

Every Cage is 4-connected

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A (δ, g) -cage is a regular graph of degree δ and girth g with the least possible order.

Fu, Huang, Rodger (1997):

Every $(3, g)$ -cage is 3-connected

Every (δ, g) -cage is δ -connected (?)

Pelayo, Marcote, Balbuena (2001):

Every cubic cage is quasi-4-connected

Jiang, Mubayi/Daven, Rodger (1998/1999):

Every (δ, g) -cage, $\delta \geq 3$, is 3-connected

Pelayo, Marcote, Balbuena, Fàbrega (2002):

Every (δ, g) -cage, $g \geq 10$, is 4-connected

Basic Definitions

- A graph $G = (V, E)$ is called **regular** if all its vertices have the same degree δ .
- A **(δ, g) -graph** is a regular graph of degree δ and girth g .
- A **(δ, g) -cage** is a (δ, g) -graph with the least possible number of vertices.
- $g \geq 3$: $f(\delta, g)$ denotes the order of a (δ, g) -cage.
- The order of every **(δ, g) -graph** is at least the so-called **Moore bound** $n(\delta, g)$. Thus,

$$f(\delta, g) \geq n(\delta, g) = \begin{cases} \frac{\delta \cdot (\delta-1)^{\frac{g-1}{2}} - 2}{\delta-2}, & \text{if } g \text{ is odd;} \\ \frac{2 \cdot (\delta-1)^{\frac{g}{2}} - 2}{\delta-2}, & \text{if } g \text{ is even.} \end{cases}$$

- A **Moore graph** is a (δ, g) -cage s.t. $f(\delta, g) = n(\delta, g)$.
- A graph $G = (V, E)$ is **connected** if every pair of vertices is joined by a path.
- A subset of vertices $S \subset V$ is said to be a **cutset** if the graph $G - S$ is not connected.
- G is **k -connected** if every cutset S has cardinality at least k .

Some Examples of Cages

- The **complete graph** $K_{\delta+1}$ is the unique $(\delta, 3)$ -cage. Certainly, it has no cutsets, and hence we can state that its connectedness is optimal.
 - The **complete bipartite graph** $K_{\delta, \delta}$ is the unique $(\delta, 4)$ -cage. Clearly, it is δ -connected.
 - Every Moore graph is a cage. For example:
Petersen, Heawood, Tutte-Coxeter, Hoff.-Singl.,...
 - The **McGee graph** is the unique $(3, 7)$ -cage. It has 24 vertices ($n(3, 7) = 22$) and is 3-connected.
 - The **Robertson graph** is the unique $(4, 5)$ -cage. It has 19 vertices ($n(4, 5) = 17$) and is 4-connected.
 - The **Robertson-Wegner graph** is a $(5, 5)$ -cage. It has 30 vertices ($n(5, 5) = 26$) and is 5-connected. (There are 4 known $(5, 5)$ -cages)
 - The **O'Keefe-Wong graph** is the unique $(6, 5)$ -cage. It has 40 vertices ($n(6, 5) = 36$) and is 5-connected.
- ★ <http://www.cs.uwa.edu.au/~gordon/data.html>

Starting known results

- ★ ([1]) If $3 \leq g_1 < g_2$, then $f(\delta, g_1) < f(\delta, g_2)$.
- ★ ([4]) Let S be a cutset of a (δ, g) -cage with $\delta \geq 3$ and $g \geq 5$. Then, the **diameter** of the subgraph of G induced by S is at least $\lfloor \frac{g}{2} \rfloor$. Furthermore, the inequality is **strict** if $d_{G[S]}(u, v)$ is maximized for exactly one pair of vertices.
- ★ ([2]) Let G be a connected graph with minimum degree $\delta \geq 3$ and girth g , where either $g = 2\ell + 1$ or $g = 2\ell + 2$. Then, G is **δ -connected** if
$$|V(G)| \leq 2(1 + \delta + \delta(\delta - 1) + \dots + \delta(\delta - 1)^{\ell-1}) - \delta.$$
- ★ ([4,5]) Every (δ, g) -cage with $\delta \geq 3$ is **3-connected**.
- ★ ([3])(Conjecture) Every (δ, g) -cage is **δ -connected**.

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- [1] P. Erdős and H. Sachs, Reguläre Graphen gegebener Tailenweite mit minimaler Knotenzahl, *Wiss. Z. Uni. Halle (Math. Nat.)* **12** (1963), 251–257.
 - [2] M.C. Balbuena, A. Carmona, J. Fàbrega and M.A. Fiol, On the order and size of s -geodetic digraphs with given connectivity, *Discrete Math.* **174** (1997), 19–27.
 - [3] H.L. Fu, K.C. Huang and C.A. Rodger, Connectivity of Cages, *J. Graph Theory* **24** (1997), 187–191.
 - [4] T. Jiang and D. Mubayi, Connectivity and Separating Sets of Cages, *J. Graph Theory* **29** (1998), 35–44.
 - [5] M. Daven and C.A. Rodger, (k, g) -cages are 3-connected, *Discrete Math.* **199** (1999), 207–215.

Strategy

G: (δ, g) -cage with $\delta \geq 4$ and $g \geq 5$.

AIM: To prove: Every cage is 4-connected.

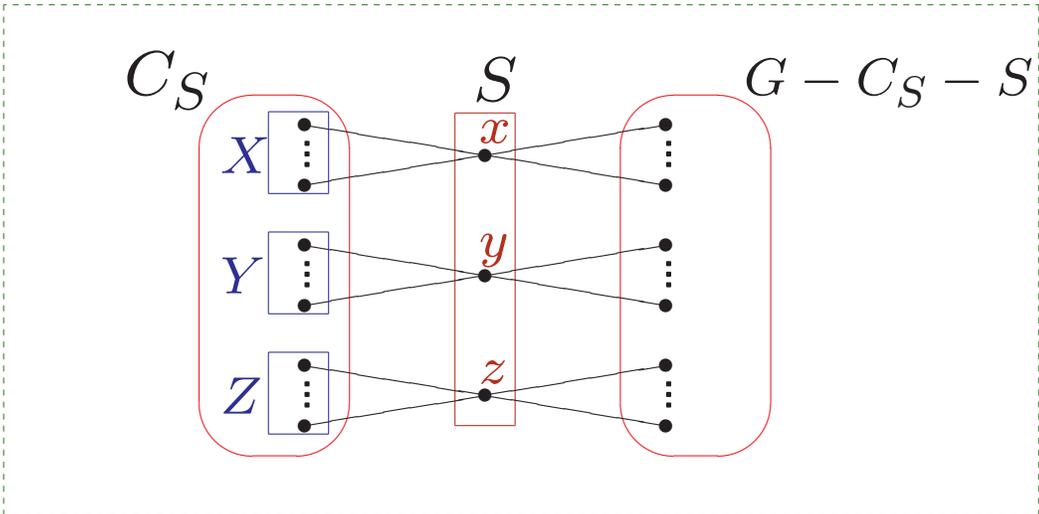
Step 1: Suppose: G has a cutset F of order 3.

AIM: Constructing a new graph G^* from G .

Step 2: $\mathcal{F} = \{F : F \text{ is a cutset of order } 3\}$:

C_F is the smallest component of $G - F$.

Step 3: $S = \{x, y, z\} \in \mathcal{F}$ s.t. C_S is an **atom**:
 $|V(C_S)| \leq |V(C_F)|$ for every $F \in \mathcal{F}$.



Constructing a new graph

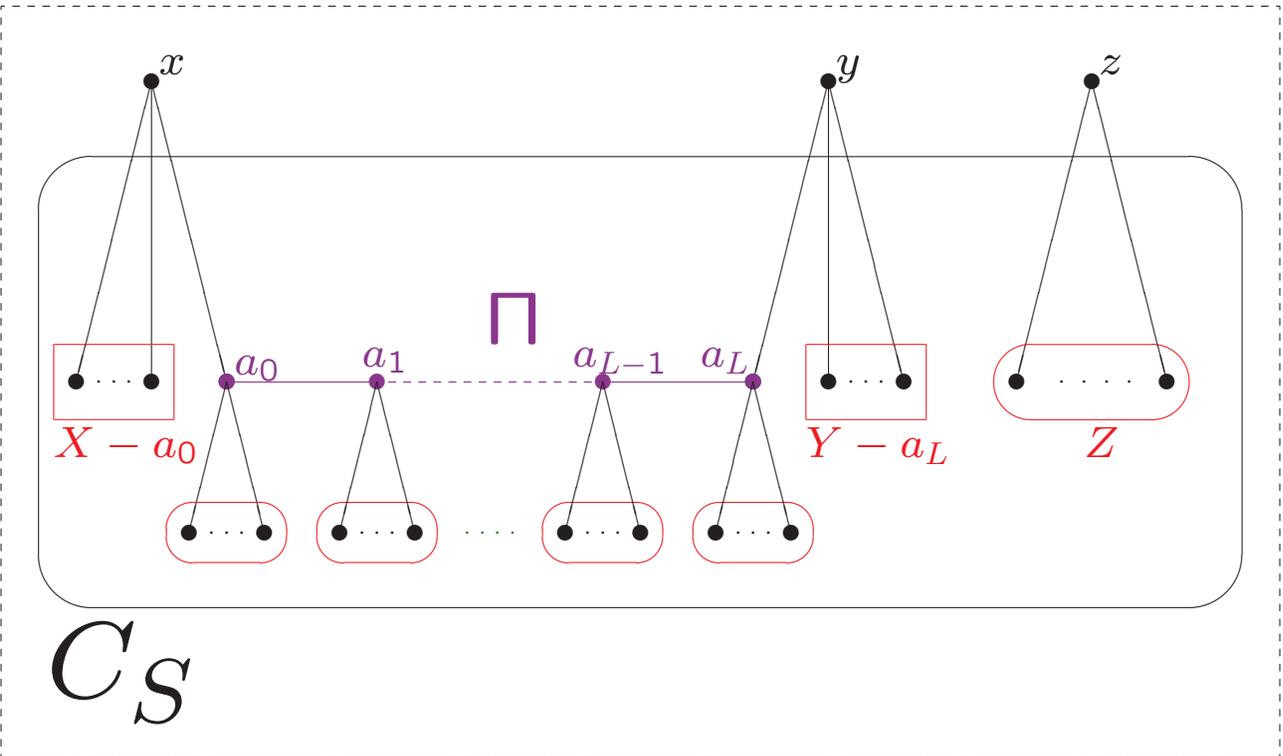
$$X = N_{C_S}(x), Y = N_{C_S}(y), Z = N_{C_S}(z)$$

$$L = \min\{d_{C_S}(X, Y), d_{C_S}(X, Z), d_{C_S}(Y, Z)\}$$

Π : Path of length L joining X and Y

Step 4 : Consider the subgraph: $H = C_S - \Pi$
and:

$$\Omega = \{v \in H : deg_H(v) = \delta - 1\}$$



Constructing a new graph

- If $\delta \geq 4$, $g \geq 10$ then:
 - ★ $3 \leq \ell - 2 \leq L \leq \ell - 1$, where $\ell = \lfloor \frac{g-1}{2} \rfloor$.
 - ★ The "subsets" of Ω are pairwise disjoint.

Step 5: Labelling of vertices in Ω in a "suitable" (and nontrivial!) way.

Step 6: Considering a "certain" one-to-one map $\sigma : \Omega \rightarrow \Omega$.

Step 7: Take a copy H' of H , and define a new graph G^* as follows:

★ $V(G^*) = V(H) \cup V(H')$.

★ $E(G^*) = E(H) \cup E(H') \cup E^+$, where:

$$E^+ = \{w(\sigma(w))' : w \in \Omega\}$$

Obtaining a contradiction

- $|V(G^*)| < |V(G)|$
- If $\delta \geq 4$, $g \geq 10$ then:
 - ★ G^* is a regular graph of degree $\delta = \delta(G)$.
 - ★ The girth of G^* is $g(G^*) \geq g(G) \geq g$.



Since G was supposed to be a cage, we have obtained a contradiction.

Epilogue

We have proved that every

- $(4, g)$ -cage is 4-connected
- (δ, g) -cage, $\delta \geq 5$, $g \geq 10$, is 4-conn.

We should prove (or disprove):

- Every (δ, g) -cage, $5 \leq g \leq 9$, is δ -connected.
- Every (δ, g) -cage is δ -edge-connected.
- Every $(4, g)$ -cage is supercon. (and q. 5-c).
- The diameter of a cage is at most $g - 1$.
- Every $(4, g)$ -cage with g even is bipartite.
- Every (δ, g) -cage is δ -connected.
- Every (δ, g) -cage is superc. (and q. $\delta + 1$ -c).

For any comments or suggestions:

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