On monophonic sets in graphs[1]

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MSC2000: 05C12,05C05

Abstract

Classical convexity can be naturally translated to graphs considering shortest paths, giving rise to the geodesic convexity. Moreover, if we consider induced paths between vertices, we get the so-called *monophonic convexity*. In this work, we present some results involving monophonic sets.

1 Introduction

In what follows, $G = (V, E)$ denotes a finite connected graph with no loops or multiple edges. The *distance* $d(u, v)$ between two vertices u and v is the length of a shortest $u - v$ path in G. The eccentricity of a vertex $u \in V$ is defined as $ecc(u) = \max\{d(u, v)|v \in V\}$. A $u-v$ path ρ is called *monophonic* if it is a chordless path. The path ρ is called a $u-v$ *geodesic* if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$. The *geodetic closed interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics. Similarly, the *monophonic closed interval* $J[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $S \subseteq V$, the *geodetic closure* $I[S]$ of S is the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in S$. The *monophonic closure* is defined as the union of all monophonic closed intervals.

The most natural convexities in a graph are *path convexities* defined by a system $\mathcal{P}$ of paths in G. Thus far, two special types of path convexities have received the most attention, the *geodesic convexity* and the *monophonic convexity*. A set $W \subseteq V$ is called *geodetically convex* (or simply *g-convex*) if $I[W] = W$, while it is said to be *geodetic* if $I[W] = V(G)$. Likewise, W is

[1]Partially supported by projects MCYT-FEDER TIC-2001-2171, MCYT-FEDER BFM2002-0557, MCYT-FEDERBFM2003-0368 and Gen. Cat. 2001SGR00224.

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called *monophonically convex* (or simply *m-convex*) if $J[W] = W$, and it is called *monophonic* if $J[W] = V(G)$.

Given a vertex set $S \subset V$, the smallest g-convex set containing S is denoted as $[S]_g$ and it is called the g-convex hull of S. Similarly, the m-convex hull of S, $[S]_m$, is defined as the minimum m-convex set containing S. Observe that $[S]_g \subseteq [S]_m$. It is also clear that both $I[S] \subseteq [S]_g$ and $J[S] \subseteq [S]_m$.

For a nonempty set W of vertices in a connected graph G, a connected subgraph of G with the minimum number of edges that contains all of W clearly must be a tree; such a tree is called a *Steiner W-tree*. The *Steiner interval* $S(W)$ of W consists of all vertices that lie on some Steiner W-tree. If $S(W) = V(G)$, then W is called a *Steiner set* for G.

In [2], it was shown that every Steiner set in a graph G is also geodetic. Unfortunately, this particular result turned out to be wrong and was disproved by Pelayo [6]. In [4] the authors study parameters related to geodetic and monophonic convexities. In this work we show some results involving monophonic convexity.

2 Every Steiner set is a monophonic set

Throughout this section, we only consider the monophonic convexity (see [3]). A subset of vertices S of a connected graph $G = (V, E)$ is a said to be a (monophonic) *hull set* if its m-convex hull $[S]_m$ covers the graph, i.e., if $[S]_m = V$. We have proved that the statement *every g-hull set is monophonic* is far from being true. We have also seen that not every Steiner set is a (geodesic) hull set [4], therefore it seems natural to ask whether *every Steiner set monophonic.* Very pleasantly, this time, the answer turns out to be affirmative.

Lemma 2.1 *Let $G = (V, E)$ be a connected graph. Let $W \subseteq V$, and let T be a Steiner W-tree in G. Then $V(T) \subseteq J[W]$.*

Theorem 2.1 *Every Steiner set of a connected graph $G = (V, E)$ is monophonic.*

A *distance-hereditary graph* is a graph in which every monophonic path is a geodesic [5]. Then, as a consequence of Theorem 2.1 we can derive the following corollary.

Corollary 2.1 *In any distance-hereditary graph, every Steiner set is geodetic.*

This last result also holds for *interval graphs*, but it is not true for *split graphs*. Both interval and split graphs are chordal. It is still an open question the problem of characterizing those classes of chordal graphs for which *every Steiner set is geodetic.*

Theorem 2.2 *In any interval graph, every Steiner set is geodetic.*

3 The contour of a graph is a monophonic set

Given a graph $G = (V, E)$, the contour of G(see [1]), $Ct(G)$, is defined as the set of vertices such that its eccentricity is greater that or equal to the eccentricity of its neighbors. In [1], it is shown that the contour of a graph is a (geodetic) hull set, and, in general, it is not a geodetic set. Going one step further, we have obtained the following result.

Theorem 3.1 *The contour of a graph is a monophonic set.*

It is interesting to notice that the obtained proof is a constructive one. As a consequence of this theorem we have the following corollary, which have already been proved in [1].

Corollary 3.1 *The contour of a distance-hereditary graph is a geodetic set.*

4 The edge problem

In this section, we focus our attention on the edges that lie in paths joining two vertices. We define the *edge intervals* of a graph as follows. The *edge (geodetic) closed interval* $I_e[u, v]$ is the set of edges of all $u - v$ geodesics. Similarly, the *edge monophonic closed interval* $J_e[u, v]$ is the set of vertices of all monophonic $u - v$ paths. For $S \subseteq V$, the *edge geodetic closure* $I_e[S]$ of S is the union of all edge closed intervals $I_e[u, v]$ over all pairs $u, v \in S$. The *edge monophonic closure*, $J_e[S]$, is defined as the union of all edge closed monophonic intervals over all pairs $u, v \in S$. In other words, we have

$$I_e[S] = \bigcup_{u,v \in S} I_e[u, v], \quad J_e[S] = \bigcup_{u,v \in S} J_e[u, v].$$

A set S of vertices for which $J_e[S] = E(G)$ is called an *edge monophonic set.* Similarly, S is called an *edge geodetic set* if $I_e[S] = E(G)$. A set $W \subseteq V$ is an *edge Steiner set* if the edges lying in some Steiner W-tree cover $E(G)$. Notice that: (1) every edge Steiner set is a Steiner set, (2) every edge geodetic set is a geodetic set, (3) every edge monophonic set is a monophonic set, and (4) every edge geodetic set is an edge monophonic set. It is easy to find examples where the converses of these statements are not true. We have obtained the following results.

Theorem 4.1 *Every edge Steiner set is an edge monophonic set.*

Corollary 4.1 *In a distance-hereditary graph, every edge Steiner set is edge geodetic.*

As in the vertex case, this last result also holds for interval graphs, but it is not true for split graphs. It is still an open question the problem of characterizing those classes of chordal graphs for which *every edge Steiner set is edge geodetic.*

Theorem 4.2 *In the class of interval graphs, every edge Steiner set is edge geodetic.*

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