Independence and domination in graph convexity spaces

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1 Convexity spaces

- *Convexity space (Levi,1951)*: A convexity space is an ordered pair $(V, \mathcal{C})$, where V is a non-empty set and $\mathcal{C}$ is a collection of V-subsets, to be regarded as convex sets, such that

  (C1) $\emptyset, V \in \mathcal{C}$.

  (C2) Arbitrary intersections of convex sets are convex.

  (C3) Every nested union of convex sets is convex.

- *Convex hull*: Given a convexity space $(V, \mathcal{C})$ and a set $S \subset V$, the smallest convex set $[S]_{\mathcal{C}} \in \mathcal{C}$ containing S is called the convex hull of S.

⊗ Let $(V, \mathcal{C})$ be a convexity space. Then, (C3) ⇔ For every $A \subseteq V$, $[A]_{\mathcal{C}} = \bigcup_{\substack{F \subseteq A \\ F finite}} [F]_{\mathcal{C}}$.

- *Interval convexities*: Let V be a set and $I : V \times V \to 2^V$ be a mapping such that $x, y \in I(x, y)$ for every $x, y \in V$. The I-closed subsets of V are subsets $C \subseteq V$ such that $I(C \times C) = C$. The set $\mathcal{C}_I$ of I-closed subsets satisfies axioms (C1), (C2) and (C3) of convexity spaces. The function I is called an interval function of the convexity space $(V, \mathcal{C}_I)$. Convexity spaces admitting an interval function are named interval convexity spaces.

- *Graph convexity space*: A graph convexity space is an ordered pair $(G, \mathcal{C})$, formed by a connected graph G, with vertex set V, and a convexity $\mathcal{C}$ on V such that $(V, \mathcal{C})$ is a convexity space satisfying the additional axiom:

  (C4) Every member of $\mathcal{C}$ induces a connected subgraph of G.

2 Independent sets and Classical parameters

- *Convexly dependent set*: A subset $A \subset V$ of a convexity space $(V, \mathcal{C})$ is called convexly dependent if $[A]_{\mathcal{C}} = [A - a]_{\mathcal{C}}$ for some $a \in A$.

- *Rank*: The rank r of a convexity space $(V, \mathcal{C})$ is the maximum cardinality of a convexly independent set.

- *Redundant set*: Given a convexity space $(V, \mathcal{C})$, a nonempty set $A \subset V$ is called redundant when it has the property:

$$[A]_{\mathcal{C}} = \bigcup_{a \in A} [A - a]_{\mathcal{C}}$$

- *Carathéodory number*: The Carathéodory number c of a convexity space $(V, \mathcal{C})$ is the maximum cardinality of an irredundant set.

- *Exchange dependent set*: Given a convexity space $(V, \mathcal{C})$ a nonempty set $A \subset V$ is called exchange dependent when, for every $x \in A$, the following holds:

$$[A - x]_{\mathcal{C}} \subseteq \bigcup_{a \in A - x} [A - a]_{\mathcal{C}}$$

- *Exchange number*: The Exchange number e of a convexity space $(V, \mathcal{C})$ is the maximum cardinality of an exchange independent set.

- *Helly dependent set*: Given a convexity space $(V, \mathcal{C})$, a finite set $A \subset V$ is called Helly dependent when it satisfies:

$$\bigcap_{a \in A} [A - a]_{\mathcal{C}} \neq \emptyset$$

- *Helly number*: The Helly number h of a convexity space $(V, \mathcal{C})$ is the maximum cardinality of a Helly independent set.

↬ Let $(V, \mathcal{C})$ be a convexity space with Caratheodory number c, Helly number h, rank r and exchange number e.

⊗ For every $A \subseteq V$, $[A]_{\mathcal{C}} = \bigcup_{\substack{F \subseteq A \\ |F| = c}} [F]_{\mathcal{C}}$.

⊗ $h = \min\{k : \ \{S_i\}_{j=1}^{l} \in \mathcal{P}(\mathcal{C}), \bigcap_{i=1}^{l} S_i = \emptyset \Rightarrow \exists \{S_{i_j}\}_{j=1}^{k} \ s.t. \ \bigcap_{j=1}^{k} S_{i_j} = \emptyset\}$.

⊗ Every convexly dependent set is both redundant and Helly dependent. Hence, $\max\{c, h\} \leq r$.

⊗ Both convex and Helly independence are hereditary properties.

⊗ Every set of cardinality at least $c + 2$ is exchange dependent. Hence, $e - 1 \leq c$.

⊗ $c \leq \max\{h, e - 1\}$.

⊗ If $(V, \mathcal{C})$ is an interval convexity space, then $c \leq e$.

↬ Let $G \square H$ the cartesian product of two graphs G and H. In all cases, we consider the geodesic convexity.

⊗ $r(G) + r(H) - 2 \leq r(G \times H) \leq r(G) + r(H)$

⊗ $h(G \square H) = \max\{h(G), h(H)\}$

⊗ $e(G \square H) = e(G) + e(H) - 1$

⊗ $c(G \square H) = e(G \square H) - 1$

3 Domination parameters

- *Closed interval/Geodetic closure*: For vertices u and v in a graph G, the closed interval $I[u, v]$ consists of u and v together with all vertices lying in a $u - v$ geodesic. For $S \subset V(G)$, $I[S]$ is the union of all closed intervals $I[u, v]$ with $u, v \in S$, and it is called the geodetic closure of S.

- *Geodetic number*: A set S of vertices of G for which $I[S] = V(G)$ is called a geodetic set of G. The geodetic number $gn(G)$ is the minimum cardinality of a geodetic set.

- *Monophonic interval/Monophonic closure*: For vertices u and v in a graph G, the (closed) monophonic interval $J[u, v]$ consists of u and v together with all vertices lying in a $u - v$ induced path. For $S \subset V(G)$, $J[S]$ is the union of all monophonic intervals $J[u, v]$ with $u, v \in S$, and it is called the monophonic closure of S.

- *Monophonic number*: A set S of vertices of G for which $J[S] = V(G)$ is called a monophonic set of G. The monophonic number $mn(G)$ is the minimum cardinality of a monophonic set.

- *Hull numbers*: A set S of vertices of G is a g-hull set if $[S] = V(G)$. A g-hull set of G of minimum cardinality is a minimum g-hull set and its cardinality is the g-hull number $ghn(G)$. Similarly is defined the m-hull number $mhn(G)$.

⊗ For any integers a, b, c, d such that $3 \leq a \leq b \leq c \leq d$, there exists a connected graph G such that:

(a) $a = mhn(G)$, $b = mn(G)$, $c = ghn(G)$, and $d = gn(G)$,
(b) $a = mhn(G)$, $b = ghn(G)$, $c = mn(G)$, and $d = gn(G)$.

⊗ $ghn(G \square H) = \max\{ghn(G), ghn(H)\}$

⊗ $\max\{gn(G), gn(H)\} \leq gn(G \square H) \leq gn(G) \cdot gn(H) - \min\{gn(G), gn(H)\}$ (both bounds are sharp)

4 Steiner sets

- *Steiner tree/Steiner interval*: For a connected graph G of order $n \geq 3$ and a set $W \subset V(G)$, a tree T contained in G is a Steiner tree with respect to W if T is a tree of minimum order with $W \subset V(T)$. The Steiner interval $S(W)$ of a vertex set W consists of all vertices in G that lie on some Steiner tree with respect to W.

- *Steiner number*: A vertex set W is a Steiner set of G if $S(W) = V(G)$. The minimum cardinality among the Steiner sets of G is the Steiner number $st(G)$.

- *Edge Steiner number*: The set $S_e(W)$ of a vertex set W consists of all edges in G that lie on some Steiner tree with respect to W. The set W is an edge Steiner set of G if $S_e(W) = E(G)$. The minimum cardinality among the edge Steiner sets of G is the edge Steiner number $st_e(G)$.

⊗ Every Steiner set is monophonic.

⊗ Every edge Steiner set is edge monophonic.

⊗ In the class of interval graphs, every (edge) Steiner set is (edge) geodetic.

⊗ For every triple a, b, c of integers with $3 \leq a \leq b \leq c$, there exists a connected graph G such that:

(a) $hn(G) = a$, $gn(G) = b$, $st(G) = c$
(b) $hn(G) = a$, $st(G) = b$, $gn(G) = c$
(c) $st(G) = a$, $hn(G) = b$, $gn(G) = c$

5 Boundary vertices

- *Extreme set*: A vertex v of a graph G is an extreme vertex if $V - v$ is convex. The extreme set $Ext(G)$ is the set of all extreme vertices of G.

⊗ The geodesic convexity of a graph is a convex geometry (*every convex set of G is the convex hull of its extreme vertices*) if and only if G is Ptolemaic.

⊗ The monophonic convexity of a graph is a convex geometry iff G is chordal.

- *Eccentricity*: The eccentricity of a vertex $u \in V(G)$ is defined as $ecc(u) = max\{d(u,v) : v \in V\}$.

- *Eccentric set*: A vertex $v \in V(G)$ is called an eccentric vertex of G if there exists another vertex $u \in V(G)$ such that $d(u,v) = ecc(u)$. The eccentric set $Ecc(G)$ of G is the set of all its eccentric vertices.

- *Contour*: A vertex v in G is a contour vertex if $ecc(v) \geq ecc(u)$, for all $u \in N(v)$. The contour of a graph G, denoted $Ct(G)$, is the set of all its contour vertices.

⊗ The contour of every graph is a g-hull set.

⊗ The contour of every graph is monophonic.

⊗ If the contour of a graph is equal to its periphery, then it is geodetic.

⊗ The contour of every chordal graph is geodetic.

- *Periphery*: A vertex $v \in V(G)$ is called a *peripheral vertex* of G if no vertex in $V(G)$ has an eccentricity greater than $ecc(v)$, that is, if the eccentricity of v is equal to the diameter of G. The periphery $Per(G)$ of G is the set of all of its peripheral vertices.

- *Boundary*: Let u, v be vertices in $V(G)$. The vertex v is said to be a *boundary vertex* of u if no neighbor of v is further away from u than v. By $\partial(u)$ we denote the set of all boundary vertices of u. A vertex v is called a *boundary vertex of* G if $v \in \partial(u)$ for some vertex $u \in V(G)$. The boundary $\partial(G)$ of G is the set of all its boundary vertices.

⊗ The boundary of every graph G is edge geodetic, i.e., every edge of G lies on some shortest path joining two vertices of $\partial(G)$.

⊗ Let G be a connected graph and $u \in V(G)$. Then $S = \{u\} \cup \partial(u)$ is a geodetic set. Moreover, if G is bipartite, then $S = \{u\} \cup \partial(u)$ is edge-geodetic.

⊗ Let $(a,b,c,d) \in \mathbb{Z}^4$ satisfying $2 \leq a \leq b, c \leq d, (a,b,c,d) \neq (2,2,2,3), (a,b,c,d) \neq (a,b,a+1,a+1)$. Then there exists a connected graph G such that

$$|Per(G)| = a, \quad |Ct(G)| = b, \quad |Ecc(G)| = c, \quad \text{and} \quad |\partial(G)| = d.$$

⊗ Let G be a connected graph with $|Ext(G)| = a$, $|Per(G)| = b$, $|Ct(G)| = c$, and $|\partial(G)| = d$. Then

(a) $0 \leq a \leq c \leq d$, and $2 \leq b \leq c \leq d$,

(b) $(a,b,c,d) \neq (a,2,2,3)$ for any value of a,

(c) $(a,b,c,d) \neq (2,2,3,3)$,

(d) $(a,b,c,d) \neq (2,3,3,3)$.

⊗ Let $(a,b,c,d) \in \mathbb{Z}^4$ be integers satisfying the above constraints. If $a \geq 2$, then there exists a connected graph G such that $|Ext(G)| = a$, $|Per(G)| = b$, $|Ct(G)| = c$, and $|\partial(G)| = d$.