On the metric dimension of some families of graphs

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Abstract

The concept of (minimum) resolving set has proved to be useful and/or related to a variety of fields such as Chemistry [3,6], Robotic Navigation [5,8] and Combinatorial Search and Optimization [7]. This work is devoted to evaluating the so-called metric dimension of a finite connected graph, i.e., the minimum cardinality of a resolving set, for a number of graph families, as long as to study its behavior with respect to the join and the cartesian product of graphs.

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1 Definitions and terminology

All graphs considered are simple, finite and connected. Given a graph $G = (V, E)$ and $u, v \in V$, $d_G(u, v)$ (or simply $d(u, v)$) denotes the distance between u and v in G, i.e., the length of a shortest $u - v$ path.

Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs. The *join* $G_1 + G_2$ is the graph whose vertex set is $V_1 \cup V_2$ and $E(G_1 + G_2) = E_1 \cup E_2 \cup \{v_1 v_2 : v_1 \in V_1, v_2 \in V_2\}$. By the *cartesian product* $G_1 \square G_2$ is meant the graph whose vertex set is $V_1 \times V_2$; two vertices (x_1, x_2) and (y_1, y_2) being adjacent in $G_1 \square G_2$ if and only if either $x_1 = y_1$ and $x_2 y_2 \in E_2$ or $x_2 = y_2$ and $x_1 y_1 \in E_1$.

A vertex v is a *boundary vertex* of a graph G if there exists a vertex u such that no neighbor of v is further away from u than v. The set of all boundary vertices of G is called its *boundary* $\partial(G)$ [3]. A vertex v is called *extreme* if the subgraph induced by its neighborhood $N(v)$ is a clique. The set of all extreme vertices of G is denoted by $Ext(G)$. Notice that $Ext(G) \subseteq \partial(G)$.

A vertex x of a graph G is said to resolve two vertices u and v of G if $d(x, u) \neq d(x, v)$. An ordered vertex set S of G is a *resolving set* (also called a *metric generator*) of G if every two distinct vertices of G are resolved by some vertex of S. A *metric basis* of G is a resolving set of minimum cardinality. The *metric dimension* $\beta(G)$ of G is the cardinality of its basis [4,9]. Given a finite resolving set $S = \{x_1, \ldots, x_s\} \subseteq V(G)$ and a vertex $t \in V(G)$, the s-pla $(d(t, x_1), \ldots, d(t, x_s))$ is called the vector of *metric coordinates* of t.

2 Metric dimension of some families

2.1 Bounds

One important question when dealing with a certain graph parameter is the problem of obtaining both a lower and an upper bound valid either for every graph or for a restricted class of graphs. This is the current state of the art on this subject for the metric dimension (including our contributions [1]).

- [2,5] If $d = diam(G)$, then: $|G| - d^{\beta(G)} \leq \beta(G) \leq |G| - d$
- [1] $\beta(G) \leq |\partial(G)|$

- [1] $\max\{\beta(G), \beta(H)\} \leq \beta(G \square H) \leq \min\{\beta(G) + |H|, \beta(H) + |G|\} - 1$
- [1] $2 \leq \beta(G) \leq \beta(H) \leq \beta(G \square H) \leq \min\{\beta(G) + |H|, \beta(H) + |G|\} - 2$
- [1] $\beta(G) + \beta(H) \leq \beta(G + H)$

- [2] $\beta(G) \leq \beta(G \square K_2) \leq \beta(G) + 1$
- [1] $\beta(G) \leq \beta(G \square P_n) \leq \beta(G) + 1$
- [1] $\beta(G \square K_n) \leq \beta(G) + n - 2$, if $n \geq 3$
- [1] $\beta(G \square C_n) \leq \begin{cases} \beta(G) + 1, & \text{if n is odd} \\ \beta(G) + 2, & \text{if n is even} \end{cases}$

- [2] $\beta(Q_n) \leq n$
- [7] $\lim\limits_{n \to \infty} \dfrac{\beta(Q_n) \cdot \log_2(n)}{n} = 2$
- [1] $\beta(Q_n) \leq n - 5$, if $n \geq 15$

2.2 Exact values

In [5], it was proved that the problem of computing the metric dimension of an arbitrary graph is NP-hard. Nevertheless, it is rather straightforward for some particular cases, such as those displayed in the following table.

name	path	cycle	complete	bicomplete	wheel	hypercube
G	P_n	C_n	K_n	$K_{r,s}$	$W_{1,r}$	$Q_r = [K_2]^r$
$|V(G)|$	$n \geq 1$	$n \geq 3$	$n \geq 2$	$r + s \geq 3$	$r + 1 = 4, 7$	2^r
$\beta(G)$	1	2	$n - 1$	$n - 2$	3	$r \ (r \leq 4)$

Next, we show the main known results on the metric dimension problem when we restrict ourselves to a certain class of graphs.

- [2] $\beta(G) = |G| - 2 \Leftrightarrow G = K_{s,t}$, or $G = K_s + \overline{K}_t$, or $G = K_s + (K_1 \cup K_t)$

- [2,4,9] If T is a tree having $\lambda(T) \geq 1$ exterior major vertices, then $\beta(T) = |Ext(T)| - \lambda(T)$
- [2,6] If $G = T + e$ is an unicyclic graph, then $\beta(T) - 2 \leq \beta(T + e) \leq \beta(T) + 1$

- [1,8] $n \notin \{3, 6\}$: $\beta(W_{1,n}) = \lfloor \frac{2n+2}{5} \rfloor$
- [1] $n \notin \{1, 2, 3, 6\}$: $\beta(K_1 + P_n) = \lfloor \frac{2n+2}{5} \rfloor$

- [5] $\beta(P_m \Box P_n) = 2$
- [5] $\beta(P_{m_1} \Box P_{m_2} \Box \ldots \Box P_{m_d}) = d$ (wrong result)

- [1] $n \geq 3$: $\beta(P_m \Box K_n) = n - 1$
- [1] $\beta(P_m \Box C_n) = \begin{cases} 2, \text{ if n odd} \\ 3, \text{ if n even (and m} \neq 1) \end{cases}$
- [1] $\beta(C_m \Box C_n) = \begin{cases} 3, \text{ if m or n odd} \\ 4, \text{ otherwise} \end{cases}$
- [1] $m \geq 4$: $\beta(K_m \Box C_n) = \begin{cases} m, \quad \text{if m} = 4 \text{ and n odd,} \\ m - 1, \text{ otherwise.} \end{cases}$
- [1] $m \leq n \Rightarrow dim(K_m \Box K_n) = \begin{cases} n - 1 \quad \text{if } 2m - 2 < n, \\ \lfloor \frac{2m+2n-2}{3} \rfloor \text{ if } 2m - 2 \geq n. \end{cases}$

2.3 Open problems

Currently, we are approaching several problems related to the metric dimension of graphs. Some of them are the following:

- Obtaining a better upper bound for the metric dimension of the cartesian product of two graphs. To be more precise, we are trying to prove (or finding a counterexample) that for all pair of graphs G, H: $\beta(G \Box H) \leq \beta(G) + \beta(H)$.
- Computing the metric dimension in the class of hypercubes (some old and new known values are shown in the table below), grid graphs and Hamming graphs.

n	2	3	4	5	6	7	8	9	10
$\beta(Q_n)$	2	3	4	4	5	6	6	7	7

- Characterizing all graphs for which $\beta(G) = |G| - 3$.

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