Geodesic Convexity and Chordal Graphs

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Abstract

A *convexity* on a finite set X is a family $\mathcal{C}$ of subsets of X (each such set called a *convex set*), which is closed under intersection and which contains both X and the empty set. The pair $(X, \mathcal{C})$ is called a convexity space. A (finite) *graph convexity* space is a pair $(G, \mathcal{C})$, formed by a finite connected graph $G = (V, E)$ and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying that every member of $\mathcal{C}$ induces a connected subgraph of G.

Given vertices u, v in a graph $G = (V, E)$ we let $d(u, v)$ denote the distance between u and v in G. A $u - v$ path ρ is called *monophonic* if it is a chordless path. The path ρ is called a $u - v$ *geodesic* if it is a shortest $u - v$ path, that is, if $|E(\rho)| = d(u, v)$. The *geodetic closed interval* $I[u, v]$ is the set of vertices of all $u - v$ geodesics and for $S \subseteq V$, the *geodetic closure* $I[S]$ of S is the union of all geodesic closed intervals $I[u, v]$ over all pairs $u, v \in S$. Similarly, the *monophonic closed interval* $J[u, v]$ is the set of vertices of all monophonic $u - v$ paths and the *monophonic closure* $J[S]$ of S is the set formed by the union of all monophonic closed intervals $J[u, v]$ over all pairs $u, v \in S$..

The most natural convexities in a graph are *path convexities* defined by a system $\mathcal{P}$ of paths in G. Thus far, two special types of path convexities have received the most attention, the *geodesic convexity* (also called *metric convexity* [13]) and the *monophonic convexity* (also known as *minimal path convexity* [5]). A set $W \subseteq V$ is called *geodetically convex* (or simply *g-convex*) if $I[W] = W$, while it is said to be *geodetic* if $I[W] = V(G)$. Likewise, W is called *monophonically convex* (or simply *m-convex*) if $J[W] = W$, and it is called *monophonic* if $J[W] = V(G)$. Given a vertex set $S \subseteq V$, the smallest g-convex set containing S is denoted as $[S]_g$ and it is called the g-convex hull of S. Similarly, the m-convex hull of S, $[S]_m$, is defined as the minimum m-convex set containing S. In the case where $[S]_g = V$ ($[S]_m = V$, resp.), S is called a *g-hull set* (*m-hull set*, resp.).

For a nonempty set W of vertices in a connected graph G, a connected subgraph of G with the minimum number of edges that contains all of W clearly must be a tree; such a tree is called a *Steiner W-tree*. The *Steiner interval* $S(W)$ of W consists of all vertices that lie on some Steiner W-tree. If $S(W) = V(G)$, then W is called a *Steiner set* for G. In [4], it was shown that every Steiner set in a graph G is also geodetic. Unfortunately, this particular result turned out to be wrong and was disproved by Pelayo [12]. In [9], the authors proved that every Steiner set is monophonic. In [10], the corresponding *edge Steiner set* problem was approached, being proved that every edge Steiner set is edge monophonic but not necessarily edge geodetic.

Given vertex set $W \subseteq V$, the *eccentricity* in W of a vertex $u \in W$ is defined as $ecc_W(u) = \max\{d(u, v) \mid v \in W\}$. In particular, $ecc_G(u) = ecc(u) = \max\{d(u, v) \mid v \in V\}$. The *periphery* of G, denoted $Per(G)$, is the set of vertices that have maximum eccentricity, i.e., the set of the so-called *peripheral* vertices. A vertex v is said to be *simplicial* in G if the subgraph induced by its neighborhood $N(v)$ is a clique. The *extreme set* of G, denoted $Ext(G)$, is the set of all its simplicial vertices. With the aim of generalizing these two definitions, the so-called *contour* of G was introduced in [2] as follows. A vertex $u \in W$ is said to be a *contour vertex* of W if

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$ecc_W(u) \geq ecc_W(v)$, for all $v \in N(u) \cap W$. The contour $Ct(W)$ of W is the set formed by all the contour vertices of W. If $W = V$, this set is called the contour of G and is denoted $Ct(G)$. Notice that $Per(G) \cup Ext(G) \subseteq Ct(G)$. In [2], it was proved that every convex set is the g-convex hull of its contour. In [10], it was proved that every m-convex set W satisfies $J[Ct(W)] = W$, and thus that the contour of the whole graph G is monophonic. In this same work [10], it was also seen that $Ct(G)$ does not need to be geodetic, unless that, for example, $Ct(G) = Per(G)$. In [3], we have introduced the so-called *expanded contour* $EC(W)$ of a set $W \subseteq V$, by slightly relaxing the definition of contour vertex, proving that every g-convex set W satisfies $I[EC(W)] = W$.

Chordal graphs form an important subclass of perfect graphs, which have been extensively studied in different ways, including within the context of convexity in graphs [1, 6, 8, 11]. Some significant subfamilies of chordal graphs are: complete graphs, trees, Ptolemaic graphs, split graphs, interval graphs and strongly chordal graphs. A convexity space is called a *convex geometry* if every convex set S is the convex hull of its extreme points (a point $p \in S$ is called an *extreme point* of S if it does not belong to the convex hull of $S - p$) [7]. In [6], it was proved that a graph G is chordal if and only if the monophonic convexity is a convex geometry, and that G is Ptolemaic if and only if the geodesic convexity is a convex geometry. In this talk we present some partial results we have obtained for a number of families inside the class of chordal graphs involving the following statements:

- Every Steiner set is geodetic.
- Every edge Steiner set is edge geodetic.
- The geodetic closure of the contour is geodetic.
- The contour is geodetic.

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