Graphs as convexity spaces: Classical and graph theoretic parameters

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CONVEXITY SPACE

⋆ V is a non-empty set, $\mathcal{C}$ is collection of V-sets.

- $(V, \mathcal{C})$ is a convexity space if:

(C1) $\emptyset \in \mathcal{C}$, $V \in \mathcal{C}$.

(C2) $\{W_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} W_i \in \mathcal{C}$.

(C3) $\{W_i\}_{i \in I} \subseteq \mathcal{C}$ s.t. $W_i \subseteq W_{i+1} \Rightarrow \bigcup_{i \in I} W_i \subseteq \mathcal{C}$.

GRAPH CONVEXITY SPACE

⋆ $G = (V, E)$ is a connected graph and $\mathcal{C} \subset 2^V$.

• $(G, \mathcal{C})$ is a GRAPH CONVEXITY SPACE if:

(C1) $\emptyset \in \mathcal{C}$, $V \in \mathcal{C}$.

(C2) $\{W_i\}_{i \in I} \subseteq \mathcal{C} \Rightarrow \bigcap_{i \in I} W_i \in \mathcal{C}$.

(C3) $\{W_i\}_{i \in I} \subseteq \mathcal{C}$ s.t. $W_i \subseteq W_{i+1} \Rightarrow \bigcup_{i \in I} W_i \subseteq \mathcal{C}$.

(C4) For every $U \in \mathcal{C}$, $\langle U \rangle_G$ is connected.

CONVEX HULL AND SOME TYPES OF DEPENDENCE

⋆ CONVEX HULL of $S \subset V$:

$$[S]_{\mathcal{C}} = \bigcap \{W : \; S \subset W, W \in \mathcal{C}\},$$

- S is convexly dependent if there exist $x \in S$ s.t. $x \in [S-x]_{\mathcal{C}}$.
- S is redundant if $[S]_{\mathcal{C}} = \bigcup_{a \in S} [S - a]_{\mathcal{C}}$.
- S is Helly dependent if $\bigcap_{a \in A} [S - a]_{\mathcal{C}} \neq \emptyset$.
- S is E-dependent if for all $x \in S$, $[S - x]_{\mathcal{C}} \subseteq \bigcup_{a \in S - x} [S - a]_{\mathcal{C}}$.

SOME CLASSICAL PARAMETERS

IN EVERY CASE, $x(G)$ is the maximum cardinality of ...

$r(G)$: ...of a convexly independent set [Rank].

$c(G)$: ...of an irredundant set [Caratheodory number].

$h(G)$: ...of a Helly independent set [Helly number].

$e(G)$: ...of a exchange independent set [Exchange number].

GEODESIC CONVEXITY

⋆ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_g \subset 2^V$.

- A $u - v$ geodesic is a $u - v$ path of minimum length.

- Closed interval: $I[u, v] = \{V(\rho) : \ \rho \ \text{ is a } u - v \text{ geodesic}\}$

- Geodetic closure: $I[S] = \bigcup_{u,v \in S} I[u, v]$

- g-convex set: $S \in \mathcal{C}_g \Leftrightarrow S = I[S]$.

- g-convex hull:

$$S \subseteq I[S] \subseteq I^2[S] \subseteq \ldots \subseteq I^r[S] = [S]_g \subseteq V$$

MONOPHONIC CONVEXITY

⋆ $G = (V, E)$ connected graph, $u, v \in V$, $S \subseteq V$, $\mathcal{C}_m \subset 2^V$.

▶ A $u - v$ monophonic path is a $u - v$ chordless path.

▶ Closed interval: $J[u, v] = \{V(\rho) : \ \rho$ is a $u{-}v$ monophonic path$\}$

▶ Monophonic closure: $J[S] = \bigcup_{u,v \in S} J[u, v]$

▶ m-convex set: $S \in \mathcal{C}_m \Leftrightarrow S = J[S]$.

▶ m-convex hull:

$$S \subseteq J[S] \subseteq J^2[S] \subseteq \ldots \subseteq J^r[S] = [S]_m \subseteq V$$

GEODESIC CONVEXITY PARAMETERS (I)

⋆ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_g)$ g-convexity space.

▶ g-convex set: $I[S] = S$

⊗ Convexity number: $con(G) = \max\{|S| : S \subsetneq V,\ S \text{ is convex}\}$

▶ $S \subsetneq I[S] \subsetneq \ldots \subsetneq I^r[S] = I^{r+1}[S] = [S]_g \subseteq V \;\Rightarrow\; gin(S) = r$

⊗ Geodetic iteration number: $gin(G) = \max\{gin(S) : S \subseteq V\}$

GEODESIC CONVEXITY PARAMETERS (II)

⋆ $G = (V, E)$ conn. graph, $S \subseteq V$, $(V, \mathcal{C}_g)$ g-convexity space.

▶ Geodetic set: $I[S] = V$

⊗ Geodetic number: $gn(G) = \min\{|S| : S \text{ is a geodetic set of } G\}$

▶ Hull set: $[S]_g = V$.

⊗ Hull number: $hn(G) = \min\{|S| : S \text{ is a hull set of } G\}$

↪ $hn(G) \leq gn(G)$

CONVEX PRODUCT SPACE

⋆ $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ connected graphs.

⋆ $(V_1, \mathcal{C}_1)$, $(V_2, \mathcal{C}_2)$ convexity spaces.

▶ Cartesian product: $G = G_1 \times G_2 = (V, E)$, where:

♦ $V = V_1 \times V_2$

♦ $uv = (u_1, u_2)(v_1, v_2) \in E \Leftrightarrow \begin{cases} u_1 = v_1 \quad \text{and } u_2 v_2 \in E_2 \\ \text{or} \\ u_1 v_1 \in E_1 \text{ and } u_2 = v_2 \end{cases}$

▶ Convex product space: $(V, \mathcal{C})$, where:

■ $\mathcal{C} = \mathcal{C}_1 \oplus \mathcal{C}_2 = \{A \times B \mid A \in \mathcal{C}_1,\ B \in \mathcal{C}_2\}$

GEODESIC CONVEXITY AND CARTESIAN PRODUCT

⋆ $G_1 = (V_1, E_1)$, $G_2 = (V_2, E_2)$ connected graphs.

⋆ $(V_1, \mathcal{C}_1)$, $(V_2, \mathcal{C}_2)$ g-convexity spaces (i.e., $\mathcal{C}_i = (\mathcal{C}_i)_g$, $i = 1, 2$).

▶ $G = G_1 \times G_2 = (V, E)$ satisfies:

1. $\mathcal{C}_g = \mathcal{C}_1 \oplus \mathcal{C}_2$

2. $S \in \mathcal{C}_g \Leftrightarrow \begin{cases} S_i = \pi_i(S) \in \mathcal{C}_i, \ i = 1, 2 \\ \wedge \\ S = S_1 \times S_2 \end{cases}$

3. $S \subseteq V \Rightarrow [S]_g = [S_1]_g \times [S_2]_g$, where $S_i = \pi_i(S)$, $i = 1, 2$.

KNOWN RESULTS (I)

⋆ $G_1 \times G_2 = (V, E)$, $(V, \mathcal{C}_g)$ g-convexity space.

- Sierksma, 1975 ?:

$$h(G_1 \times G_2) = \max\{h(G_1), h(G_2)\}$$

- Chartrand, Harary and Zhang 2000, 2002:

$$|V_1| \geq 2 \Rightarrow hn(G_1 \times K_2) = hn(G_1), \quad gn(G_1 \times K_2) \geq gn(G_1)$$

- Canoy and Garces, 2002:

$$con(G_1 \times G_2) = \max\{|V_2| \cdot con(G_1), |V_1| \cdot con(G_2)\}$$

KNOWN RESULTS (II)

Sierksma,1975 and Soltan,1981:

- $c(G) = c(G_1) + c(G_2) - k$

  where $k \in \{0, 1, 2\}$ is the number of factors for which $e \leq c$.

- $e(G) = \max\{e_1, c_1\} + \max\{e_2, c_2\} - 1$

  $$e(G) = c_1 + sign(e_1 - c_1 - 1) + c_2 + sign(e_2 - c_2 - 1) + 1$$

↪ $e - 1 \leq c$

↪ $c + \text{sign}(e - c - 1) + 1 = \max\{e, c\}$

RANK

⊛ $S \subseteq V$ is convexly indep. iff for each $x \in S$, $x \notin [S - x]$.

⊛ $r(G) = \max\{|\, S \,| : \; S \subseteq V$ is convx. indep$\}$

▶ $S \subseteq V$ is <u>weakly convexly independent</u>:

there exist at most $x \in S$ such that $x \in [S - x]$

▶ $r^1(G) = \max\{|\, S \,| : \; S \subseteq V$ is weakly convx. indep$\}$

$\hookrightarrow$ $r \leq r^1 \leq r + 1$

$\Rrightarrow$ $\boxed{r(G_1 \times G_2) = r^1(G_1) + r^1(G_2) - 2}$

CARATHEODORY NUMBER

⊛ $c(G) = \max\{|S| : \bigcup_{a \in S} [S-a]_{\mathcal{C}} \subsetneq [S]_{\mathcal{C}}\}$

▶ $S \subseteq V$ is <u>weakly irredundant</u>:

there exist at least $x \in S$ such that $\bigcup_{a \in S-x} [S-a]_{\mathcal{C}} \subsetneq [S]_{\mathcal{C}}$

▶ $c^1(G) = \max\{|S| : S \subseteq V \text{ is weakly irredundant}\}$

↪ $c \leq c^1 \leq c + 1$

↪ $c^1 = c + \text{sign}(e - c - 1) + 1 = \max\{e, c\}$

⇛ $\boxed{c(G_1 \times G_2) = c^1(G_1) + c^1(G_2) - 2}$

EXCHANGE NUMBER

⊛ $S \subseteq V$ is E-independent iff there exist $x \in S$ such that

$$[S-x]_{\mathcal{C}} \subsetneq \bigcup_{a \in S-x} [S-a]_{\mathcal{C}}$$

⊛ $e(G) = \max\{|\, S \,| : \; S \subseteq V \;\; \text{is E} - \text{independent}\}$

↪ $e - 1 \leq c \leq c^1 \leq c + 1$

↪ $c^1 = c + \text{sign}(e - c - 1) + 1 = \max\{e, c\}$

⇛ $$\boxed{e(G_1 \times G_2) = c^1(G_1) + c^1(G_2) - 1 = c(G_1 \times G_2) + 1}$$

HULL NUMBER

⊛ $S \subseteq V$ is a hull set if $[S]_g = V$.

⊛ $hn(G) = \min\{|S| : S \text{ is a hull set of } G\}$

$\Rightarrow$ $$\boxed{hn(G_1 \times G_2) = \max\{hn(G_1), hn(G_2)\}}$$

GEODETIC NUMBER

⊛ $S \subseteq V$ is a geodetic set if $I[S] = V$.

⊛ $gn(G) = \min\{|S| : S$ is a geodetic set of $G\}$

∗ $g_1 = gn(G_1)$, $g_2 = gn(G_2)$:

⇛ $$\max\{g_1, g_2\} \leq gn(G_1 \times G_2) \leq g_1 g_2 - \min\{g_1, g_2\}$$

↪ Both bounds are tight.