On the metric dimension of graphs

Ignacio M. Pelayo
U.P.C., Barcelona, Spain

C. Hernando, M. Mora, C. Seara, David R. Wood
U.P.C., Barcelona, Spain
J. Cáceres, M.L. Puertas
U.A., Almeria, Spain

7th International Colloquium on Graph Theory, France, 2005

RESOLVING SETS

⋆ $G = (V, E)$ is a connected graph.

▶ A vertex x of G RESOLVES a pair of vertices u, v of G if:

$$d(x, u) \neq d(x, v).$$

▶ A vertex subset $S \subset V$ is a RESOLVING SET of G if:

every two distinct vertices of G are resolved by some vertex of S.

METRIC DIMENSION

⋆ $S \subset V$ is a resolving set of $G = (V, E)$.

▶ S is a metric basis if it is a minimum resolving set.

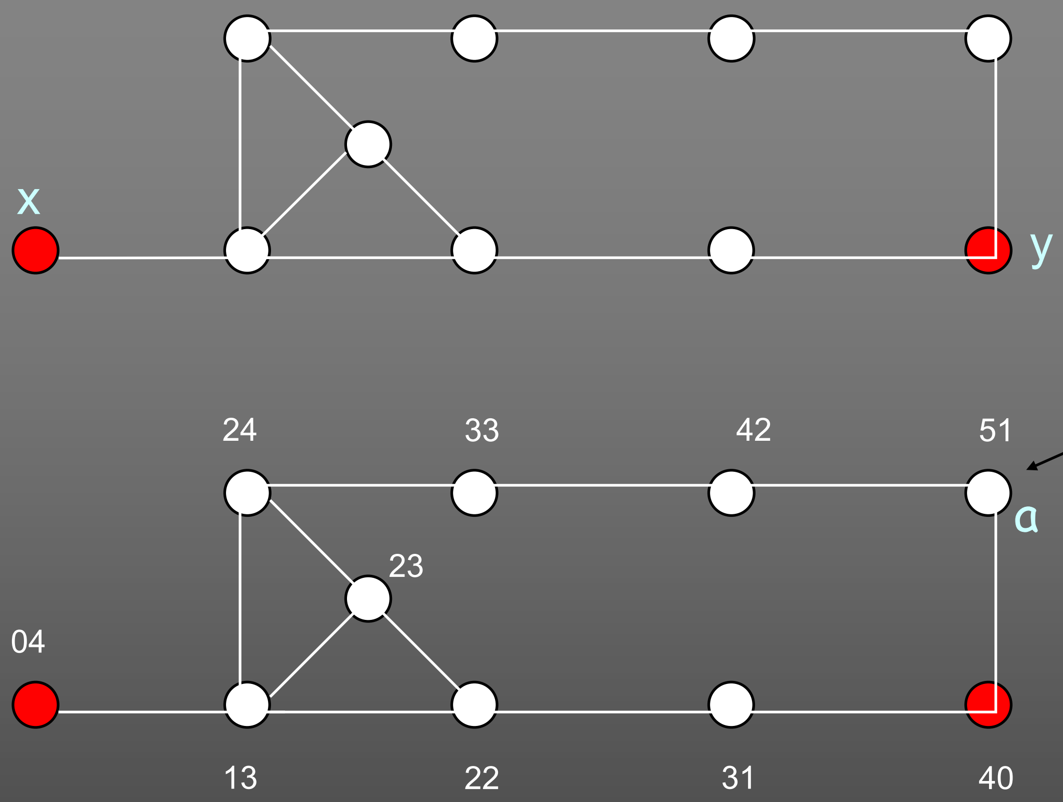
⋆ $S = \{u_1, u_2, \dots, u_r\}$ is a metric basis of G.

▶ The metric dimension of G is: $\beta(G) = r$.

⋆ x is a vertex of G.

▶ The metric coordinates of x are:

$$(x_1, \dots, x_r), \text{ where } x_i = d(u_i, x).$$



d(x,a)=5

d(y,a)=1

β(G)=2

S={x,y} is a metric basis

SOME BASIC KNOWN RESULTS

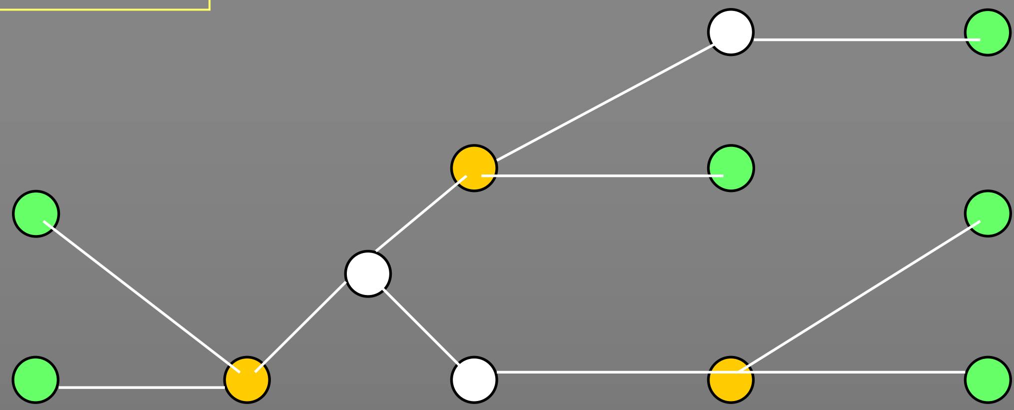
- The problem of computing the metric dimension of an arbitrary graph is NP-hard..

name	path	cycle	complete	bicomplete	wheel	hypercube
G	P_n	C_n	K_n	$K_{r,s}$	$W_{1,r}$	$Q_r = [K_2]^r$
$\lvert V(G)\rvert$	$n \geq 1$	$n \geq 3$	$n \geq 2$	$r+s \geq 3$	$r+1 = 4,7$	2^r
$\beta(G)$	1	2	$n-1$	$n-2$	3	r $(r \leq 4)$

- If T is a tree s.t. $\lambda(T) \geq 1$, then $\beta(T) = |Ext(T)| - \lambda(T)$.

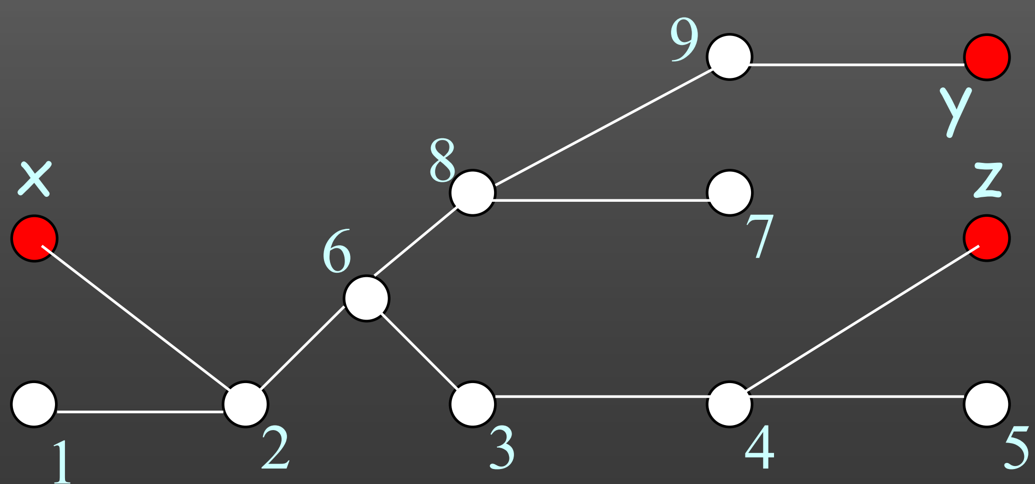
- If $G = T + e$, then $\beta(T) - 2 \leq \beta(T+e) \leq \beta(T) + 1$.

- $r \notin \{3,6\}$: $\beta(W_{1,r}) = \lfloor \frac{2r+2}{5} \rfloor$.



leaves

exterior major vertices



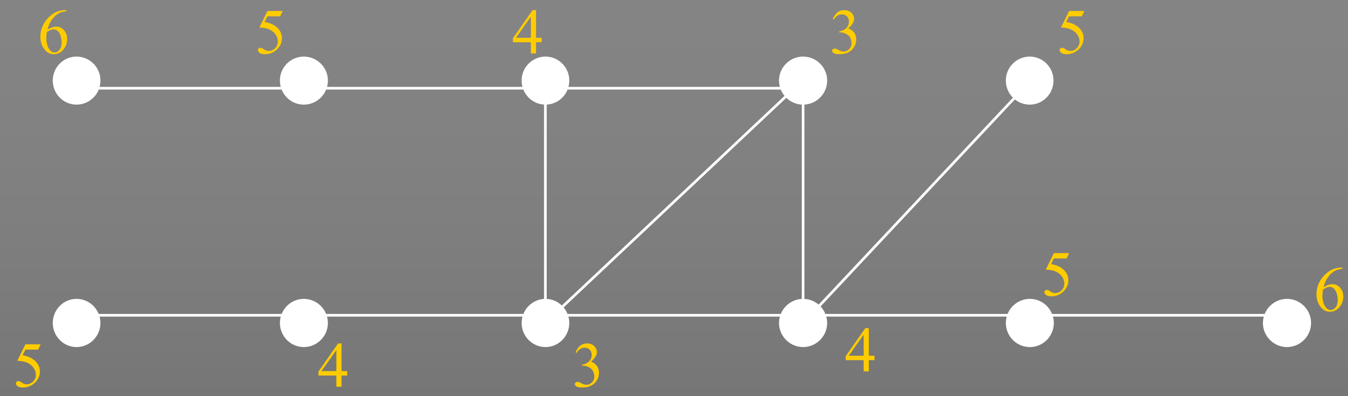
1	255
2	144
3	342
4	451
5	562
6	233
7	435
8	324
9	415
x	055
y	506
z	560

BOUNDARY VERTICES

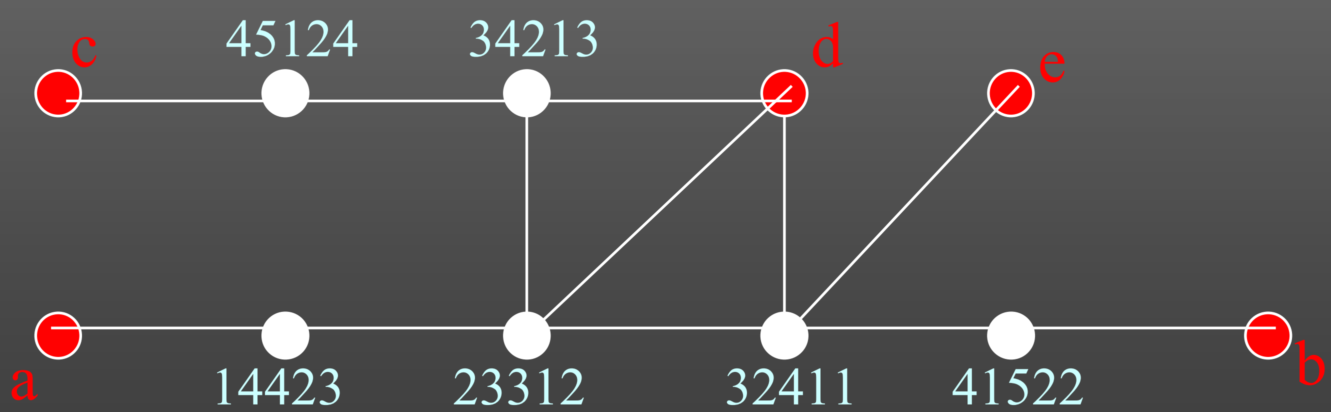
- ⋆ u, v are two vertices of a graph G.
- ▶ v is a boundary vertex of u if:

$$\forall\ w \in N(v),\ d(u,w) \leq d(u,v).$$

- ▶ The set of all boundary vertices of u is its boundary $\partial(u)$.
- ▶ The boundary of G is the set $\partial(G) = \cup_{u \in V(G)} \partial(u)$.
- ■ THEOREM: **The boundary $\partial(G)$ is a resolving set of G.**
- ■ COROLLARY: $\beta(G) \leq |\partial(G)|$.



$$\partial(G) = \{a,b,c,d,e\}$$



BOUNDS AND CHARACTERIZATIONS

- $|G| - d^{\beta(G)} \leq \beta(G) \leq |G| - d \ (d = diam(G))$.

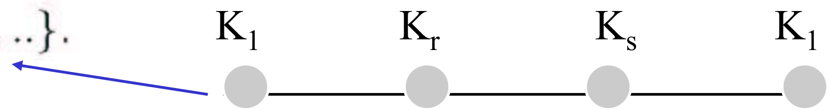
- $\beta(G) = 1 \Leftrightarrow G = P_n$.

- $\beta(G) = |G| - 1 \Leftrightarrow G = K_n$.

- $t \geq 2$: $\beta(G) = |G| - 2 \Leftrightarrow G \in \{K_{s,t}, K_s + \overline{K}_t, K_s + (K_1 \cup K_t)\}$.

- $d \geq 3$, $\beta(G) = |G| - d \Rightarrow \exists\ a \in Per(G)$ without twins.

- $\beta(G) = |G| - 3 \Leftrightarrow G \in \{.., ..., ..\}$.



CARTESIAN PRODUCT

⋆ $G_1 = (V_1, E_1)$ $G_2 = (V_2, E_2)$ are two connected graphs.

▶ The cartesian product $G_1 \square G_2$ is the graph with:.

◊ $V(G_1 \square G_2) = V_1 \times V_2$.

◊ (u_1, u_2) is adjacent to (v_1, v_2) iff $\begin{cases} u_1 = v_1 \text{ and } u_2v_2 \in E_2 \\ \quad\quad \textit{or} \\ u_1v_1 \in E_1 \text{ and } u_2 = v_2 \end{cases}$

■ $d((u_1, u_2), (u_1, u_2)) = d_{G_1}(u_1, v_1) + d_{G_2}(u_2, v_2)$.

■ THEOREM: $\beta(G_1 \square G_2) \geq \max[\beta(G_1), \beta(G_2)]$.

SOME KNOWN RESULTS ON THE C.P.

- $\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1.$

- $\beta(P_m \square P_n) = 2.$

- $\lim_{n \to \infty} \beta(Q_n) \cdot \frac{\log_2(n)}{n} = 2$

- $\beta(P_{m_1} \square P_{m_2} \square \ldots \square P_{m_d}) = d$ (wrong result: DAM, **70** (1996)).

CARTESIAN PRODUCTS [BOUNDS]

- $\max[\beta(G), \beta(H)] \leq \beta(G \Box H)]$.
- $|G| \geq 3, |H| \geq 3$: $\beta(G \Box H) \leq \min\{\beta(G) + |H|, \beta(H) + |G|\} - 2$
- $\beta(G \Box K_n) \leq \max\{n-1, 2 \cdot \beta(G)\}$.
- $\beta(G \Box P_n) \leq \beta(G) + 1$.
- $\beta(G \Box C_n) \leq \beta(G) + 2$.
- $\beta(G \Box T) \leq \beta(G) + |Ext(T)| - 1$.
- For all $k \geq 1$ and $n \geq 2$ there is a k-connected graph $G_{n,k}$ for which $\beta(G_{n,k}) \leq 2k$ and $\beta(G_{n,k} \Box G_{n,k}) \geq n$.
- For all $k \geq 1$ there in no function f such that: $\beta(G \Box H) \leq f(\beta(G), \beta(H))$, for all k-connected graphs G and H.

CARTESIAN PRODUCTS [EXACT VALUES]

- $m \leq n \Rightarrow dim(K_m \square K_n) = \begin{cases} n-1 & \text{if } 2m-2 < n, \\ \lfloor \frac{2m+2n-2}{3} \rfloor & \text{if } 2m-2 \geq n. \end{cases}$
- $m \geq 4$: $\beta(K_m \square C_n) = \begin{cases} m, & \text{if m = 4 and n odd,} \\ m-1, & \text{otherwise.} \end{cases}$
- $\beta(C_m \square C_n) = \begin{cases} 3, & \text{if m or n is odd} \\ 4, & \text{otherwise.} \end{cases}$
- $\beta(C_m \square P_n) = \begin{cases} 2, & \text{if m odd} \\ 3, & \text{if m even (and n} \neq 1) \end{cases}$

$G \backslash H$	K_n	C_n	P_n
K_m	$n-1, \lfloor \frac{2m+2n-2}{3} \rfloor$	$m-1, m$	$m-1$
C_m		$3, 4$	$2, 3$
P_m			2

CARTESIAN PRODUCT OF A CYCLE BY A PATH

- $m \geq 3, n \geq 2$: $\beta(C_m) = 2$, $\beta(P_n) = 1$
- $\max[\beta(G), \beta(H)] \leq \beta(G \square H)]$.
- $\beta(G \square P_n) \leq \beta(G) + 1$.

$\Rightarrow$ $2 = \beta(C_m) \leq \beta(C_m \square P_n) \leq \beta(C_m) + 1 = 3$.

♦ $\beta(C_m \square G) = 2 \Leftrightarrow G$ is a path and m is odd.

<u>THEOREM</u>: $\beta(C_m \square P_n) = \begin{cases} 2, & \text{if m odd} \\ 3, & \text{if m even (and } n \neq 1) \end{cases}$

DOUBLY RESOLVING SETS

- Two vertices v, w of $G \neq K_1$ are DOUBLY RESOLVED by a pair of vertices x, y of G if:

$$d(v,x) - d(w,x) \neq d(v,y) - d(w,y).$$

- A set $S \subseteq V$ is a DOUBLY RESOLVING SET of G if every pair of distinct vertices of G are doubly resolved by two vertices in S. $\psi(G)$ is the minimum cardinality of a doubly resolving set.

- $1 \leq \beta(G) \leq \psi(G) \leq |G| - 1$
- $2 \cdot \beta(G \Box G) \geq \psi(G)$
- $\beta(G \Box H) \leq \beta(G) + \psi(H) - 1$
- $\beta(G \Box H) \leq \beta(G) + 2\beta(H \Box H) - 1$

name	path	odd cycle	even cycle	complete	tree
G	P_n	C_n	C_n	K_n	T_n
$\|V(G)\|$	$n \geq 2$	$n \geq 3$	$n \geq 4$	$n \geq 3$	$n \geq 2$
$\beta(G)$	1	2	2	$n-1$	$\|Ext(T_n)\| - \lambda(T_n)$
$\psi(G)$	2	2	3	$n-1$	$\|Ext(T_n)\|$

C. P. OF AN ODD CYCLE BY A PATH

- $\psi(C_m) = \begin{cases} 2, & \text{if m odd} \\ 3, & \text{if m even} \end{cases}$

- $\psi(P_n) = 2$

- $\beta(G) \leq \beta(G \Box P_n) \leq \beta(G) + \psi(P_n) - 1 = \beta(G) + 1$

- $\beta(G) \leq \beta(C_m \Box G) \leq \beta(G) + \psi(C_m) - 1 \leq \begin{cases} \beta(G) + 1, & \text{if m odd} \\ \beta(G) + 2, & \text{if m even} \end{cases}$

$\Rightarrow$ m odd $\Rightarrow$ $\beta(C_m \Box P_n)) = 2.$

http://es.arXiv.org/find/math/1/PELAYO/0/1/0/past/3/0