On the Steiner, hull and geodetic numbers of graphs[*]

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Abstract

Given a graph G and a subset $W \subseteq V(G)$, a *Steiner W-tree* is a tree of minimum order that contains all of W. Let $S(W)$ denote the set of all vertices in G that lie on some Steiner W-tree; we call $S(W)$ the *Steiner interval* of W. If $S(W) = V(G)$, then we call W a *Steiner set* of G. The minimum order of a Steiner set of G is called the *Steiner number* of G.

Given two vertices u, v in G, a shortest $u - v$ path in G is called a $u - v$ *geodesic*. Let $I[u, v]$ denote the set of all vertices in G lying on some $u - v$ geodesic, and let $J[u, v]$ denote the set of all vertices in G lying on some induced $u - v$ path. Given a set $S \subseteq V(G)$, let $I[S] = \bigcup_{u,v \in S} I[u, v]$, and let $J[S] = \bigcup_{u,v \in S} J[u, v]$. We call $I[S]$ the *geodetic closure* of S and $J[S]$ the *monophonic closure* of S. If $I[S] = V(G)$, then S is called a *geodetic set* of G. If $J[S] = V(G)$, then S is called a *monophonic set* of G. The minimum order of a geodetic set in G is called the *geodetic number* of G.

In this paper, we explore the relationships both between Steiner sets and geodetic sets and between Steiner sets and monophonic sets. We thoroughly study the relationship between the Steiner number and the geodetic number, and address the questions: in a graph G when must every Steiner set also be geodetic and when must every Steiner set also be monophonic. In particular, among others we show that every Steiner set in a connected graph G must also be monophonic, and that every Steiner set in a connected interval graph H must be geodetic.

Keywords: chordal graph; convexity, geodesic, geodetic set, geodetic number, monophonic path, monophonic set, monophonic number, Steiner set, Steiner number.

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