Every Cubic Cage is quasi 4-connected

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A *(δ, g)-cage* is a regular graph of degree δ and girth g with the least possible order.

It was proved in [3] (see pg 6):

Every $(3, g)$-cage is 3-connected

The same authors conjectured that:

All (δ, g)-cages are maximally connected

It was independently showed in [4,5] that:

Every (δ, g)-cage, with $\delta \geq 3$ is 3-connected

In this paper, we prove that:

Every $(3, g)$-cage, $g \geq 5$, is quasi 4-connected,

which can be seen as a further step towards the proof of the aforementioned *conjecture*.

Basic Definitions

- A graph $G = (V, E)$ is called regular if all its vertices have the same degree δ.
- A (δ, g)-graph is a regular graph of degree δ and girth g.
- A (δ, g)-cage is a (δ, g)-graph with the least possible number of vertices.
- For every $g \geq 3$, $f(\delta, g)$ denotes the order of a (δ, g)-cage.
- Simply counting the numbers of vertices in the distance partition with respect to a vertex yields the so-called Moore bound *n(δ,g)*, which is a lower bound on the order $f(\delta, g)$ of a cage. For example:

$$f(3, g) \geq n(3, g) = \begin{cases} 3 \cdot 2^{\frac{g-1}{2}} - 2, & \text{if g is odd;} \\ 2 \cdot 2^{\frac{g}{2}} - 2, & \text{if g is even.} \end{cases}$$

- A Moore graph is a (δ, g)-cage s.t. $f(\delta, g) = n(\delta, g)$.
- A cubic cage is a $(3, g)$-cage.
- For every $v \in V$, $N(v)$ denotes the neighbourhood of v, that is, the set of all vertices adjacent to v.

Connectedness Definitions

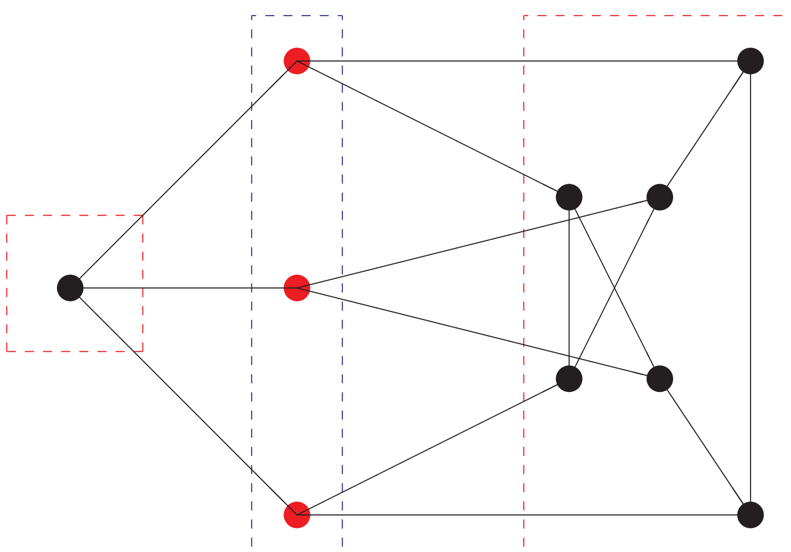
- A graph $G = (V, E)$ is connected if every pair of vertices is joined by a path.
- A subset of vertices $S \subset V$ is said to be a cutset if the graph $G - S$ is not connected.
- G is k-connected if every cutset S has cardinality at least k.
- The connectivity κ of G is the maximum integer k such that G is k-connected.
- G is maximally connected if $\kappa = \delta$.
- A minimum cutset S is called trivial if $S = N(v)$ for some $v \in V$
- A graph G is superconnected if it is maximally connected and every minimum cutset is trivial.
- A $(3, g)$-graph G is quasi 4-connected if it is superconnected and, for every minimum cutset S, the nonconnected graph $G - S$ has precisely two components.
- A $(3, g)$-graph G is cyclically 4-edge-connected if it is 3-edge-connected and, for every minimum edge-cutset S, the nonconnected graph $G - S$ has at most one cyclic component.

First Examples of Cubic Cages

- The complete graph K_4 is the unique $(3,3)$-cage. Certainly, it has no cutsets, and hence we can state that its connectedness is optimal.

- The complete bipartite graph $K_{3,3}$ is the unique $(3,4)$-cage. Observe that it is superconnected, but not quasi 4-connected. In fact, this graph is the only cubic cage which is not quasi 4-connected.

- The Petersen graph P is the unique $(3,5)$-cage. It is easy to see that this graph is not only superconnected but also quasi 4-connected (see figure below).



More Cubic Cages

- The Heawood graph is the unique $(3, 6)$-cage. It is bipartite, has 14 vertices and diameter 3.

- The Mcgee graph is the unique $(3, 7)$-cage. It has 24 vertices and diameter 4.

- The Tutte-Coxeter graph is the unique $(3, 8)$-cage. It is bipartite, has 30 vertices and diameter 4.

- There are 18 (nonisomorphic) $(3, 9)$-cages, all of them having 58 vertices and diameter 6. The first $(3, 9)$-cage was found by Biggs and Hoare.

- There are 3 $(3, 10)$-cages, which have 70 vertices. These graphs where found by O'Keefe and Wong.

- McKay and Myrvold showed that a $(3, 11)$-cage must have 112 vertices. However, although there is only one such graph known (found by Balaban), there may still be others to be found.

- The point/line incidence graph of the generalized hexagon of order 2 is the unique $(3, 12)$-cage. It is bipartite, has 126 vertices and diameter 6.

- For $g \geq 13$, no other graph is sure to be a cubic cage. For more information on cubic cages, see:

⋆ http://www.cs.uwa.edu.au/∼ gordon/data.html

Starting known results

- ([2]) Let G be a connected graph with minimum degree $\delta \geq 3$ and diameter D. Then, G is super-connected if $D \leq 2\lfloor \frac{g-3}{2} \rfloor$.

- ([1]) If $3 \leq g_1 < g_2$, then $f(\delta, g_1) < f(\delta, g_2)$.

⋆ ([4]) Let S be a cutset of a (δ, g)-cage with $\delta \geq 3$ and $g \geq 5$. Then, the diameter of the subgraph of G induced by S is at least $\lfloor \frac{g}{2} \rfloor$. Furthermore, the inequality is strict if $d_{G[S]}(u, v)$ is maximized for exactly one pair of vertices.

- ([4,5]) Every (δ, g)-cage with $\delta \geq 3$ is 3-connected.

- (Conjecture) Every (δ, g)-cage is δ-connected.

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[1] P. Erdös and H. Sachs, Reguläre Graphen gegebener Taillenweite mit minimaler Knotenzahl, *Wiss. Z. Uni. Halle (Math. Nat.)* **12** (1963), 251–257.

[2] J. Fàbrega and M.A. Fiol, Maximally Connected Digraphs, *J. Graph Theory* **13** (1989), 657–668.

[3] H.L. Fu, K.C. Huang and C.A. Rodger, Connectivity of Cages, *J. Graph Theory* **24** (1997), 187–191.

[4] T. Jiang and D. Mubayi, Connectivity and Separating Sets of Cages, *J. Graph Theory* **29** (1998), 35–44.

[5] M. Daven and C.A. Rodger, (k, g)-cages are 3- connected, *Discrete Math.* **199** (1999), 207–215.

New results

Consider the set $\mathcal{F}$ of all nontrivial minimum cutsets of a $(3,g)$-cage G, with $g \geq 6$. Assume $\mathcal{F} \neq \emptyset$.

For every $F \in \mathcal{F}$, let C_F denote a smallest component of $G - F$. Let $S = \{x, y, z\} \in \mathcal{F}$ such that:

$$|V(C_S)| \leq |V(C_F)|, \text{ for every } F \in \mathcal{F}, \qquad (1)$$

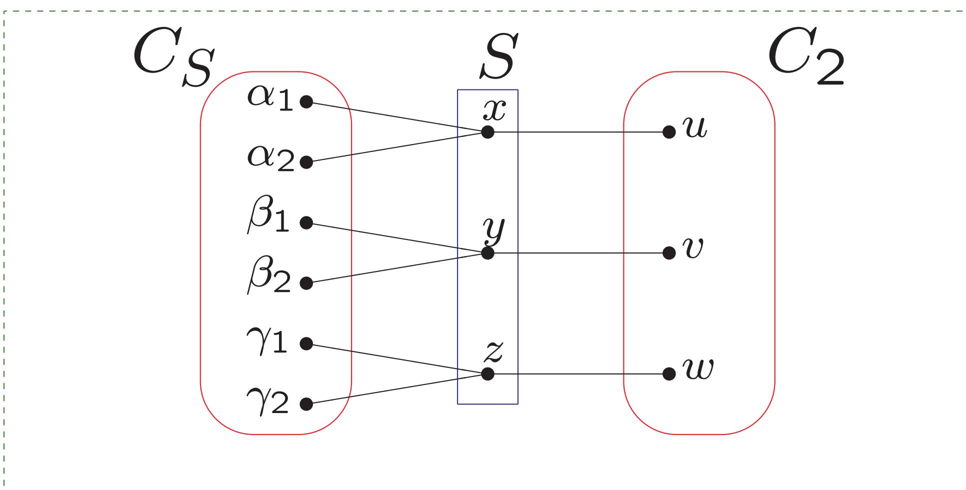
and consider the subsets of C_S:

$$X = N_{C_S}(x),\ Y = N_{C_S}(y), \text{ and } Z = N_{C_S}(z).$$

and: $L = \min\{d_{C_S}(X,Y), d_{C_S}(X,Z), d_{C_S}(Y,Z)\}$.

Lemma: If S satisfies (1), then: $|X| = |Y| = |Z| = 2$.

Moreover: $2|V(C_S)| \leq |V(G)| - 6$, $1 \leq L \leq \lfloor \frac{g-7}{2} \rfloor$. ■



Starting from both the mentioned known results and this lemma, we have obtained the main new results of this work, which are showed next.

Proposition: Every $(3, g)$-cage is superconnected.

Outline of the Proof: For $5 \leq g \leq 9$, this statement is directly derived from either the known result by Fàbrega and Fiol or the lemma. Suppose G is a nonsuperconnected $(3, g)$-cage with $g \geq 10$ and $d_{C_S}(\alpha_1, \beta_1) = L$. Consider the only path Π of length L in C_s joining α_1 to β_1, and the set:

$$\Omega = [N_{C_1}(V(\Pi)) \setminus V(\Pi)] \cup \{\alpha_2, \beta_2\} \cup \{\gamma_1, \gamma_2\},$$

which is proved to have cardinality $L + 5$. Next, take the subgraph $\tilde{C}_S = C_S - \Pi$ and construct a new graph G^* as follows. Let $\tilde{C}'_S$ be a copy of $\tilde{C}_S$ such that $V(\tilde{C}_S) \cap V(\tilde{C}'_S) = \emptyset$, and let φ be an isomorphism between $\tilde{C}_S$ and $\tilde{C}'_S$ such that $\varphi(\gamma_1) = \gamma'_2$, $\varphi(\gamma_2) = \gamma'_1$, and for all $v \in V(\tilde{C}_S) \setminus \{\gamma_1, \gamma_2\}$, $\varphi(v) = v'$. Let G^* be the graph such that:

$$V(G^*) = V(\tilde{C}_S) \cup V(\tilde{C}'_S),\ E(G^*) = E(\tilde{C}_S) \cup E(\tilde{C}'_S) \cup \{w\varphi(w) : w \in \Omega\}$$

To finalize, we prove that G^* is a $(3, g)$-graph. Since $|V(G^*)| < |V(G)|$, the desired contradiction is obtained. ∎

Theorem: Every $(3, g)$-cage with $g \geq 5$ is quasi 4-connected.

Outline of the Proof: Suppose G to be superconnected but not quasi 4-connected. This means that there exists a vertex v such that $N(v)$ is a cutvertex and $G - N(v)$ has three components. This fact allows us to derive that $\{v\} \cup N(v)$ is a cutset such that its induced subgraph $G[\{v\} \cup N(v)]$ has diameter 2. So, we get a contradiction (see page 6, result $\star$). In consequence, every $(3, g)$-cage must be quasi 4-connected. ∎

Epilogue

Quasi 4-connected graphs are so-called because they exhibit many of the properties of 4-connected graphs (see joint work of T. Politof and A. Satyanarayana). For this reason, it is reasonable to put forward this work as a further step towards the proof of the conjecture:

- Every cage is maximally connected

Moreover, this new result can also be considered as a very significant step towards the proof of another known conjecture:

- Every cubic cage is cyclically 4-edge-connected

Apart from this last statement, we are currently trying to prove another partial results related to the first conjecture:

- Every cubic cage is edge-superconnected.
- Every (δ, g)-cage,$\delta \geq 4$, is 4-edge-connected (proved).
- Every $(4, g)$-cage is maximally connected.
- Every $(4, g)$-cage is edge-superconnected.
- The diameter of every (δ, g)-cage is at most $g - 1$.

For any comments or suggestions:

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