Limited broadcast domination: upper bounds and complexity[Ⓢ]

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Abstract. Limited dominating broadcasts were proposed as a variant of dominating broadcasts, where the broadcast function is upper bounded by a constant k. The minimum cost of such a dominating broadcast is the dominating k-broadcast number. We present a unified upper bound on this parameter for any value of k, in terms of both k and the order of the graph. We also prove that the decision problem of determine if a graph has a dominating k-broadcast function with cost at most s is NP-complete.

Keywords. Broadcast, domination, graphs.

Domination in graphs has shown as a extremely fruitful concept, since it was originally defined in the late fifties [1] and named in the early sixties [10]. A dominating set of a graph G is a vertex set S such that any vertex not in S has at least one neighbor in S. Multiple variants of domination have been defined over the past fifty years, putting the focus on different aspects. One of them is the idea of broadcasting, firstly introduced in [9] and taken up more recently in [5]. This model reflects the idea of several broadcast stations, with associated transmission powers, that can broadcast messages to places at distance greater than one. We recall the formal definition from [5].

Definition 1. *A broadcast on G is a function $f\colon V(G) \to \{0, 1, \ldots, diam(G)\}$. A vertex $v \in V(G)$ with $f(v) > 0$ is a f-dominating vertex and the vertex set $V_f^+ = \{v \in V(G) \colon f(v) > 0\}$ is the f-dominating set. An f-dominating vertex v is said to f-dominate every vertex u with $d(u, v) \leq f(v)$. A dominating broadcast on G is a broadcast f such that every vertex in G is f-dominated. The cost of a f-dominating broadcast is $\sigma(f) = \sum_{v\in V_f^+} f(v)$ and the dominating broadcast number $\gamma_B(G)$ is the minimum value of $\sigma(f)$ over all dominating broadcast of G.*

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Dunbar et all. suggested in [4], as an open problem, considering the broadcast dominating problem with limited broadcast power, defined as follows.

Definition 2. *Let G be a graph and let $k \geq 1$ be an integer. For any function $f\colon V(G) \to \{0,1,\dots,k\}$, let $V_f^+ = \{v \in V(G)\colon f(v) \geq 1\}$. We say that f is a dominating k-broadcast if for every $u \in V(G)$ there exists $v \in V_f^+$ such that $d(u,v) \leq f(v)$. The cost of a dominating k-broadcast f is $\omega(f) = \sum_{u \in V(G)} f(u)$ and the dominating k-broadcast number of G is*

$$\gamma_{B_k}(G) = \min\{\omega(f)\colon f \text{ is a dominating } k\text{-broadcast on } G\}.$$

We have studied this parameter in [3], for the particular case $k = 2$. Among other results, we have obtained a general upper bound in terms of the order of the graph.

Theorem 1. *For every graph G of order n,*

$$\gamma_{B_2}(G) \leq \lceil 4n/9 \rceil.$$

We have also characterized in [3] the family of trees that attain the bound. Let T_9 be the tree shown in Figure 1.1, with nine vertices and central vertex x. Consider the family $\mathcal{F}$ of trees consisting on $m \geq 1$ copies of T_9 in addition to $m-1$ edges, any of them joining central vertices of two copies.

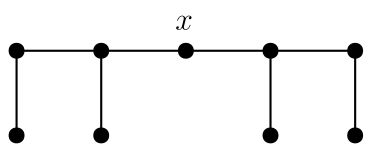

Figure 1.1: $\gamma_{B_2}(T_9) = \lceil 4n/9 \rceil = 4$

Proposition 1. *A tree T with n vertices satisfies $\gamma_{B_2}(T) = \lceil 4n/9 \rceil$ if and only if $T \in \mathcal{F} \cup \{P_1, P_2, P_4\}$.*

In this work we extend these results to the general case $k \geq 3$. We follow the ideas developed in [7, 8] and we obtain a general upper bound for the dominating k-broadcast number of any graph.

Theorem 2. *Let G be a graph of order n and let $k \geq 3$ be an integer. Then,*

$$\gamma_{B_k}(G) \leq \begin{cases} \lceil \frac{n}{3} \rceil & \dots \quad rad(G) \leq k \\ \\ \lceil \frac{k+2}{k+1} \frac{n}{3} \rceil & \dots \quad rad(G) > k \end{cases}$$

If G has radius at most k, then the equality $\gamma_{B_k}(G) = \gamma_B(G)$ can be easily deduced from the definition of dominating k-broadcast, so the first part of the bound comes from [7, 8].

The following examples illustrate the tightness of the bound. On the one hand, for every $k \geq 3$ and $r \leq k$, the path P_n, where $n = 2r + 1$, satisfies $rad(P_n) = r \leq k$ and $\gamma_{B_k}(P_n) = \lceil n/3 \rceil$.

On the other hand, the tree T_k shown in Figure 1.2 has $3k + 3$ vertices and radius $k + 1$. Moreover it satisfies $\gamma_{B_k}(T) = \lceil \frac{k+2}{k+1} \frac{n}{3} \rceil$.

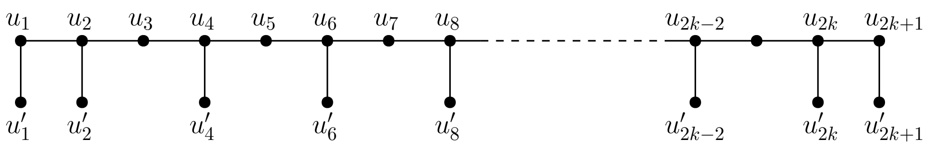

Figure 1.2: T_k has $n = 3k + 3$ vertices and satisfies $\gamma_{B_k}(T_k) = \lceil \frac{k+2}{k+1} \frac{n}{3} \rceil$.

Finally we focus on the computational point of view of this parameter. In [3] we showed that the decision problem of determining if a graph G has a dominating 2-broadcast with cost at most s is NP-complete. However it can be done in linear time in the family of trees, by applying a general technique developed in [2].

The classical DOMINATING SET PROBLEM is a well-known NP-complete decision problem, in the same way that some others domination related problems. On the other hand, a polynomial algorithm to compute an optimal broadcast domination function of a graph G was quite surprisingly obtained in [6]. We consider now the following decision problem.

DOMINATING k-BROADCAST PROBLEM INSTANCE: A graph G of order n and integers $k \geq 3$, $s \geq 2$. QUESTION: Does G have a dominating k-broadcast with cost $\leq s$?

Using a reduction of 3-SAT to DOMINATING k-BROADCAST PROBLEM, in a similar way to the classical reduction of 3-SAT to DOMINATING SET PROBLEM, we have obtained:

Theorem 3. *DOMINATING k-BROADCAST PROBLEM is NP-complete.*

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