Every Cubic Cage is quasi 4-connected

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Extended Abstract

Throughout this paper, $G = (V, E)$ denotes a (δ, g)-graph with vertex set V and edge set E, that is, a regular simple graph of degree δ and girth g. For every $v \in V$, $N(v)$ denotes the neighbourhood of v. If $S \subset V$, then $N(S) = \cup_{v \in S} N(v)$. If H is a subgraph of G, then $N_H(S) = N(S) \cap V(H)$. The subgraph of G induced by S is denoted $G[S]$. For $X, Y \subset V$, $d(X, Y) = \min\{d(x, y) : x \in X, y \in Y\}$, where $d(x, y)$ denotes the distance between x and y. The *diameter* D of G is the maximum distance over all pairs of vertices in G. A graph G is called *connected* if every pair of vertices is joined by a path. If $S \subset V$ and $G - S$ is not connected, then G is said to be a *cutset*. A (*connected*) *component* of a nonconnected graph G is a maximal connected subgraph of G. A connected graph is called *k-connected* if every cutset has cardinality at least k. The *connectivity* κ of a connected graph G is defined as the maximum integer k such that G is k-connected. The *minimum* cutsets are those having cardinality κ. A minimum cutset S is called *trivial* if $S = N(v)$ for some $v \in V$.

A (δ, g)-graph G is *superconnected* if $\kappa = \delta$ and every minimum cutset is trivial (see [4]). A superconnected $(3, g)$-graph is *quasi* 4-*connected* if for each minimum cutset $S = N(v)$, $G - S$ has precisely two components. Quasi 4-connected graphs, which exhibit many of the properties of 4-connected graphs, offer a true refinement of the strict vertex connectivity (see [1]).

Let $f(\delta, g)$ denote the smallest integer ν such that there exists a (δ, g)-graph having ν vertices. A (δ, g)-*cage* is a (δ, g)-graph with $f(\delta, g)$ vertices. This type of graphs has been intensely studied since introduced by Tutte in [7] (see [8] for a survey). Most of the work carried out so far has focused on the existence problem, whereas very little is known about structural properties. Recently, several authors have approached the problem of studying the connectivity of cages. In the first parper on this issue (see [5]), Fu et al. proved that every (δ, g)-cage is 2-connected. In addition, they conjectured that all (δ, g)-cages are δ-connected and proved this statement for $\delta = 3$. Subsequently, it has been proved that every (δ, g)-cage is 3-connected (see [3, 6]).

This paper puts forward a further contribution towards the proof of the mentioned *conjecture* showing that every $(3, g)$-cage with $g \geq 5$ is quasi 4-connected. This statement has been proved taking into account the following known results.

Theorem 0.1 *([4]) Let G be a connected graph with minimum degree $\delta \geq 3$ and diameter D. Then G is superconnected if $D \leq 2\lfloor \frac{g-3}{2} \rfloor$.*

Theorem 0.2 *([5]) If $\delta \geq 3$ and $3 \leq g_1 < g_2$, then $f(\delta, g_1) < f(\delta, g_2)$.*

Theorem 0.3 *([6]) Let S be a cutset of a (δ, g)-cage with $\delta \geq 3$ and $g \geq 5$. Then, the diameter of $G[S]$ is at least $\lfloor \frac{g}{2} \rfloor$. Furthermore, the inequality is strict if $d_{G[S]}(u, v)$ is maximized for exactly one pair of vertices.*

The only $(3, g)$-cages with $g = 3$ and $g = 4$ are K_4 and $K_{3,3}$ respectively. Certainly, the complete graph K_4 has not cutsets. It is also clear that the complete bipartite graph $K_{3,3}$ is superconnected but not quasi 4-connected. For this reason, we henceforth assume that $g \geq 5$.

Let $G = (V, E)$ be a $(3, g)$-cage with girth $g \geq 5$. Let $\mathcal{F}$ denote the set of all nontrivial minimum cutsets of G. The main goal of this work is to show that $\mathcal{F} = \emptyset$, i.e., that G is superconnected. To this end, suppose on the contrary that $\mathcal{F} \neq \emptyset$. For every $F \in \mathcal{F}$, let C_F denote a smallest component of $G - F$. Notice that, as $g \geq 5$, the graph C_F must be cyclic and thus $|C_F| \geq g$. Let $S = \{x, y, z\}$ denote a nontrivial minimum cutset of G satisfying the following condition:

$$|V(C_S)| \leq |V(C_F)|, \text{ for every } F \in \mathcal{F}. \quad (1)$$

In the rest of this work, we use the following notation: $C_1 = C_S$, $C_2 = (G - S) \setminus C_1$, $X = N_{C_1}(x)$, $Y = N_{C_1}(y)$, $Z = N_{C_1}(z)$ and $L = \min\{d_{C_1}(X,Y), d_{C_1}(X,Z), d_{C_1}(Y,Z)\}$.

Lemma 0.1 *If $S \in \mathcal{F}$ satisfies (1), then $|X| = |Y| = |Z| = 2$.*

Proof. Suppose $X = \{\alpha_1\}$. Consider the set $F = \{\alpha_1, y, z\}$, which clearly is a minimum cutset satisfying $|V(C_F)| < |V(C_1)|$. In consequence, it must be trivial. Let β_1 be a vertex such that $F = N(\beta_1)$. Observe that $\beta_1 \in V(C_1)$, because otherwise $\beta_1 = x$, contradicting Theorem 0.3. Suppose $N_{C_1}(y) = \{\beta_1\}$ [resp. $N_{C_1}(z) = \{\beta_1\}$] and consider the set $F' = \{\alpha_1, \beta_1, z\}$ [resp. $F' = \{\alpha_1, \beta_1, y\}$]. Observe that F' is a minimum cutset satisfiying $|V(C'_F)| < |V(C_1)|$. Therefore, F' must be trivial, which is a contradiction because $N_{C_1}(\beta_1) = \{\alpha_1\}$. We can thus assume that $|N_{C_1}(y)| = |N_{C_1}(z)| = 2$, i.e., $N_{C_1}(y) = \{\beta_1, \beta_2\}$, $N_{C_1}(z) = \{\beta_1, \gamma_2\}$. Finally, take the set $F'' = \{\alpha_1, \beta_2, \gamma_2\}$, which is a nontrivial minimum cutset since $g \geq 5$ and $|C_1| \geq 5$. As $|V(C''_F)| < |V(C_1)|$, we arrive again at a contradiction. ■

From this result, we can conclude that for every nontrivial minimum cutset S satisfying (1), the only possible structure is that displayed in Figure 2, where $4 \leq |N_{C_1}(S)| \leq 6$ and $|N_{C_2}(S)| = 3$. Starting from this fact, the following statements are proved.

Theorem 0.4 *Let G be a $(3, g)$-cage with $g \geq 5$. Then, G is superconnected.*

Outline of the proof. This proof is based on the following facts:

- The McGee cage, which is the unique $(3, 7)$-cage has diameter $D = 4$.
- There are 18 different $(3, 9)$-cages, all of them having diameter $D = 6$ (see [2]).
- If G is not superconnected:

  (i) $|V(C_2)| \geq |V(C_1)| + 3$; (ii) if $g \geq 8$, then $L \geq 1$; (iii) $L \leq \ell - 3$, where $\ell = \lfloor \frac{g-1}{2} \rfloor$.

For $g = 7, 9$, we have that $(3, g)$-cages are superconnected because of the first two above points and Theorem 0.1. From now on, assume that G is not superconnected. As $L \leq \ell - 3$, it follows that $\ell \geq 3$, that is, $g \geq 7$. Therefore, any $(3, g)$-cage with $g \leq 6$ must be superconnected. For $g \geq 8$, we have $1 \leq L \leq \ell - 3$, from where $\ell \geq 4$ and hence $g \geq 9$. As a consequence, the $(3, 8)$-cage must be superconnected. Finally, if $g \geq 10$, we define a 3-regular connected graph G^* with less vertices than G as follows: $V(G^*) = V(\tilde{C}_1) \cup V(\tilde{C}'_1)$, where $\tilde{C}_1$ is a certain subgraph of C_1, and $\tilde{C}'_1$ is a copy of $\tilde{C}_1$; $E(G^*) = E(\tilde{C}_1) \cup E(\tilde{C}'_1) \cup E^+$, where E^+ is a set of edges joining vertices of $\tilde{C}_1$ and $\tilde{C}'_1$ in a suitable way. As for the girth of G^*, we prove that $g(G^*) \geq g$, arriving at a contradiction with the fact that G is a $(3, g)$-cage, taking into account Theorem 0.2. ■

Theorem 0.5 *Let G be a $(3, g)$-cage with $g \geq 5$. Then, G is quasi 4-connected.*

Proof. By Theorem 0.4 G is superconnected. When $g = 5$, G is the Petersen graph, which is quasi 4-connected. Assume henceforth $g \geq 6$. If G is not quasi 4-connected, then there exists a cutset $S = N(v)$ for some vertex v, such that $G - N(v)$ has more than two components. Under this assumption, the subgraph induced by $\{v\} \cup N(v)$ is a separating set of diameter 2. So, from Theorem 0.3 it follows that $2 \geq \lfloor g/2 \rfloor \geq 3$, which is impossible. ■

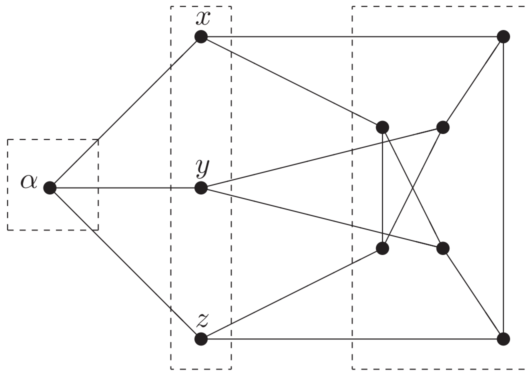

Figure 1: The Petersen graph is quasi 4-connected

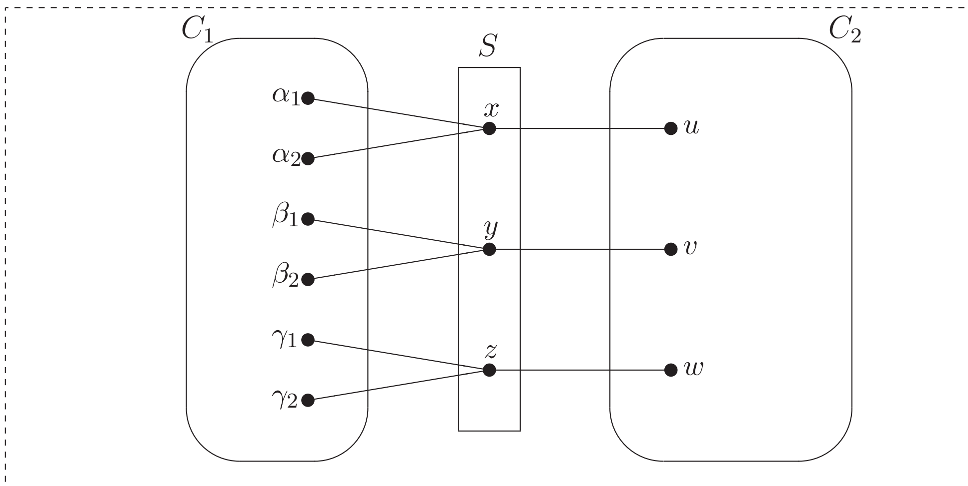

Figure 2: Structure of a nonsuperconnected $(3, g)$-cage with $g \geq 5$.

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