Locating domination in bipartite graphs and their complements[S]

Carmen Hernando, Mercè Mora, Ignacio M. Pelayo

Departament de Matemàtiques, Universitat Politècnica de Catalunya.
E-mail: {carmen.hernando,merce.mora,ignacio.m.pelayo}@upc.edu

---

Abstract. A set S of vertices of a graph G is *distinguishing* if the sets of neighbors in S for every pair of vertices not in S are distinct. A *locating-dominating set* of G, *LD-set* for short, is a dominating distinguishing set. The *location-domination number* of G, $\lambda(G)$, is the minimum cardinality of an LD-set. In this work we study relationships between $\lambda(G)$ and $\lambda(\overline{G})$ for bipartite graphs. The main result is the characterization of connected bipartite graphs G satisfying $\lambda(\overline{G}) = \lambda(G) + 1$. To this aim, we define an edge-labeled graph G^S associated with a distinguishing set S that turns out to be very helpful.

Keywords. Locating-domination, bipartite graphs, complement, distinguishing sets

---

1 Introduction

Let $G = (V, E)$ be a simple, finite graph. The *neighborhood* of a vertex $u \in V$ is $N(u) = \{v : uv \in E\}$. We say that wo vertices u and v are *twins* if $N(u) = N(v)$ or $N(u) \cup \{u\} = N(v) \cup \{v\}$. A set $S \subseteq V$ is *distinguishing* if $N(u) \cap S \neq N(v) \cap S$ for every pair of different vertices $u, v \notin S$. A *locating-dominating set*, *LD-set* for short, is a distinguishing dominating set. The *location-domination number* of G, denoted by $\lambda(G)$, is the minimum cardinality of a locating-dominating set. A locating-dominating set of cardinality $\lambda(G)$ is called an *LD-code* [3]. LD-codes and the location-domination parameter have been intensively studied during the last decade (see [2] and its references).

The *complement* of G, denoted by $\overline{G}$, is the graph on the same vertices such that two vertices are adjacent in $\overline{G}$ if and only if they are not adjacent in G. This work is devoted to approach the relationship between $\lambda(G)$ and $\lambda(\overline{G})$ for connected bipartite graphs. It follows immediately from the definitions that a set $S \subseteq V$ is distinguishing in G if and only if it is distinguishing in $\overline{G}$. A straightforward consequence of this fact are the following results.

[S] Partially supported by projects MTM2014-60127-P, MTM2015-63791-R, Gen. Cat. DGR 2014SGR46. This project has received funding from the European Union's Horizon 2020 research and innovation programme under the Marie Skłodowska-Curie grant agreement No 734922.

Proposition 1 ([2])**.** *If $S \subseteq V$ is an LD-set of a graph $G = (V, E)$, then S is an LD-set of $\overline{G}$ if and only if S is a dominating set of $\overline{G}$;*

Proposition 2 ([1])**.** *For every graph G, $|\lambda(G) - \lambda(\overline{G})| \leq 1$.*

According to the preceding inequality, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$ for every graph G, all cases being feasible for some connected graph G. We intend to determine graphs such that $\lambda(\overline{G}) > \lambda(G)$, that is, we want to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$. This problem was completely solved in [2] for the family of block-cactus. In this work, we carry out a similar study for bipartite graphs. For this purpose, we first introduce in Section 2 the graph associated with a distinguishing set. This graph turns out to be very helpful to derive some properties related to LD-sets and the location-domination number of G, and will be used to state the main results in Section 3.

2 The graph associated with a distinguishing set

Let $G = (V, E)$ be a graph of order n and let $S \subseteq V$ be a distinguishing set of G.

Definition 1. *The graph associated with S, denoted by G^S, is the edge-labeled graph defined as follows:*

i) $V(G^S) = V \setminus S$;

ii) *If $x, y \in V(G^S)$, then $xy \in E(G^S)$ if and only if the sets of neighbors of x and y in S differ in exactly one vertex $u(x, y) \in S$;*

iii) *The label $\ell(xy)$ of edge $xy \in E(G^S)$ is the only vertex $u(x, y) \in S$ described in the preceding item.*

We can represent the graph G^S with the vertices lying on $|S| + 1$ levels, from bottom (level 0) to top (level $|S|$), in such a way that vertices with exactly k neighbors in S are at level k. There are at most $\binom{|S|}{k}$ vertices at level k. An edge of G^S has its endpoints at consecutive levels. Moreover, if $e = xy \in E(G^S)$, with $\ell(e) = u \in S$, and x is at exactly one level higher than y, then the neighborhood of x is S is obtained by adding vertex u to the neighborhood of y is S. Hence, x and y have the same neighborhood in $S \setminus \{u\}$. Therefore, the existence of an edge in G^S with label $u \in S$ means that $S \setminus \{u\}$ is not a distinguishing set. The converse is not necessarily true: it is possible that $S \setminus \{u\}$ were not distinguishing because it does not distinguish u from another vertex not in S. See Figure 1.1 for an example of an LD-set-associated graph. Next proposition states some properties of this graph.

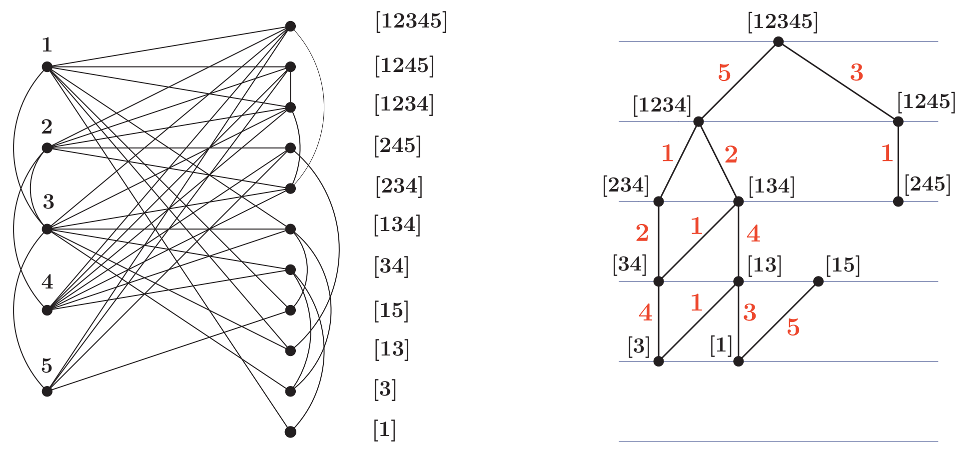

Figure 1.1: A graph G (left) and the graph G^S associated with the distinguishing set $S = \{1, 2, 3, 4, 5\}$ (right). The neighbors in S of each vertex are those enclosed in brackets.

Proposition 3. *Let $S \subseteq V$ be a distinguishing set of size r of a graph G of order n. Then, the graph G^S satisfies the following conditions.*

i) *G^S is a bipartite graph of order $n - r$ and incident edges have different labels.*

ii) *If $xy \in E(G^S)$ and $\ell(xy) = u \in S$, then x and y are not distinguished by $S \setminus \{u\}$.*

iii) *All the vertices belonging to a connected component of the subgraph of G^S induced by the edges with label in S' have the same neighborhood in $S \setminus S'$. item[iv)] The set S is a distinguishing set of $\overline{G}$ and $G^S = \overline{G}^S$. Moreover, the representation by levels of the graph $\overline{G}^S$ is obtained by turning upside down the representation of the graph G^S.*

3 The bipartite case

This section is devoted to solve the equation $\lambda(\overline{G}) = \lambda(G) + 1$ when we restrict ourselves to bipartite graphs. As we have seen in Section 1, $\lambda(\overline{G}) \in \{\lambda(G) - 1, \lambda(G), \lambda(G) + 1\}$, and for every pair r, s, $3 \le r \le s$, it is easy to give bipartite graphs such that $\lambda(\overline{G}) = \lambda(G) - 1$ (the bi-star $K_2(r, s)$) and bipartite graphs such that $\lambda(\overline{G}) = \lambda(G)$ (the biclique $K_{r,s}$). We want to analyze now the case

$\lambda(\overline{G}) = \lambda(G)+1$. In the sequel, $G = (V, E)$ stands for a bipartite connected graph of order $n \geq 3$, being U and W their stable sets and $1 \leq |U| = r \leq s = |W|$.

Proposition 4. *G satisfies $\lambda(\overline{G}) \leq \lambda(G)$ if any of the following conditions holds:*

i) $1 \leq r \leq 2$ *or* $3 \leq r = s$*;*

ii) *G has an LD-code with vertices at both stable sets;*

iii) $r < s$ *and W is an LD-code of G.*

Corollary 1. *If $\lambda(\overline{G}) = \lambda(G) + 1$, then U is the only LD-code of G and $3 \leq r < s \leq 2^r - 1$.*

Theorem 1. *Let $3 \leq r < s$. Then, $\lambda(\overline{G}) = \lambda(G) + 1$ if and only if the following conditions hold:*

i) *W has no twins;*

ii) *There exists a vertex $w \in W$ such that $N(w) = U$;*

iii) *For every vertex $u \in U$, the graph G^U associated with U has at least two edges with label u.*

By Proposition 3, and taking into account that G is a bipartite graph, the third condition of the preceding theorem is equivalent to the existence of at least two pairs of twins in W in the graph $G - u$, for every vertex $u \in U$.

Corollary 2. *If $r \geq 3$ and $\lambda(\overline{G}) = \lambda(G) + 1$, then $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$.*

Theorem 2. *For every pair (r, s), $r, s \in \mathbb{N}$, such that $3 \leq r$ and $\frac{3r}{2} + 1 \leq s \leq 2^r - 1$, there exists a bipartite graph $G(r, s)$ such that $\lambda(\overline{G}) = \lambda(G) + 1$.*

References

[1] C. Hernando, M. Mora, I. M. Pelayo. Nordhaus-Gaddum bounds for locating domination, *Eur. J. Combin.*, **36** (2014), 1–6.

[2] C. Hernando, M. Mora, I. M. Pelayo. On global location-domination in graphs, *Ars Math. Contemp.* **8** (2015), no. 2, 365–379.

[3] D. F. Rall, P. J. Slater. On location-domination numbers for certain classes of graphs, *Congr. Numer.* **45** (1984), 97–106.