



# Searching for geodetic boundary vertex sets

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## Abstract

A vertex  $v$  is a boundary vertex of a connected graph  $G$  if there exists a vertex  $u$  such that no neighbor of  $v$  is further away from  $u$  than  $v$ . We obtain a number of properties involving different types of boundary vertices: peripheral, contour and eccentric vertices, including a realization theorem that not only corrects a wrong statement detected in [2], but also improves it. We also prove that the boundary vertex set  $\partial(G)$  of any graph  $G$  is geodetic, that is, every vertex in  $G$  lies on some shortest path joining two boundary vertices. A vertex  $v$  belongs to the contour  $Ct(G)$  of  $G$  if no neighbor of  $v$  has an eccentricity greater than those of  $v$ . We study the geodeticity of the contour  $Ct(G)$  and other related sets.

*Keywords:* boundary, chordal graph, contour, eccentricity, geodesic convexity, geodetic set, perfect graph, periphery

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## 1 Introduction

Given a connected graph  $G = (V, E)$  and  $u, v \in V$ ,  $d(u, v)$  denotes the distance between  $u$  and  $v$  in  $G$ . A  $u$ - $v$  path  $\rho$  is called a  $u$ - $v$  geodesic if it is a shortest  $u$ - $v$  path. The geodesic interval  $I[u, v]$  is the set of vertices of all  $u$ - $v$  geodesics. For  $S \subseteq V$ , the geodesic closure of  $S$  is  $I[S] = \bigcup_{u,v \in S} I[u, v]$ .

A (finite) graph convexity space is a pair  $(G, \mathcal{C})$ , formed by a finite connected graph  $G = (V, E)$  and a family  $\mathcal{C}$  of subsets of  $V$  (each such set called a convex set) which is closed under intersection, which contains both  $V$  and the empty set, and such that every convex set induces a connected subgraph of  $G$ . In this work, we consider only the so-called geodesic convexity  $\mathcal{C}_g$  defined as follows. A vertex set  $W \subseteq V$  is called convex if  $I[W] = W$ . Given  $A \subseteq V$ , the smallest convex set containing  $A$  is denoted  $[A]$  and is called the convex hull of  $A$ . The set  $A \subseteq V$  is called a hull set if  $[A] = V$ , and geodesic if  $I[A] = V$ .

Given a graph convexity space  $(G, \mathcal{C})$  and a convex set  $W \subseteq V(G)$ , a vertex  $v \in W$  is called an extreme vertex of  $W$  if the set  $W \setminus \{v\}$  is also convex. The convexity  $\mathcal{C}$  is called a convex geometry if it satisfies the so-called Krein-Milman property: *Every convex set is the convex hull of its extreme vertices*. Certainly, this condition can be seen as a rebuilding method, that allows us to recover any convex set from its extreme points, by means of the convex hull operator. Under this point of view, the interest of any similar property is that a small subset of any convex set keeps all the information of the whole set.

A graph is called Ptolemaic if it is both distance-hereditary and chordal, i.e., if every chordless path is a geodesic and every cycle of length at least four has a chord. In [3], it was proved that the geodesic convexity of a graph is a convex geometry if and only if it is Ptolemaic. So, we could think of extending this property in two different ways. On the one hand, recovering convex sets on wider graph classes, and on the other, using an operator simpler than the convex hull one  $[ ]$ , as the geodesic closure operator  $I$  is. In both cases, finding new vertex sets playing a similar role to that of extreme vertices is necessary.

Concerning the first mentioned extension of the Krein-Milman property, Cáceres et al. [1] obtained a similar property to this one, valid for every graph, by considering, instead of the extreme vertices, the so-called contour vertices (see Section 2). As for the second generalization, consisting in using the geodesic interval operator  $I$ , a number of results have been recently obtained [1,5,6]. For example, it has been proved that in the class of distance-hereditary graphs, every convex set is the geodesic closure of its contour vertices [1,6].

The rest of this work is organized as follows. In Section 2, we focus our

attention on several types of *boundary vertices* [2]: extreme, peripheral, contour and eccentric vertices. We show, among other facts, that one of the main results in [2] does not hold, and establish a realization theorem that not only corrects the mentioned wrong statement but also improves it. In Section 3, we approach the problem of finding geodetic sets consisting of boundary vertices, proving that the boundary  $\partial(G)$  of any graph  $G$  is geodetic and presenting some sufficient conditions to guarantee the geodeticity of either the contour  $Ct(G)$  or its geodetic closure  $I[Ct(G)]$ . Finally, Section 4 is devoted to investigate the geodeticity of the contour for several classes of perfect graphs.

## 2 Boundary vertices

Let  $G = (V, E)$  be a graph and  $u, v \in V$ . The vertex  $v$  is said to be a *boundary vertex* of  $u$  if no neighbor of  $v$  is further away from  $u$  than  $v$  [2]. A vertex  $v$  is called a *boundary vertex of  $G$*  if it is the boundary vertex of some vertex  $u \in V$ . The boundary  $\partial(G)$  of  $G$  is the set of all of its boundary vertices:

$$\partial(G) = \{v \in V \mid \exists u \in V \text{ s.t. } \forall w \in N(v) : d(u, w) \leq d(u, v)\}.$$

Given  $W \subseteq V$ , the *eccentricity* in  $W$  of a vertex  $u \in W$  is defined as  $ecc_W(u) = \max\{d_G(u, v) \mid v \in W\}$ . In particular,  $ecc_G(u) = ecc(u) = \max\{d(u, v) \mid v \in V\}$ . Given  $u, v \in V$ , the vertex  $v$  is called an *eccentric vertex* of  $u$  if no vertex in  $V$  is further away from  $u$  than  $v$ , that is, if  $d(u, v) = ecc(u)$ . A vertex  $v$  is called a *eccentric vertex of  $G$*  if it is the eccentric vertex of some vertex  $u \in V$ . The eccentricity  $Ecc(G)$  of  $G$  is the set of all of its eccentric vertices:

$$Ecc(G) = \{v \in V \mid \exists u \in V \text{ s.t. } ecc(u) = d(u, v)\}.$$

A vertex  $v \in V$  is called *peripheral* if no vertex in  $V$  has an eccentricity greater than  $ecc(v)$ , that is, if the eccentricity of  $v$  is exactly equal to the diameter  $D(G)$  of  $G$ . The periphery  $Per(G)$  of  $G$  is the set all of its peripheral vertices:

$$Per(G) = \{v \in V \mid ecc(u) \leq ecc(v), \forall u \in V\} = \{v \in V \mid ecc(v) = D(G)\}.$$

A vertex  $v \in V$  is called a *contour vertex* of  $G$  if no neighbor vertex of  $v$  has an eccentricity greater than  $ecc(v)$ . The contour  $Ct(G)$  of  $G$  [1] is the set of all its contour vertices:

$$Ct(G) = \{v \in V \mid ecc(u) \leq ecc(v), \forall u \in N(v)\}.$$

**Proposition 2.1** *Let  $G = (V, E)$  be a connected graph. Then,*

- (i)  $Per(G) \subseteq Ct(G) \cap Ecc(G)$ ,  $Ecc(G) \cup Ct(G) \subseteq \partial(G)$ .
- (ii) If  $|Per(G)| = |Ct(G)| = 2$ , then either  $|\partial(G)| = 2$  or  $|\partial(G)| \geq 4$ .
- (iii) If  $|Ecc(G)| = |Per(G)| + 1$ , then  $|\partial(G)| > |Ecc(G)|$ .

**Corollary 2.2** If  $2 \leq |Per(G)| = a$ ,  $|Ct(G)| = b$ ,  $|Ecc(G)| = c$ ,  $|\partial(G)| = d$ :

$$\begin{cases} 2 \leq a \leq \min\{b, c\} \leq \max\{b, c\} \leq d, \\ (a, b, c, d) \neq (2, 2, 2, 3), \\ (a, b, c, d) \neq (a, b, a + 1, a + 1). \text{ [**]} \end{cases}$$

In [2], Chartrand et al. included the following realization theorem:

**Theorem 2.3** [2] For each triple  $a, c, d$  of integers with  $2 \leq a \leq c \leq d$ , there is a connected graph  $G$  such that  $|Per(G)| = a$ ,  $|Ecc(G)| = c$ , and  $|\partial(G)| = d$ .

Notice that, as a consequence of the constraint [\*\*] displayed in Corollary 2.2, it is immediately derived that this result is false for the case  $a + 1 = c = d$ . Next, we present a realization theorem that not only corrects this mistake, but also essentially improves and completely solves the posed question.

**Theorem 2.4** Let  $(a, b, c, d) \in \mathbb{Z}^4$  satisfying the constraints in Corollary 2.2. Then, there exists a connected graph  $G = (V, E)$  satisfying:

$$|Per(G)| = a, \quad |Ct(G)| = b, \quad |Ecc(G)| = c, \quad |\partial(G)| = d.$$

### 3 Boundary vertex sets as geodetic sets

In [1], Cáceres et al. proved the following statement:

**Theorem 3.1** [1] Let  $G = (V, E)$  be a connected graph and  $W \subseteq V$  a convex set. Then,  $W$  is the convex hull of its contour vertices.

The contour of a graph needs not to be geodetic. For example, in Figure 1, we show two graphs whose contour set is  $\{u, v, w\}$  and  $I[\{u, v, w\}] = V \setminus \{z\}$ . As for the eccentricity, it is easy to design a graph  $G$  such that  $I[Ecc(G)] \subsetneq V(G)$ .

**Theorem 3.2** The boundary  $\partial(G)$  of every graph  $G = (V, E)$  is geodetic.

At this point, we have studied the geodeticity of  $I[Ct(G)]$ . Notice that the graph  $G$  in Figure 1(a) satisfies:  $Ct(G) = \{u, v, w\}$ ,  $I[Ct(G)] = V(G) \setminus \{z\}$ ,  $\partial(G) = V(G) \setminus \{a\}$ . Hence,  $\partial(G) \not\subseteq I[Ct(G)]$ .

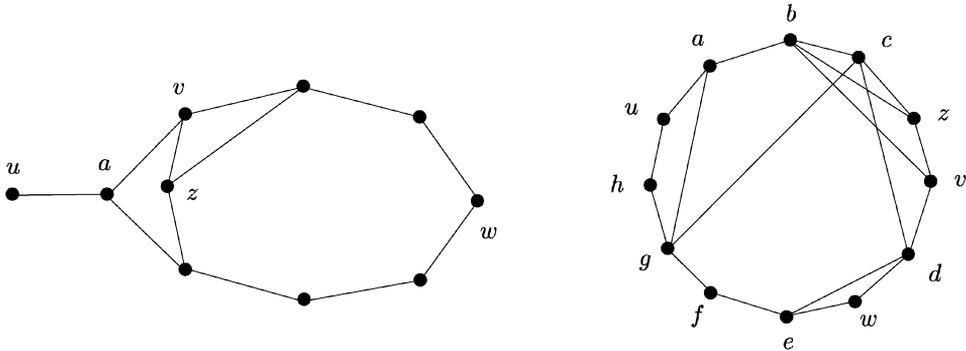


Fig. 1. Two graphs whose contour is not geodetic.

**Proposition 3.3** *Let  $G$  be a connected graph. If  $Ecc_G(Ct(G)) \subseteq I[Ct(G)]$ , then  $I^2[Ct(G)] = V$ . That is,  $I[Ct(G)]$  is a geodetic set.*

Starting from this fact, we are currently trying to prove that either, for every graph  $G$ ,  $Ecc_G(Ct(G)) \subseteq I[Ct(G)]$  or else find a counterexample.

**Remark 3.4** We know of no example of a graph  $G$  having a contour  $Ct(G)$  such that its geodetic closure be not geodetic. We leave it as an open problem as to whether  $I^2[Ct(G)] = V(G)$  for every connected graph  $G$ .

Given a set of vertices  $W \subseteq V$ , the *geodetic iteration number*  $gin(W)$  of  $W$  is defined as the minimum integer  $k \geq 0$  such that  $I^k[W] = I[I^{k-1}[W]] = [W]$ .

**Definition 3.5** An integer sequence  $(d_1, \dots, d_s)$  of size  $s$  satisfying  $d_1 > \dots > d_s$  is called the *eccentricity sequence* of a set of vertices  $W \subseteq V$  if

$$\{k \in \mathbb{N} : k = ecc(v), \text{ for some } v \in W\} = \{d_1, \dots, d_s\}.$$

**Theorem 3.6** *Let  $G = (V, E)$  be a connected graph whose contour has an eccentricity sequence of size  $s$ . Then,  $I^s[Ct(G)] = V$ .*

**Corollary 3.7** *Let  $G = (V, E)$  be a connected graph.*

- (i) *If  $Ct(G) = Per(G)$ , then  $I[Ct(G)] = V$ .*
- (ii) *If  $|Ct(G)| = |Per(G)| + 1$ , then  $I^2[Ct(G)] = V$ ,*

## 4 Geodeticity of the contour of perfect graphs

In this section, we study the *geodetic* behavior of different boundary-type sets into the class of perfect graphs. We have found examples of perfect graphs for which neither  $Per(G)$  nor  $Ecc(G)$  are geodetic sets. We have also been able to design a few perfect graphs having a non-geodetic contour (see Figure 2).

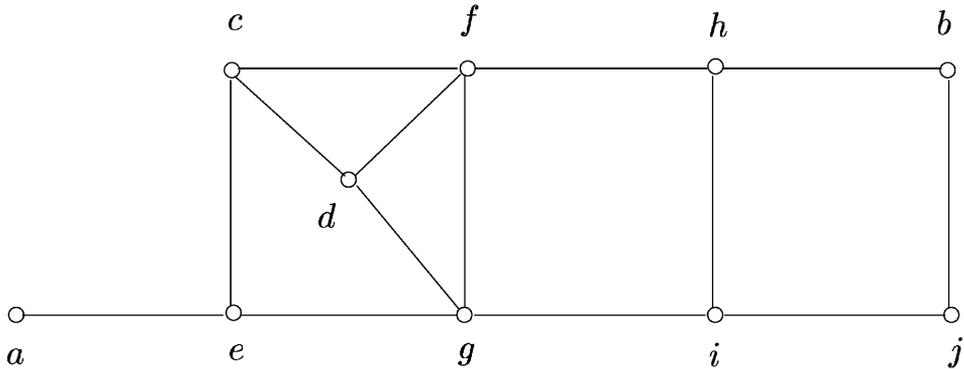


Fig. 2. Permutation graph  $G$  satisfying  $Ct[G] = \{a, b, c\}$  and  $I[Ct(G)] = V \setminus \{d\}$ .

However,  $Ct(G)$  holds this property in several perfect graph classes. In a previous work we obtained the following theorem.

**Theorem 4.1** [6] *The contour of every distance-hereditary graph is geodetic.*

In this work, we have proved a similar statement for the class of chordal graphs. The proof is mainly based on the well known characterization of these graphs as intersection graphs of a family of subtrees of a tree (see [4,7]).

**Theorem 4.2** *The contour of every chordal graph is geodetic.*

In Figure 3, the state of the art on this topic is displayed. Notice that the main issue to be approached is the case of bipartite graphs.

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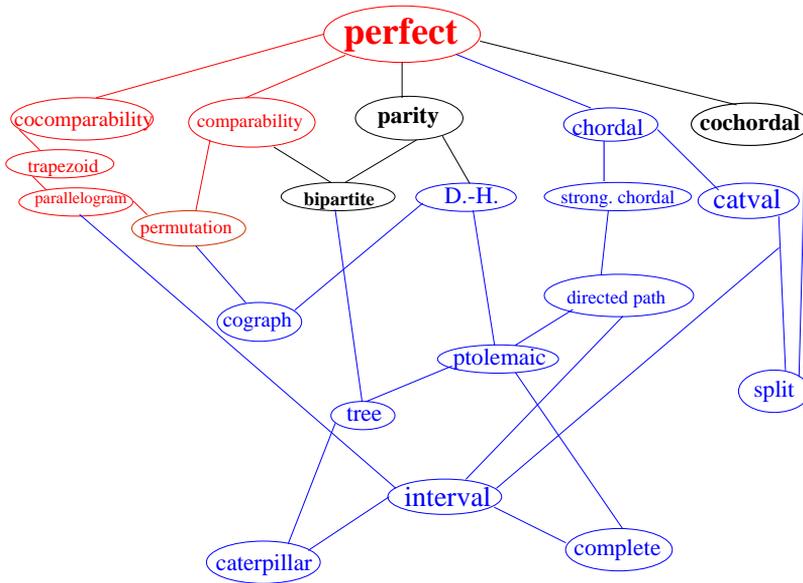


Fig. 3. Hasse diagram of the perfect family. Open problems appear in boldface.

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