 with the g-convex sets for G_2.

Theorem 1.4 *Let $G_1 = ((V_1, E_1), \mathcal{C}_1)$ and $G_2 = ((V_2, E_2), \mathcal{C}_2)$, where the elements of $\mathcal{C}_i$ are the g-convex sets in G_i. Let $G = (V, E)$ be the Cartesian product graph $G_1 \times G_2$ and let $\mathcal{C}$ be the set of g-convex sets in G. Then*

1. *$S \in \mathcal{C} \Leftrightarrow S_i \in \mathcal{C}_i$, $i \in \{1, 2\}$, and $S = S_1 \times S_2$ (where $S_i = \pi_i(S)$).*

2. *$\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2$.*

Proof. Part (1) of the previous proposition tells us that the g-convex subsets of V are precisely the Cartesian products $A_1 \times A_2$ where $A_i \in \mathcal{C}_i$ for $i = 1, 2$, since every g-convex set S must equal the Cartesian product of convex sets given for $[S]_\mathcal{C}$ in the proposition and since the proposition shows that every such Cartesian product equals its closure. Therefore item 1 holds. Item 2 is simply the observation that $\mathcal{C}_1 \oplus \mathcal{C}_2$ produces the very same convex sets as does $\mathcal{C}$. ■

While fundamental, the result of Theorem 1.4 is not surprising. But neither should it be entirely obvious. For example, in $\mathbf{R} \times \mathbf{R}$, the most well known notion of convexity is that of Euclidean convexity, yet the operator $\oplus$ applied to geodesic convexity spaces induced on the factor sets yields a very different convexity space, one in which the bounded nonempty convex sets are Cartesian products of bounded intervals of real numbers, i.e. horizontally and vertically aligned rectangles. Intuitively what is going on is that the standard distance metric in Cartesian products of graphs is the taxicab metric, whereas the familiar metric in $\mathbf{R} \times \mathbf{R}$ is the Euclidean metric.

2 Hull Numbers and Geodetic Numbers of Cartesian Products

Given a graph G, a subset S of $V(G)$ is called a *g-hull set* for G if $[S]_g = V(G)$. The *hull number* of a graph G, denoted by $hn(G)$, is the minimum cardinality of a g-hull set of $V(G)$. We are in a position to give a simple proof of a generalization of the following known result.

Proposition 2.1 *([14]) If $|V(G)| \geq 2$ then $hn(G \times K_2) = hn(G)$.*

Theorem 2.2 $hn(G \times H) = \max\{hn(G), hn(H)\}$

Proof. Let $A = \{a_1, \cdots, a_p\}$ be a minimum g-hull set for G and let $B = \{b_1, \cdots, b_q\}$ be a minimum g-hull set for H, where without loss of generality we can assume that $p \geq q$. Consider any $S \subseteq V(G \times H)$. By Proposition 1.3, $[S]_g = [S_1]_g \times [S_2]_g$, so S is a g-hull set for $G \times H$ if and only if S_1 is a g-hull set for G and S_2 is a g-hull set for H. The choice $S = \{(a_1, b_1), (a_2, b_2), ..., (a_q, b_q), (a_{q+1}, b_q)), (a_{q+2}, b_q), ..., (a_p, b_q)\}$ satisfies this requirement, with $|S| = p$, so $hn(G) \leq p$. Also, $hn(G \times H) \geq p$, since S must have at least $hn(G)$ many elements for its projection S_1 to have at least $hn(G)$ elements. Thus $hn(G \times H) = p$, completing the proof. ■

Given a graph G, a subset S of $V(G)$ is called a *geodetic set* for G if $I[S] = V(G)$. The *geodetic number* of a graph G, denoted by $gn(G)$, is the minimum cardinality of a geodetic set of $V(G)$. Since $I[S]$ need not be a convex set, whereas $[S]_g$ is necessarily convex, it turns out that the determination of the geodetic number of the Cartesian product of two graphs is more interesting than the determination of the hull number. Not surprisingly, the following lower bound result is related to the hull number for Cartesian products. The special case where $H = K_2$ has already appeared ([15]).

Proposition 2.3 *Let G and H be graphs. Then $gn(G \times H) \geq \max\{gn(G), gn(H)\}$, with equality when G, H are complete graphs.*

Proof. If S is a minimum geodetic set in $G \times H$ then by item 3 of Proposition 1.3, $V(G \times H) = I[S] \subseteq I[S_1] \times I[S_2]$. Therefore S_1, S_2 are geodetic sets in G, H respectively, so $gn(G \times H) = |S| \geq \max\{|S_1|, |S_2|\} \geq \max\{gn(G), gn(H)\}$, proving the lower bound. Consider complete graphs G, H with vertex sets $V(G) = \{u_1, u_2, \cdots u_p\}$ and $V(H) = \{v_1, v_2, \cdots, v_q\}$, where without loss of generality $p \geq q$. Then $gn(G) = p$ and $gn(H) = q$. Let $S = \{(u_1, v_1), (u_2, v_2), \cdots, (u_q, v_q), (u_{q+1}, v_q), (u_{q+2}, v_q), \cdots, (u_p, v_q)$ It is straightforward to verify that S is geodetic set for $G \times H$. Hence, $gn(G \times H) \leq |S| \leq p = \max\{gn(G), gn(H)\} \leq gn(G \times H)$, so equality holds. ■

Definition 2.4 *Let G, H be graphs and $G \times H$ the Cartesian product of G and H. Given $x \in V(H)$, the subgraph of $G \times H$ induced by $\{(u, x) \mid u \in V(G)\}$ is isomorphic to G. We call it the copy of G corresponding to x. Similarly, given $u \in V(G)$, the subgraph of $G \times H$ induced by $\{(u, x) \mid x \in V(H)\}$ is isomorphic to H, and we call it the copy of H corresponding to u.*

Given a path P and two vertices x, y on P, we use $P[x, y]$ to denote the portion of P between x and y, inclusive.

Theorem 2.5 *Let G, H be graphs with $gn(G) = p$ and $gn(H) = q$ where $p \geq q \geq 2$. Then*

$$gn(G \times H) \leq pq - q.$$

Proof. Let $A = \{a_1, \cdots, a_p\}$ be a geodetic set of G and $B = \{b_1, \cdots, b_q\}$ a geodetic set of H. Let $S = A \times B - \{(a_1, b_1), (a_2, b_2), \cdots, (a_q, b_q)\}$. Then $|S| = pq - q$. We show that S is a geodetic set of $G \times H$.

Let (u, x) be any vertex in $G \times H$, where $u \in V(G)$ and $x \in V(H)$. We prove that $(u, x) \in I_{G\times H}(S)$. Since A is a geodetic set of G, there exist $a_i, a_j \in A$, where $i \leq j$, such that u lies on a shortest a_i, a_j-path P in G. Similarly, since B is a geodetic set of H, there exist $b_k, b_l \in B$, where $k \leq l$, such that x lies on a shortest b_k, b_l-path Q in H.

We consider different cases.

Case 1. $i = j$ and $k = l$.

In this case, $u = a_i$ and $x = b_k$. If $i \neq k$, then $(u, x) \in S$. Suppose then $i = k$. Let $h \in [q] - \{i\}$; such h exists since $q \geq 2$. Let L be a shortest path between (a_h, b_i) and (a_i, b_i) in the copy of G corresponding to b_i and let L' be a shortest path between (a_i, b_i) and (a_i, b_h) in the copy of H corresponding to a_i. Then $L \cup L'$ is a path in $G \times H$ with length $d_G(a_h, a_i) + d_H(b_i, b_h)$ between (a_h, b_i) and (a_i, b_h). that contains (u, x). By Proposition 1.1, $d_{G\times H}((a_h, b_i), (a_i, b_h)) = d_G(a_h, a_i) + d_H(b_i, b_h)$. Hence $L \cup L'$ is a shortest path between (a_h, b_i) and (a_i, b_h) in $G \times H$. Furthermore, $(a_h, b_i), (a_i, b_h) \in S$ since $i \neq h$. Hence $(u, x) \in I_{G\times H}(S)$.

Case 2 $i < j, k = l$.

In this case, $x = b_k$. Suppose first that $k \notin \{i, j\}$. Then $(a_i, b_k), (a_j, b_k) \in S$. Recall that P is a shortest a_i, a_j-path in G containing u. The copy of P in the copy of G corresponding to b_k is a shortest path between (a_i, b_k) and (a_j, b_k) in $G \times H$ that contains (u, x). Hence $(u, x) \in I_{G\times H}(S)$.

Suppose next that $k = i$. Let L be a copy of P in the copy of G corresponding to b_i. Note that L contains (u, x). Let L' be a shortest path between (a_i, b_i) and (a_i, b_j) in the copy of H corresponding to a_i. By arguments similar to those in Case 1, $L \cup L'$ is a shortest path between (a_j, b_i) and (a_h, b_i) in $G \times H$ that contains (u, x). Since $(a_j, b_i), (a_i, b_i) \in S$, $(u, x) \in I_{G\times H}(S)$.

The subcase where $j = k$ can be handled in the same fashion.

Case 3 $i = j, k < l$

This case can be handled in the same fashion as Case 2.

Case 4 $i < j, k < l$

We consider three subcases.

Subcase 3.1 $i = k$ or $j = l$.

Recall that P is a shortest path between a_i and a_j in G that contains u and Q is a shortest path between b_k and b_l in G that contains x. Let L_1 be a copy of P in the copy of G correspoding to b_l. Let L_2 be a copy of Q in the copy of H corresponding to u. Let L_3 be a copy of P in the copy of G corresponding to x. Let L_4 be a copy of Q in the copy of H correpsonding to a_j. It is straightforward to verify, using Proposition 1.1 that $L_1[(a_i, b_l), (u, b_l)] \cup L_2[(u, b_l), (u, x)] \cup L_3[(u, x), (a_j, x)] \cup L_4[(a_j, x), (a_j, b_k)]$ is a shortest path between $(a_i, b_l), (a_j, b_k)$ in $G \times H$ that contains (u, x). Since $(a_i, b_l), (a_j, b_k) \in S$, we have $(u, x) \in I_{G\times H}(S)$.

Subcase 3.3 $i \neq k, j \neq l$

Let L_1 be a copy of P in the copy of G correspoding to b_k. Let L_2 be a copy of Q in the copy of H corresponding to u. Let L_3 be a copy of P in the copy of G corresponding to x. Let L_4 be a copy of Q in the copy of H correpsonding to a_l. It is straightforward to verify, using Proposition 1.1 that $L_1[(a_i, b_k), (u, b_k)] \cup L_2[(u, b_k), (u, x)] \cup L_3[(u, x), (a_j, x)] \cup L_4[(a_j, x), (a_j, b_l)]$ is a shortest path between

$(a_i, b_k), (a_j, b_l)$ in $G \times H$ that contains (u, x). Since $(a_i, b_k), (a_j, b_l) \in S$, we have $(u, x) \in I_{G\times H}(S)$. ■

Next, we show that inequality in Theorem 2.5 is best possible.

Construction 2.6 Given positive integers p, t, let D_p^t be a graph obtained as follows. Take p vertices $x_1, \cdots, x_p$, take $\binom{p}{2}$ groups of vertices $W_{i,j}$ for $i, j \in [p], i < j$, where the $W_{i,j}$'s are pairwise disjoint and also disjoint from $\{x_1, \cdots, x_p\}$ and each $W_{i,j}$ consists of t vertices. Next, for each pair $i, j \in [p], i < j$, join each of the t vertices in $W_{i,j}$ to both x_i and x_j. Finally, we add a new vertex z and join it to all the other vertices. The resulting graph is D_p^t.

Lemma 2.7 *Let p, t be positive integers such that $t > p$. Let $G = D_p^t$ be defined as in Construction 2.6. Then $gn(G) = p$.*

Proof. We keep the notation used in Construction 2.6. First, note that G has diameter 2. Since $S = \{x_1, \cdots, x_p\}$ is clearly a geodetic set of G, $gn(G) \le p$. Let S' be an arbitrary geodetic set of G. We need to prove that $|S'| \ge p$.

If S' contains all of $x_1, \cdots, x_p$, then $|S'| \ge p$. So, we may assume that $x_l \notin S'$ for some $l \in [p]$. let $i \in [p] - \{l\}$. Without loss of generality suppose $i < l$. Let w be any vertex in $W_{i,l}$. Suppose $u \notin S'$, then since S' is a geodetic set of G there exist $u, v \in S'$ such that w lies on a shortest u, v-path P in G. Such a path has length at least 2. Since G has diameter 2, P has length exactly 2. So u, v are two neighbors of w at distance two from each other in G. Since x_i, x_l, z are the only neighbors of w in G while z is at distance 1 from any other vertex, $\{u, v\} = \{x_i, x_l\}$. This contradicts our assumption that $x_l \notin S'$. Hence, S' must contain all of $W_{i,l}$. In particular, this implies that $|S'| \ge |W_{i,l}| = t > p$. ■

Proposition 2.8 *Let p, q, t be positive integers such that $t > pq - q$. Let $G = D_p^t$ as defined in Construction 2.6. Let $H = K_q$. Then $gn(G) = p$, $gn(H) = q$, and $gn(G \times H) \ge pq - q$.*

Proof. We keep the notation used in Construction 2.6. That $gn(G) = p$ follows readily from Lemma 2.7, while $gn(H) = q$ is trivial. Let S be a smallest geodetic set of $G \times H$. Suppose that $|S| < pq - q$, we derive a contradiction.

Suppose $V(H) = V(K_q) = \{v_1, \cdots, v_q\}$. For each $i \in [q]$, let G_i denote the copy of G corresponding to v_i. Since $|S| < (p-1)q$, by the pigeonhole principle, there exists some $l \in [q]$ such that $|S \cap V(G_l)| < p - 1$. For each vertex $u \in V(G)$, we use $u^{(l)}$ to denote its image in G_l. Recall the notation used in Construction 2.6. Since $|S \cap V(G_l)| \le p - 2$, there exist $i, j \in [p]$ such that $x_i^{(l)}, x_j^{(l)} \notin S$. Consider the set $W_{i,j}$. For each $w \in W_{i,j}$, let $A_w = \{w^{(1)}, \cdots, w^{(q)}\}$, i.e. A_w is the set of the q images of w in $G \times H$. We show that $S \cap A_w \ne \emptyset$ for each $w \in W_{i,j}$. If $w^{(l)} \in S$, then there is nothing to prove. So, we may assume that $w^{(l)} \notin S$. Then there exist vertices $a, b \in S$ such that $w^{(l)}$ lies on a shortest a, b-path P in $G \times H$. Since G has diameter 2 by Lemma 2.7 and $H = K_q$, $G \times H$ has diameter 3. Hence P has length at most 3. In particular, this implies that either a or b is a neighbor of $w^{(l)}$ in $G \times H$. Without loss of generality, suppose a is a neighbor of $w^{(l)}$ in $G \times H$. Note that the closed neighborhood of $w^{(l)}$ in $G \times H$ is $A_w \cup \{x_i^{(l)}, x_j^{(l)}, z^{(l)}\}$ and $a, b \notin \{x_i^{(l)}, x_j^{(l)}\}$ since $x_i^{(l)}, x_j^{(l)} \notin S$. If $a \ne z^{(l)}$ then $a \in A_w$ and thus $S \cap A_w \ne \emptyset$. So we may assume that $a = z^{(l)}$. Since z is a dominating vertex in G, $a = z^{(l)}$ is at distance at most 2 from any other vertex in $G \times H$. In particular, this implies that P has length 2, in which case b is also a neighbor of $w^{(l)}$. Hence, $b \in A_w$. Therefore $S \cap A_w \ne \emptyset$.

We have shown that for each $w \in W_{i,j}$, $S \cap A_w \neq \emptyset$. Since $W_{i,j}$ contains t different w's, and the $A'_w s$ are clearly pairwise disjoint for different w's, we have $|S| \geq t > pq - q$, contradicting $|S| < pq - q$. The contradiction completes the proof. ■

We summarize Proposition 2.3, Theorem 2.5, and Proposition 2.8 as follows.

Proposition 2.9 *Let G and H be graphs. We have*

$$\max\{gn(G), gn(H)\} \leq gn(G \times H) \leq gn(G) \cdot gn(H) - \min\{gn(G), gn(H)\}.$$

Furthermore, both inequalities above are best possible.

3 Classical Invariants and Cartesian Products

One "classical" convexity invariant studied in the literature as applied to arbitrary convexity spaces is that of the *rank* of a convexity space [58], a notion aptly named because of its similarities to the rank of a matrix or matroid. We present a newly defined variation, called the *weak rank*, and show how it applies to the determination of the rank of a graph. First we introduce some definitions concerning "dependent" and "independent" sets, as one might expect.

Consider a convexity space (X,C) and a subset S of X and an element s of S. We say that s is *(convexly) dependent in S* if $s \in [S-s]_{\mathcal{C}}$, and call *$s$ (convexly) independent in S* otherwise. S is called *convexly dependent* if $x \in [S-x]_{\mathcal{C}}$ for at least one $x \in S$ (i.e. some element is dependent in S) and called *convexly independent* otherwise. S is called *strongly convexly dependent* if $x \in [S-x]_{\mathcal{C}}$ for at least two elements $x \in S$, and called *weakly convexly independent* otherwise, i.e., when at most one exceptional such element x exists in S. We use the abbreviations *sc-dependent* and *wc-independent* for the terms "strongly convexly independent" and "weakly convexly independent", respectively. When wc-independent S is such that a particular element $s \in S$ is dependent in S we say that *s is the sole dependent in S*.

Observe that (s is dependent in S) $\Leftrightarrow s \in [S-s]_{\mathcal{C}} \Leftrightarrow [S]_{\mathcal{C}} = [S-s]_{\mathcal{C}}$. Also observe that convex independence is a hereditary property, as is wc-independence. That is, every subset of a convexly independent set is convexly independent, and every subset of a wc-independent set is wc-independent. Lastly observe that if s is the sole dependent in S then $S-s$ is convexly independent, since for each $x \in S-s$ it is the case that $[(S-s)-x]_{\mathcal{C}} \subseteq [S-x]_{\mathcal{C}}$, so $x \notin [(S-s)-x]_{\mathcal{C}}$ since $x \notin [S-x]_{\mathcal{C}}$.

Definition 3.1 *The* rank *of a graph G is the maximum cardinality of a convexly independent set, and is denoted by $r = r(G)$. The* weak rank *of a graph G is the maximum cardinality of a wc-independent set, and is denoted by $r^1 = r^1(G)$.*

Proposition 3.2 $r \leq r^1 \leq r+1$.

Proof. Certainly, $r \leq r^1$. For proving the upperbound, suppose for contradiction that a wc-independent set S of cardinality exceeding $r+1$ exists. By maximality of r, the set S must be convexly dependent, so let s be the sole dependent in S. Then $S-s$ is convexly independent and yet has cardinality exceeding r, a contradiction, completing the proof. ■

We introduced the concept of the weak rank of a set because at the end of this section we can make a connection between the rank of a Cartesian product of graphs and the weak ranks of the factor graphs. Meanwhile, we still develop one more claim toward that goal.

Proposition 3.3 *Consider any connected graphs G and H and any convexly independent subset S of $V(G \times H)$. Then each $(a, b) \in S$ must be of one or both of the following types.*
Type 1: $a \notin \pi_1(S - (a, b))$ and $a \notin [S_1 - a]$.
Type 2: $b \notin \pi_2(S - (a, b))$ and $b \notin [S_2 - a]$.

Proof. Consider any $(a, b) \in S$. Suppose for contradiction that there exist elements (a, b') and (a', b) of $[S - (a, b)]$. Then there would exist an $(a, b') - (a', b)$ geodesic in $G \times H$ that passes through (a, b), contradicting that $(a, b) \notin [S - (a, b)]$. Therefore either no such b' exists or no such a' exists. Suppose first that no such b' exists, i.e. $a \notin \pi_1([S - (a, b)])$. By Proposition 1.3, $a \notin [\pi_1(S - (a, b))]$, from which (a, b) is of Type 1. By symmetry, the remaining case (in which no such a' exists) requires that (a, b) is of Type 2. ■

Finally, we can now state and prove the connection between the rank of a Cartesian product of graphs and the weak ranks of the factor graphs.

Theorem 3.4 *For any nontrivial connected graphs G_1 and G_2, $r(G_1 \times G_2) = r^1(G_1) + r^1(G_2) - 2$.*

Proof. For $i = 1, 2$, let S_i be a largest wc-independent set of vertices in G_i, and select a particular element x_i of S_i, being sure to select the sole dependent in S_i if S_i happens to fail to be convexly independent. Then let $S = \{(a, x_2) \mid a \in S_1 - x_1\} \cup \{(x_1, b) \mid b \in S_2 - x_2\}$. Clearly $|S| = r^1(G_1) + r^1(G_2) - 2$. We show that S is convexly independent, establishing that $r(G_1 \times G_2) \geq r^1(G_1) + r^1(G_2) - 2$. To show that S is convexly independent, it suffices to show for each $a \in S_1 - x_1$ that $(a, x_2) \notin [S - (a, x_2)]$, i.e. that each (a, x_2) is independent in S, since by symmetry it follows that each (x_1, b) is independent in S as well, so that each element of S is independent in S. Now, by Proposition 1.3, $\pi_1([S - (a, x_2)])$ equals $[\pi_1(S - (a, x_2))]$. Since (a, x_2) is the only vertex of S having a in the first coordinate, this latter set in turn equals $[S - a]$. Since S_1 is wc-independent and a is not its sole dependent, we have that $a \notin [S - a] = \pi_1([S - (a, x_2)])$, proving that $(a, x_2) \notin [S - (a, x_2)]$, completing our proof that $r(G_1 \times G_2) \geq r^1(G_1) + r^1(G_2) - 2$.

Next we show that $r(G_1 \times G_2) \leq r^1(G_1) + r^1(G_2) - 2$. Let

$$T_1 = \{\textit{1st coordinates of the Type 1 vertices of } S\}$$

and let

$$T_2 = \{\textit{2nd coordinates of vertices of } S \textit{ that are of Type 2 but not Type 1}\},$$

so $|T_1| + |T_2| = r(G_1 \times G_2)$.

First suppose that $|S_1 - T_1| = 0$. Then $T_2 = \emptyset$ since no element of $T_1 = S_1$ can appear more than once among the first coordinates of vertices of S. Then S_1 is convexly independent, giving us that $r(G_1 \times G_2) \leq r(G_1) + 0 \leq r^1(G_1) + r^1(G_2) - 2$, since any 2-subset of $V(G_2)$ forms a wc-independent set. Similarly, $r(G_1 \times G_2) \leq r^1(G_1) + r^1(G_2) - 2$ if $|S_2 - T_2| = 0$. So, we can assume that there exist vertices $x_1 \in S_1 - T_1$ and $x_2 \in S_2 - T_2$.

Now, $T_1 \cup \{x_1\}$ is wc-independent, since each first coordinate of a Type 1 vertex is independent in S_1 (by the meaning of "Type 1"), so is also independent in $T_1 \cup \{x_1\}$. Therefore $|T_1| \leq r^1(G_1) - 1$. Similarly, $|T_2| \leq r^1(G_2) - 1$, from which $r(G_1 \times G_2) = |T_1| + |T_2| \leq r^1(G_1) + r^1(G_2) - 2$, completing the proof. ■

We recast the result of Theorem 3.4 in the context of related results concerning other classical convexity invariants.

S is *redundant* if $[S]_{\mathcal{C}} = \bigcup_{a \in S} [S - a]_{\mathcal{C}}$.

Observe that every convex dependent set is redundant, since for every $a \in A$, $a \in [A-a]_{\mathcal{C}} \Leftrightarrow [A]_{\mathcal{C}} = [A-a]_{\mathcal{C}}$. Irredundancy is not an hereditary property.

A subset S of a convexity space $(X, \mathcal{C})$ is said to be *weakly irredundant* if it is not strongly redundant, i.e., if there exists some element x in S satisfying the following condition:

$$\bigcup_{a \in S-x} [S-a]_{\mathcal{C}} \not\supseteq [S]_{\mathcal{C}}.$$

Certainly, every irredundant set is weakly irredundant, since

$$[S]_{\mathcal{C}} \supseteq \bigcup_{a \in S} [S-a]_{\mathcal{C}} = [S-x]_{\mathcal{C}} \cup \left(\bigcup_{a \in S-x} [S-a]_{\mathcal{C}} \right).$$

The *Caratheodory number* $c = c(G)$ is the maximum cardinality of an irredundant set. Observe that $c \leq r$. To show a connection between the result of Theorem 3.4 and a result of Sierksma [53] concerning the Caratheodory number, we introduce the notion of the *weak irredundancy number* c^1 of a convexity space $(V, \mathcal{C})$ as the maximum cardinality of a weakly irredundant set.

Proposition 3.5

1. *Every weakly irredundant set is weakly convexly independent.*
2. $c^1 \leq r^1$.

Proof. Let S be a non-weakly convexly independent set. Let $y, z \in S$, $y \neq z$, s.t. $[S-y]_{\mathcal{C}} = [S-z]_{\mathcal{C}} = [S]_{\mathcal{C}}$. Hence, for every $x \in S$, $[S]_{\mathcal{C}} = \bigcup_{a \in S-x} [S-a]_{\mathcal{C}}$. In consequence, S is not a weak redundant set, proving item 1. Item 2 is an immediate consequence. ■

Proposition 3.6 $c \leq c^1 \leq c+1$.

Proof. Certainly, $c \leq c^1$. For proving the upperbound, suppose S is a set of cardinality $c+2$, this means that $S-x$ is redundant, for every $x \in S$. Hence,

$$[S]_{\mathcal{C}} = \bigcup_{a \in S} [S-a]_{\mathcal{C}} = [S-x]_{\mathcal{C}} \cup \left(\bigcup_{a \in S-x} [S-a]_{\mathcal{C}} \right) = \left(\bigcup_{a \in S-x} [S-x-a]_{\mathcal{C}} \right) \cup \left(\bigcup_{a \in S-x} [S-a]_{\mathcal{C}} \right).$$

In consequence, as $[S-x-a]_{\mathcal{C}} \subset [S-a]_{\mathcal{C}}$ for every $x \in S$, we derive that S is not a weakly irredundant set. ■

The previous result is much like Proposition 3.2 concerning weak rank. The next result is the weak irredundancy number counterpart of Theorem 3.4. We had proved this next result independently, only to realize that it is equivalent to the theorem of Sierksma [53] listed immediately after. Consequently, we omit the proof, but call attention to the obvious connections between these results concerning rank and Caratheodory/irredundancy numbers.

Theorem 3.7 $c(G_1 \times G_2) = c^1(G_1) + c^1(G_2) - 2$.

Theorem 3.8 *([53])* $c(G) = c(G_1) + c(G_2) - k$, *where* $k \in \{0, 1, 2\}$ *is the number of factors for which* $e \leq c$.

A set of vertices S is called *exchange-dependent*, or just *e-dependent*, if $[S-x]_{\mathcal{C}} \subseteq \bigcup_{a \in S-x} [S-a]_{\mathcal{C}}$ for all $x \in S$, and *exchange independent* otherwise. Notice than a set is strongly redundant if and only if it is both redundant and exchange dependent. The *exchange number*, $e = e(G)$, is the maximum cardinality of an exchange independent set.

Theorem 3.9 *([56], [55])* $e(G) = \max\{e_1, c_1\} + \max\{e_2, c_2\} - 1 = e(G) = c_1 + sign(e_1 - c_1 - 1) + c_2 + sign(e_2 - c_2 - 1) + 1$

Theorem 3.10 *([56])* $e(G_1 \times G_2) = c^1(G_1) + c^1(G_2) - 1 = c(G_1 \times G_2) + 1$

Proof. Use the previous theorem, and prove that: $c^1 = c + \text{sign}(e - c - 1) + 1 = \max\{e, c\}$. ∎

4 Concluding Remarks

While we have defined the rank, Caratheodory number and exchange number only in the context of applying them to graphs, all three are traditionally understood (using the same definitions given here) as applying to arbitrary convexity spaces.

Results given here concerning Cartesian products of two graphs G_1, G_2 generalize easily to Cartesian products of arbitrarily many graphs. For example, more generally than we bothered to state in Proposition 2.3, it is true for all graphs $G_1, ..., G_k$ that $gn(G_1 \times G_2 ... \times G_k) \geq \max\{gn(G_i)\}$.

Here are two more miscellaneous known result concerning convexity paremeters applied directly to Cartesian products of graphs.

Theorem 4.1 *([53]) Let* $(X, \mathcal{C}) = (X_1 \times X_2, \mathcal{C}_1 \oplus \mathcal{C}_2)$. *Then:* $h = \max\{h_1, h_2\}$, *where* $h_i = h(X_i)$, *where* h *denotes the Helly number of the corresponding convexity space.*

Theorem 4.2 *([11])* $con(G_1 \times G_2) = \max\{|V_2| \cdot con(G_1), |V_1| \cdot con(G_2)\}$, *where* $con(G)$, *called the* convexity number, *is the maximum cardinality of a proper g-convex set of* $V(G)$.

Whereas $e - 1 \leq c$, we know of no example graphs G for which $c(G) > e(G)$. We leave it as an open problem as to whether $e - 1 \leq c \leq e$ holds in general for all graphs.

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