

# Dominating location in graphs <sup>1</sup>

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[CanaDAM 2011](#)<sup>3</sup>

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<sup>1</sup> Joint work with J. Cáceres, C. Hernando, M. Mora, M. L. Puertas.

<sup>2</sup> UPC, Catalunya, Spain, Europe, The Earth, Solar System, Milky Way.

<sup>3</sup> UVic, BC, Canada, America, The Earth, Solar System, Milky Way.

$G = (V, E)$  is a simple finite connected graph. For every  $u \in V$ ,

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▷ A set  $\Omega$  of vertices in a graph  $G$  is a *dominating set* if any of the following equivalent statements is satisfied:

- Every vertex of  $V(G) \setminus \Omega$  has a neighbour in  $\Omega$ .
- $N[\Omega] = \bigcup_{w \in \Omega} N[w] = V(G)$ .
- For every  $u \in V(G) \setminus \Omega$ ,  $N(u) \cap \Omega = N[u] \cap \Omega \neq \emptyset$
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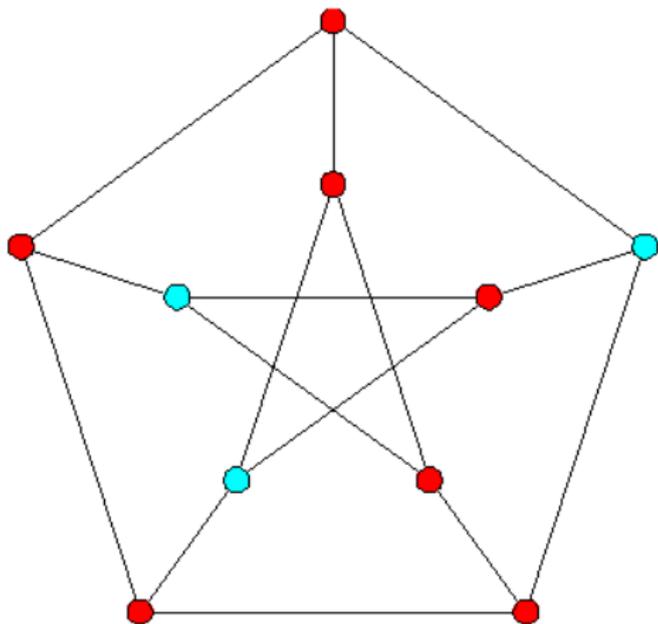
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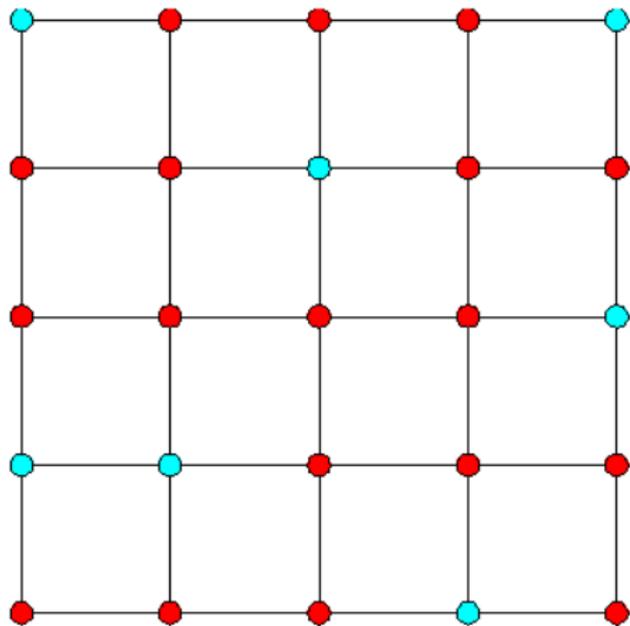
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- ▷ Dominating sets of order  $\gamma(G)$  are called  *$\gamma$ -sets*.



$\gamma(P) = 3$ , since **blue vertices** form a minimum dominating set.



$\gamma(P_5 \square P_5) = 7$ , as **blue vertices** form a minimum dominating set.

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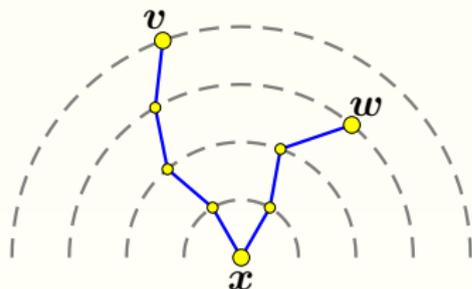
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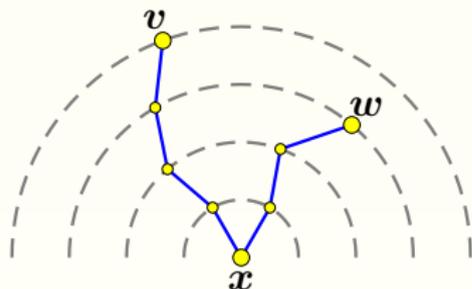
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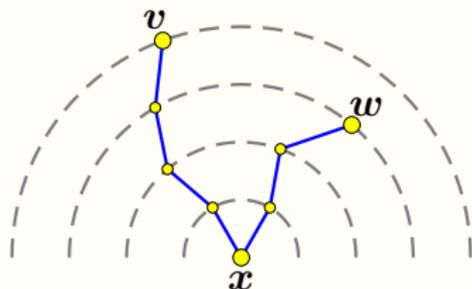
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- $S \subseteq V$  is a *locating set* (also called a *resolving set*) of  $G$  if every pair  $v, w \in V$  are resolved by some vertex  $x \in S$ . [SI76, HaMe76]

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- Let  $S = \{u_1, \dots, u_k\}$  be a locating set. The ordered set:

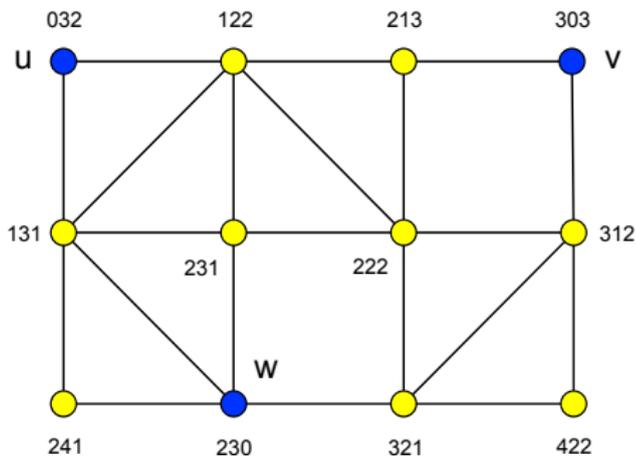
$$[d(x, u_1), \dots, d(x, u_k)]$$

is the vector of *metric coordinates* of  $x \in V$  with respect to  $S$ .

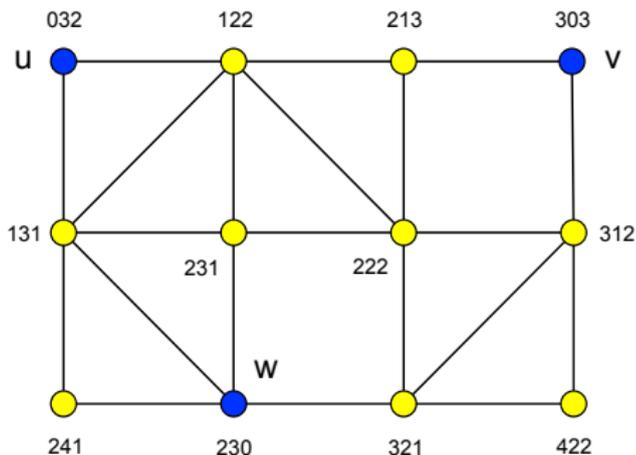


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▷ In this graph,  $\{u, v, w\}$  is a metric basis.

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- If  $\beta \geq 2$  [HeMoIMPSeWo10]:

$$\beta + D \leq n \leq \left( \left\lfloor \frac{2D}{3} \right\rfloor + 1 \right)^\beta + \beta \sum_{i=1}^{\lceil D/3 \rceil} (2i - 1)^{\beta-1}$$

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Moreover,

- \* Both bounds are tight.
- \* The set of all graphs satisfying  $n = \beta + D$  has been completely characterized.

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- ▷ A set  $D$  of vertices in a graph  $G$  is a *metric-locating-dominating set*<sup>4</sup> if it is both locating and dominating. [HeOe04]

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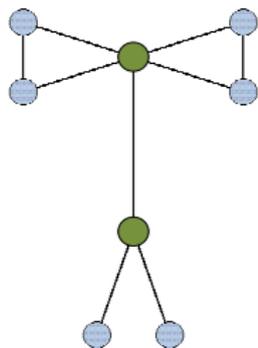
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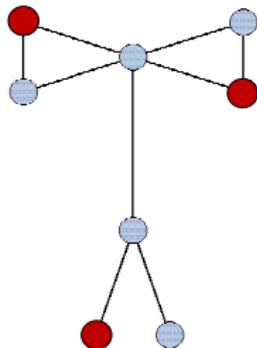
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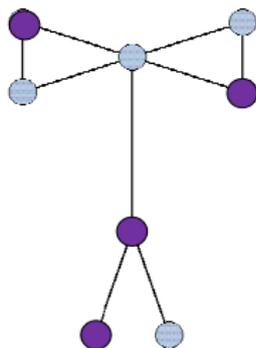
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$$\gamma(G) = 2$$



$$\beta(G) = 3$$



$$\eta(G) = 4$$

$$\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 4 \leq \gamma(G) + \beta(G) = 5$$

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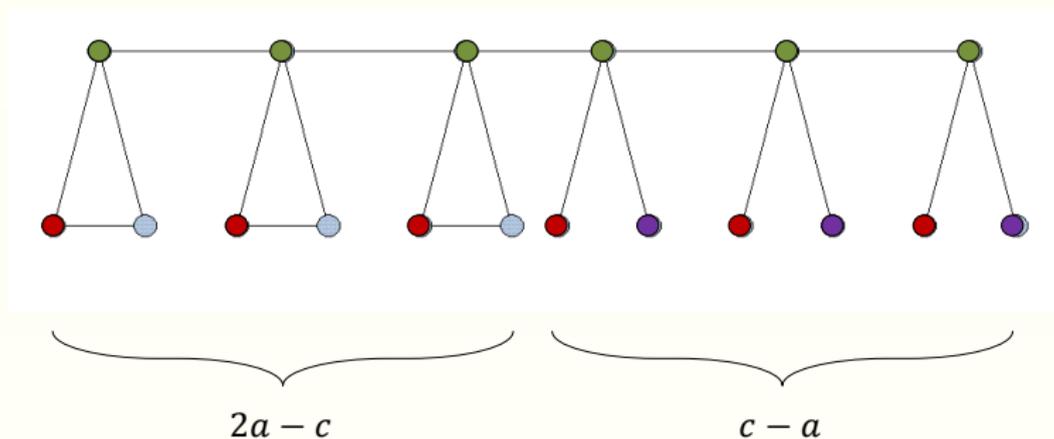
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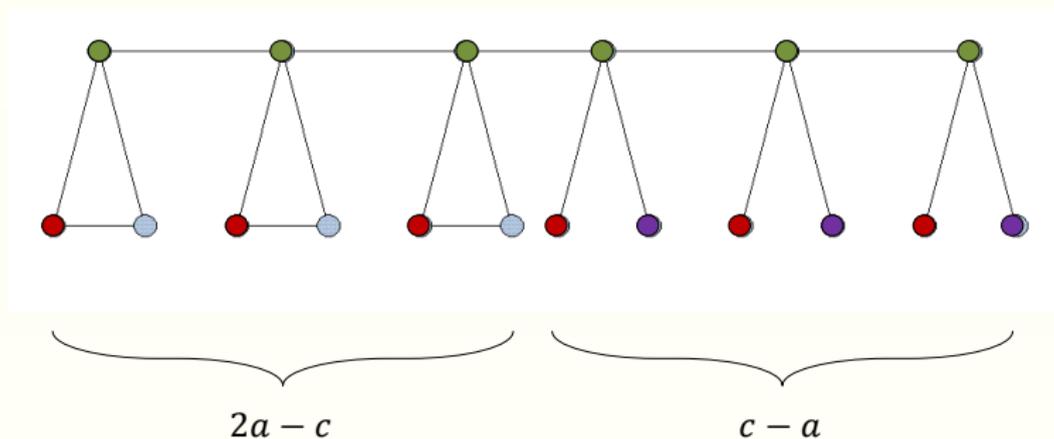
there always exists a graph  $G$  such that

$$\gamma(G) = a, \beta(G) = b \text{ and } \eta(G) = c,$$

except for the case:  $1 = b < a < c = a + 1$ .

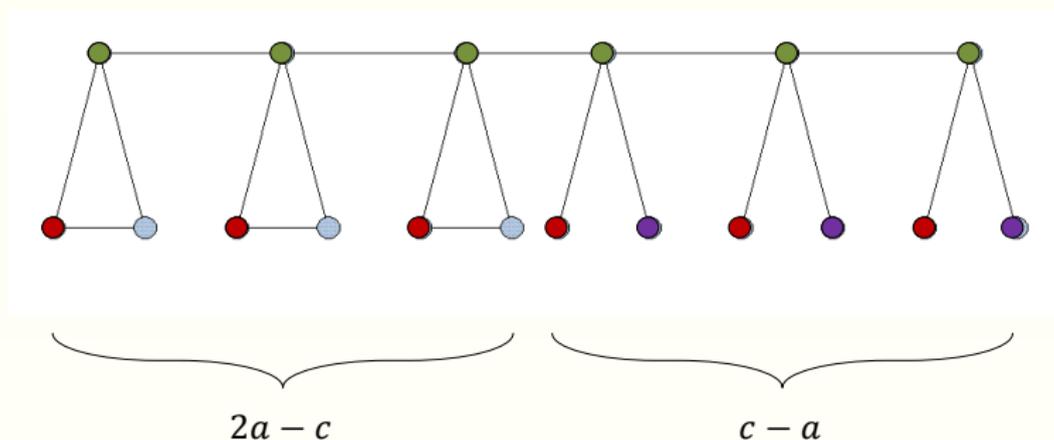


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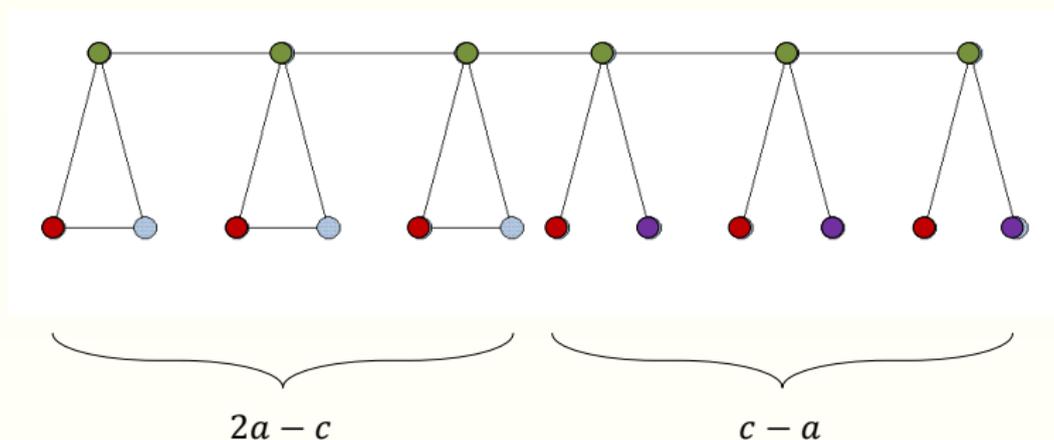
\* Green vertices:  $\gamma$ -set  $\rightarrow$   $\gamma = a$



Case:  $2 \leq a = b < c \leq 2a$

\* Green vertices:  $\gamma$ -set  $\rightarrow \boxed{\gamma = a}$

\* Red vertices: Minimum locating set  $\rightarrow \boxed{\beta = b = a}$



*Case:  $2 \leq a = b < c \leq 2a$*

- \* **Green** vertices:  $\gamma$ -set  $\rightarrow \boxed{\gamma = a}$
- \* **Red** vertices: Minimum locating set  $\rightarrow \boxed{\beta = b = a}$
- \* **Red + purple** vertices:  $\eta$ -set  $\rightarrow \boxed{\eta = (2a - c) + 2(c - a) = c}$



- ▷ A set  $D$  of vertices in a graph  $G$  is a *locating-dominating set* if for every two vertices  $u, v \in V(G) \setminus D$ ,

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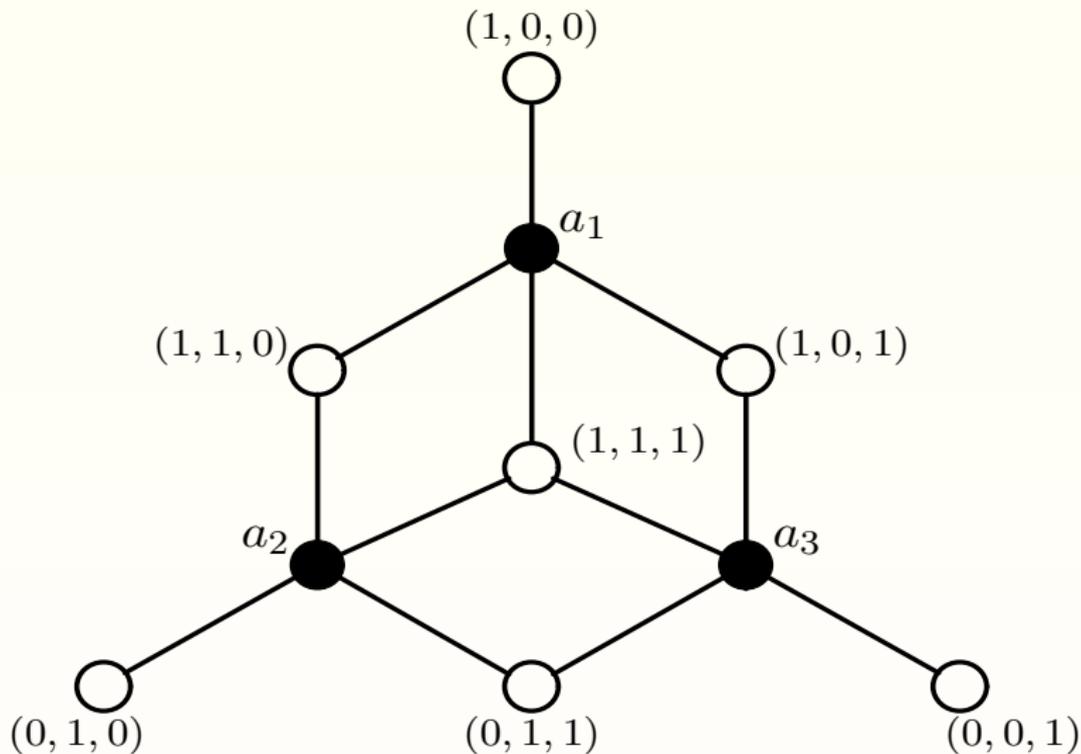
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⇒ Every locating-dominating set is both locating and dominating.  
Hence,

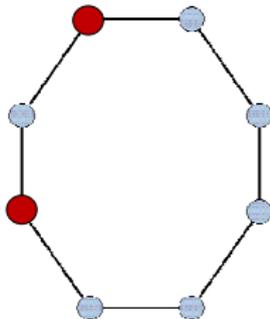
$$\max\{\gamma(G), \beta(G)\} \leq \eta(G) \leq \min\{\lambda(G), \gamma(G) + \beta(G)\}$$

and both bounds are tight.

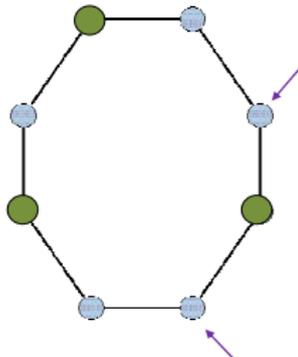


In all cases, digit **0** means "greater than 1"

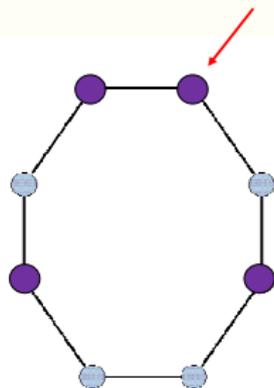
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$$\beta(G) = 2$$



$$\gamma(G) = \eta(G) = 3$$



$$\lambda(G) = 4$$

$$\max\{\gamma(G), \beta(G)\} = 3 \leq \eta(G) = 3 \leq \min\{\lambda(G), \gamma(G) + \beta(G)\} = 4$$

| $n$ | $\delta$ | $\Delta$ | $D$ | $G$        | $\gamma$ | $\beta$ | $\eta$ | $\lambda$ |
|-----|----------|----------|-----|------------|----------|---------|--------|-----------|
| 2   | 1        | 2        | 1   | $P_2$      | 1        | 1       | 1      | 1         |
| 3   | 1        | 2        | 2   | $P_3$      | 1        | 1       | 2      | 2         |
| 3   | 2        | 2        | 1   | $C_3$      | 1        | 2       | 2      | 2         |
| 4,5 | 2        | 2        | 2   | $C_4, C_5$ | 2        | 2       | 2      | 2         |
| 6   | 2        | 2        | 3   | $C_6$      | 2        | 2       | 3      | 3         |
| 5   | 3        | 4        | 2   | $W_{1,4}$  | 1        | 2       | 2      | 2         |
| 6   | 3        | 5        | 2   | $W_{1,5}$  | 1        | 2       | 3      | 3         |
| 7   | 3        | 6        | 2   | $W_{1,6}$  | 1        | 3       | 3      | 3         |

## Domination parameters of some basic graphs

| G                             | $\gamma$                    | $\beta$                        | $\eta$                         | $\lambda$                      |
|-------------------------------|-----------------------------|--------------------------------|--------------------------------|--------------------------------|
| $P_n, n > 3$                  | $\lceil \frac{n}{3} \rceil$ | 1                              | $\lceil \frac{n}{3} \rceil$    | $\lceil \frac{2n}{5} \rceil$   |
| $C_n, n > 6$                  | $\lceil \frac{n}{3} \rceil$ | 2                              | $\lceil \frac{n}{3} \rceil$    | $\lceil \frac{2n}{5} \rceil$   |
| $K_n, n > 1$                  | 1                           | $n - 1$                        | $n - 1$                        | $n - 1$                        |
| $K_{1,n-1}, n > 2$            | 1                           | $n - 2$                        | $n - 1$                        | $n - 1$                        |
| $K_{r,n-r}, n - r \geq r > 1$ | 2                           | $n - 2$                        | $n - 2$                        | $n - 2$                        |
| $W_{1,n-1}, n > 7$            | 1                           | $\lfloor \frac{2n}{5} \rfloor$ | $\lceil \frac{2n-2}{5} \rceil$ | $\lceil \frac{2n-2}{5} \rceil$ |

**Domination parameters** of some basic families

$T$  is a tree having  $\ell(T)$  leaves,  $s(T)$  support vertices, domination number  $\gamma$ , metric-location-domination number  $\eta$  and location-domination number  $\lambda$ .

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- $\eta(T) = \gamma(T) + \ell(T) - s(T)$  [HeOe04]

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- $\eta(T) = \gamma(T) + \ell(T) - s(T)$  [HeOe04]

- $\frac{n+\ell(T)-s(T)+1}{3} \leq \lambda(T) \leq \frac{n+\ell(T)-s(T)}{2}$  [BIChMaMoSe07]

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- $\frac{n+\ell(T)-s(T)+1}{3} \leq \lambda(T) \leq \frac{n+\ell(T)-s(T)}{2}$  [BIChMaMoSe07]

- $\eta \leq \lambda \leq 2\eta - 2$  [HeOe04]

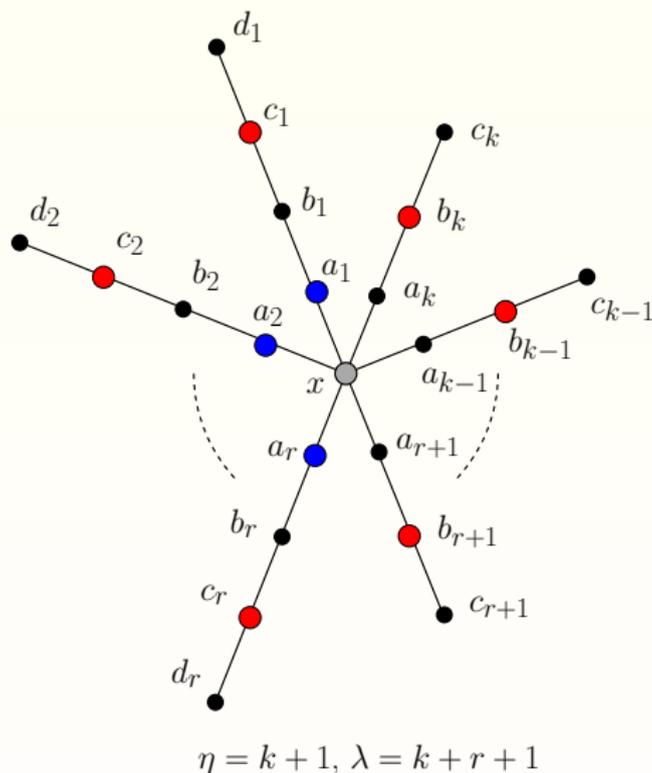
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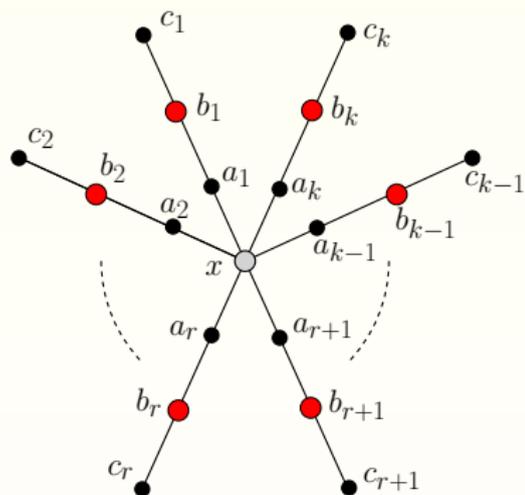
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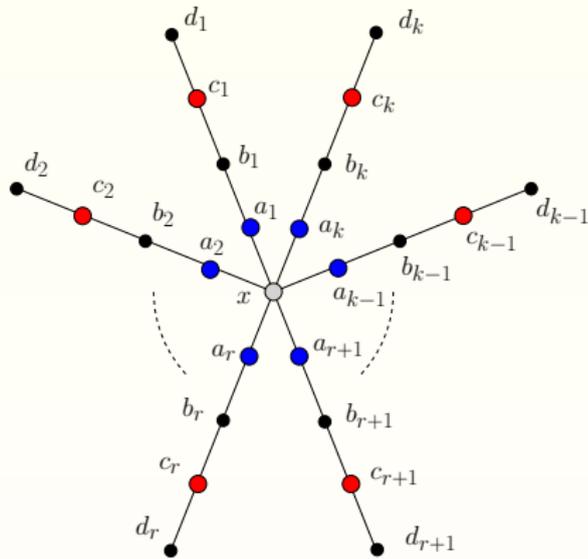
$\implies$  Given two integers  $a, b$  such that  $3 \leq a \leq b \leq 2a - 2$ , there always exists a tree  $G$  such that  $\eta(G) = a$  and  $\lambda(G) = b$ .



Spider with  $k$  legs,  $r$  of them having 4 edges, and the rest 3 edges.



$$\eta = \lambda = k + 1$$



$$\eta = k + 1, \lambda = 2k$$

LEFT: Spider with  $k$  legs, all of them having 3 edges.

RIGHT: Spider with  $k$  legs, all of them having 4 edges.

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\* In all cases, both bounds are tight.

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- $\lambda + \lfloor \frac{3D+1}{5} \rfloor \leq n \leq \lambda + 2^\lambda - 1$  is similarly proved.



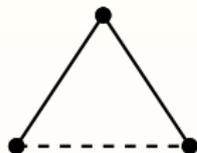
- $\eta(G) = n - 1 \Leftrightarrow \lambda(G) = n - 1 \Leftrightarrow G = K_n$  or  $G = K_{1,n-1}$

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- $\lambda(G) = n - 2 \Leftrightarrow \eta(G) = n - 2 \Leftrightarrow G \in \cup_{i=1}^7 F_i$  [HeOe04]

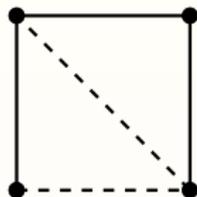
| $i$ | $F_i$   | $\beta$                                | $\eta$ | $\lambda$ |
|-----|---|--|--------|-----------|
| 1   | $\overline{K}_m + K_1 + K_1 + \overline{K}_k$             | $\begin{cases} n-4 \\ n-3 \end{cases}$ | $n-2$  | $n-2$     |
| 2   | $K_m + K_1 + \overline{K}_k, m \geq 2, k \geq 2$          | $n-3$                                  | $n-2$  | $n-2$     |
| 3   | $K_1 + \overline{K}_m + K_1 + \overline{K}_k, m \geq 2$   | $n-3$                                  | $n-2$  | $n-2$     |
| 4   | $(K_1 + \overline{K}_m) + K_1 + \overline{K}_k, m \geq 2$ | $n-3$                                  | $n-2$  | $n-2$     |
| 5   | $K_{s,t}, 2 \leq s \leq t$                                | $n-2$                                  | $n-2$  | $n-2$     |
| 6   | $K_s + \overline{K}_t, 2 \leq s, 2 \leq t$                | $n-2$                                  | $n-2$  | $n-2$     |
| 7   | $K_1 + K_s + K_t, 1 \leq s, 2 \leq t$                     | $n-2$                                  | $n-2$  | $n-2$     |

Table: All graphs satisfying  $\eta(G) = n - 2$ .

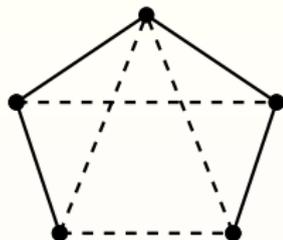
- $\eta(G) = 1 \Leftrightarrow \lambda(G) = 1 \Leftrightarrow G = P_2$
- $\lambda(G) = 2 \Rightarrow \eta(G) = 2$ . [converse false]



$$n = 3$$



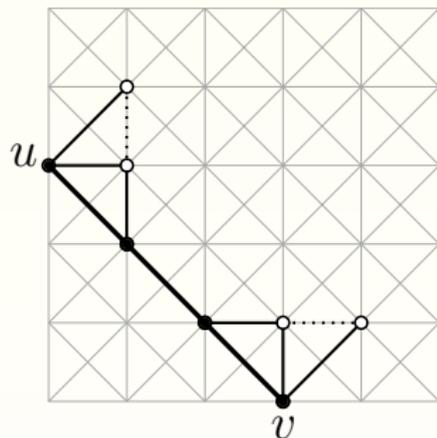
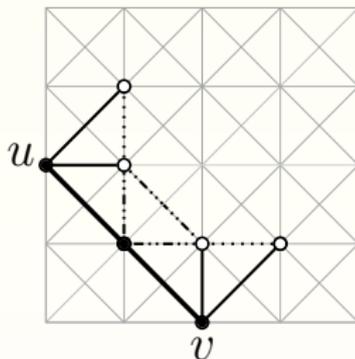
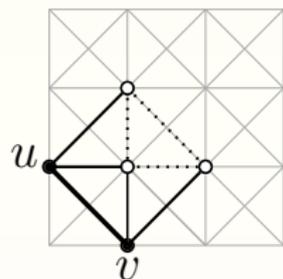
$$n = 4$$



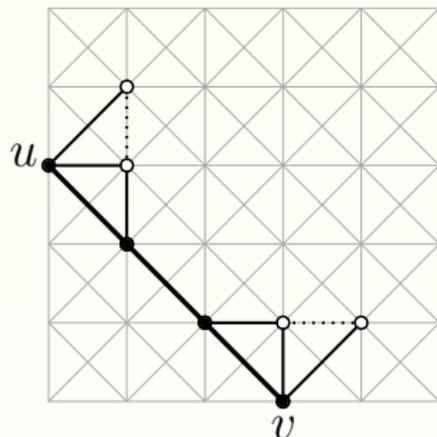
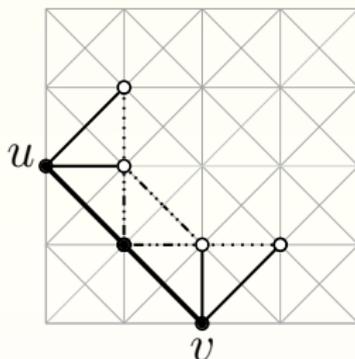
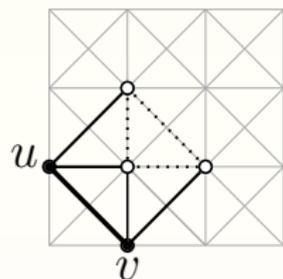
$$n = 5$$

There are 16 graphs s.t.  $\lambda = 2$  (notice that  $\lambda = 2 \Rightarrow n \leq 5$ )

There are 51 graphs satisfying  $\eta = 2$

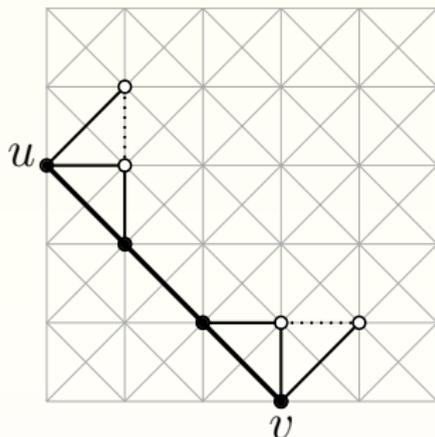
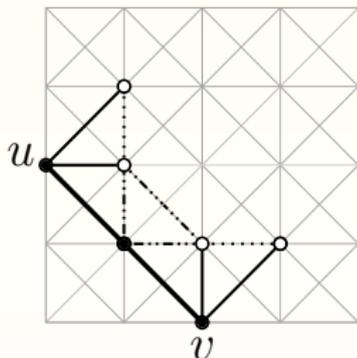
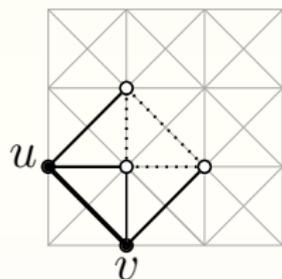


There are 51 graphs satisfying  $\eta = 2$



$$\triangleright \eta = 2 \Rightarrow n \leq \eta + \eta \cdot 3^{\eta-1} = 8$$

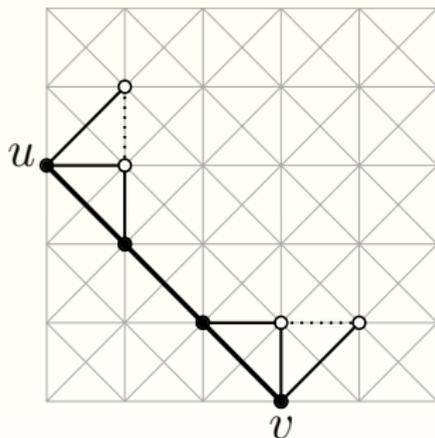
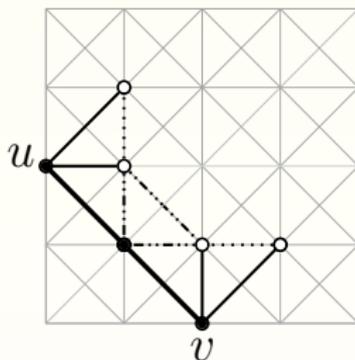
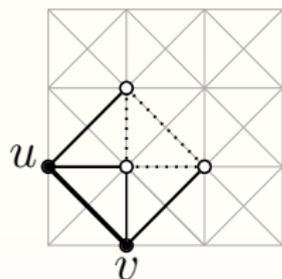
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▷  $\eta = 2 \Rightarrow n \leq \eta + \eta \cdot 3^{\eta-1} = 8$

▷ Every graph verifying  $\beta \leq 2$  can be embedded into the king grid.

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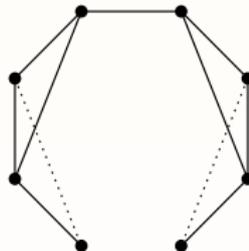
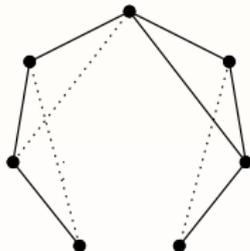
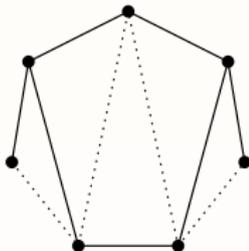
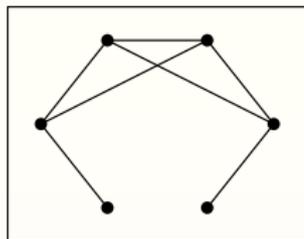
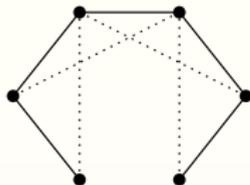
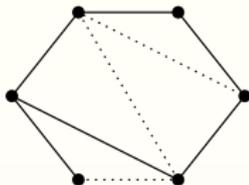
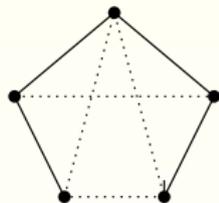
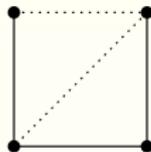
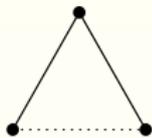


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▷ Every graph verifying  $\beta \leq 2$  can be embedded into the king grid.

▷ If  $\{u, v\}$  is an  $\eta$ -set, then it is dominant, and hence  $d(u, v) \leq 3$ .

There are 51 graphs satisfying  $\eta = 2$

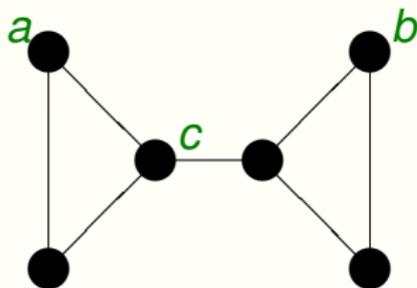




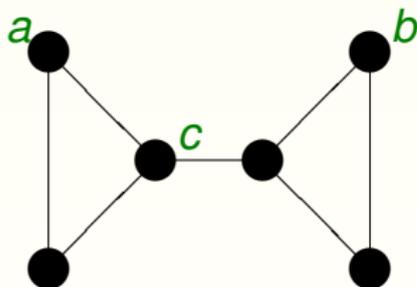
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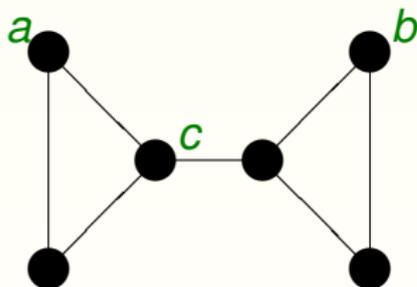


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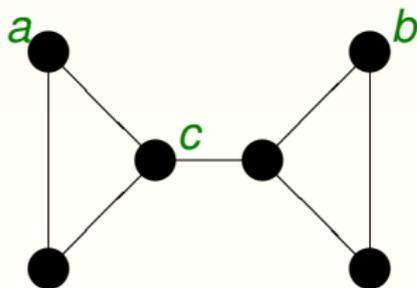
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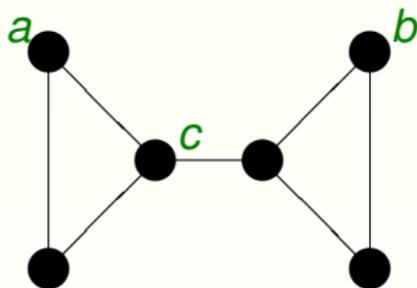
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- $\{a, b\}$  is an  $\eta$ -set
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- $n = 6, D = 3$ .
- $n - 4 = 2 = \eta(G) < \lambda(G) = 3 = n - 3$

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- In [BICHMaMoSe07], the authors obtained all trees attaining the upperbound of  $\frac{n+\ell(T)-s(T)+1}{3} \leq \lambda(T) \leq \frac{n+\ell(T)-s(T)}{2}$ . Almost nothing is known about the lowerbound.

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- In [HeOe04], it was proved that  $\frac{\lambda}{\eta}$  can not be upperbounded by a constant. Proving or disproving that, for some constant  $c$ ,

$$\eta \leq \lambda \leq c \cdot \eta^2.$$

- In [BICHMaMoSe07], the authors obtained all trees attaining the upperbound of  $\frac{n+\ell(T)-s(T)+1}{3} \leq \lambda(T) \leq \frac{n+\ell(T)-s(T)}{2}$ . Almost nothing is known about the lowerbound.
- The only significant result involving the Cartesian product operator is the following one:

$$\lambda(K_n \square K_m) = \begin{cases} n-1 & \text{if } 2m-1 < n \\ \lfloor \frac{2}{3}(n+m-1) \rfloor + 1 & \text{if } n \leq 2m-1 \text{ and } n+m = 3k+2 \\ \lfloor \frac{2}{3}(n+m-1) \rfloor & \text{in any other case} \end{cases}$$