Locating Partitions in Graphs [1]

Ignacio M. Pelayo[2]

[2]Universitat Politècnica de Catalunya, Barcelona, Spain

7th European Congress of Mathematics

Berlin, Deutschland

[1]Joint work with **Carmen Hernando** and **Mercè Mora**.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex u not in D has **at least** a neighbor in D, i.e., $\boxed{N(u) \cap D \neq \emptyset}$.

▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex u not in D has **at least** a neighbor in D, i.e., $\boxed{N(u) \cap D \neq \emptyset}$.

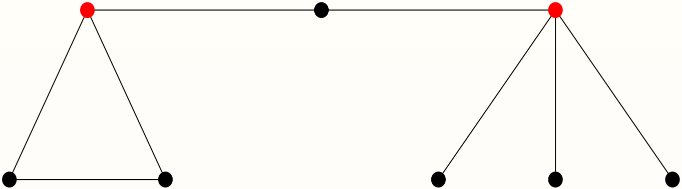
▷ The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

- A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex u not in D has **at least** a neighbor in D, i.e., $\boxed{N(u) \cap D \neq \emptyset}$.
- The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.
- A dominating set of cardinality $\gamma(G)$ is called a γ*-code*.

- A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex u not in D has **at least** a neighbor in D, i.e., $\boxed{N(u) \cap D \neq \emptyset}$.

- The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

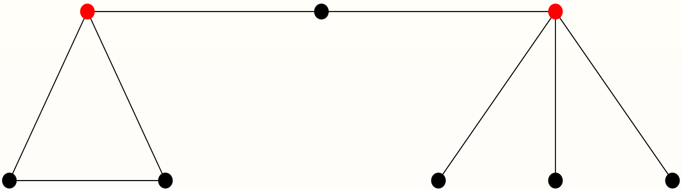
- A dominating set of cardinality $\gamma(G)$ is called a γ*-code*.



▷ A set $D \subset V(G)$ of a graph G is a *dominating set* if every vertex u not in D has **at least** a neighbor in D, i.e., $\boxed{N(u) \cap D \neq \emptyset}$.

▷ The *domination number* of G, denoted by $\gamma(G)$, is the minimum cardinality of a dominating set of G.

▷ A dominating set of cardinality $\gamma(G)$ is called a γ*-code*.



- $\boxed{\gamma(G) = 2}$ (red vertices form a γ-code).

[2] ML-set for short.

▷ A set $S \subset V(G)$ of a graph G is a *metric-locating set* [2] if for every pair $v, w \in V$, $\boxed{d(x,v) \neq d(x,w)}$, for some vertex $x \in S$.

[2] ML-set for short.

- A set $S \subset V(G)$ of a graph G is a *metric-locating set* [2] if for every pair $v, w \in V$, $\boxed{d(x,v) \neq d(x,w)}$, for some vertex $x \in S$.

- The *metric dimension* of G, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of G.

[2] ML-set for short.

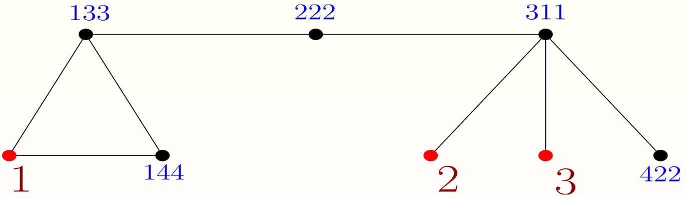
- A set $S \subset V(G)$ of a graph G is a *metric-locating set* [2] if for every pair $v, w \in V$, $\boxed{d(x,v) \neq d(x,w)}$, for some vertex $x \in S$.
- The *metric dimension* of G, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of G.
- A locating set of cardinality $\beta(G)$ is called a *metric basis*.

[2] ML-set for short.

- A set $S \subset V(G)$ of a graph G is a *metric-locating set*[2] if for every pair $v, w \in V$, $\boxed{d(x,v) \neq d(x,w)}$, for some vertex $x \in S$.

- The *metric dimension* of G, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of G.

- A locating set of cardinality $\beta(G)$ is called a *metric basis*.

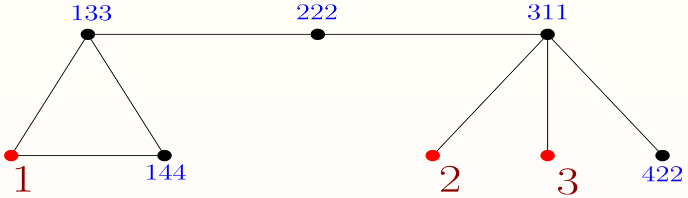


[2] ML-set for short.

- A set $S \subset V(G)$ of a graph G is a *metric-locating set* [2] if for every pair $v, w \in V$, $\boxed{d(x, v) \neq d(x, w)}$, for some vertex $x \in S$.

- The *metric dimension* of G, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of G.

- A locating set of cardinality $\beta(G)$ is called a *metric basis*.



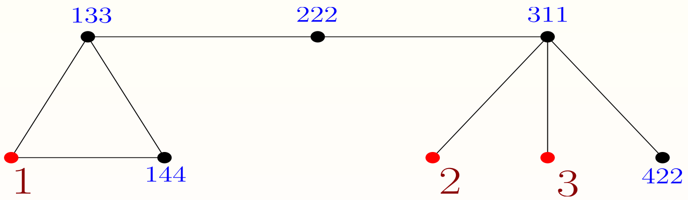
- $\boxed{\beta(G) = 3}$ ($S = \{1, 2, 3\}$ is a metric basis).

[2] ML-set for short.

▷ A set $S \subset V(G)$ of a graph G is a *metric-locating set* [2] if for every pair $v, w \in V$, $\boxed{d(x,v) \neq d(x,w)}$, for some vertex $x \in S$.

▷ The *metric dimension* of G, denoted by $\beta(G)$, is the minimum cardinality of an ML-set of G.

▷ A locating set of cardinality $\beta(G)$ is called a *metric basis*.



- $\boxed{\beta(G) = 3}$ ($S = \{1, 2, 3\}$ is a metric basis).
- Note that vertices 222 and 422 are not dominated by S.

[2] ML-set for short.

[3] MLD-set for short.

- A set $S \subset V$ is an *metric-locating-dominating set* [3] if it is both **dominating** and **locating**.

[3] MLD-set for short.

- A set $S \subset V$ is an *metric-locating-dominating set* [3] if it is both **dominating** and **locating**.
- The *MLD number* $\eta(G)$ is the minimum cardinality of an MLD-set.

[3] MLD-set for short.

- A set $S \subset V$ is an *metric-locating-dominating set* [3] if it is both **dominating** and **locating**.

- The *MLD number* $\eta(G)$ is the minimum cardinality of an MLD-set.

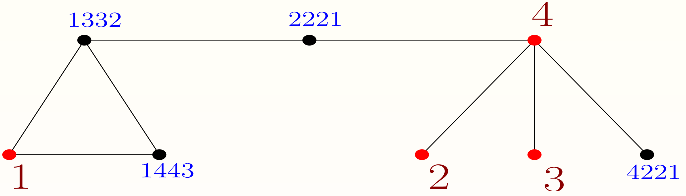
- An MLD-set of cardinality $\eta(G)$ is called an η-*code*.

[3] MLD-set for short.

- A set $S \subset V$ is an *metric-locating-dominating set*[3] if it is both **dominating** and **locating**.

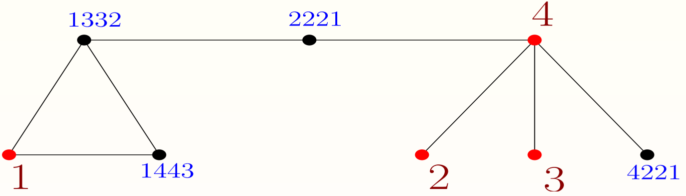
- The *MLD number* $\eta(G)$ is the minimum cardinality of an MLD-set.

- An MLD-set of cardinality $\eta(G)$ is called an η-*code*.



[3] MLD-set for short.

- ▷ A set $S \subset V$ is an *metric-locating-dominating set* [3] if it is both **dominating** and **locating**.
- ▷ The *MLD number* $\eta(G)$ is the minimum cardinality of an MLD-set.
- ▷ An MLD-set of cardinality $\eta(G)$ is called an η-*code*.



- $\boxed{\eta(G) = 4}$ ($S = \{1, 2, 3, 4\}$ is an η-code).

[3] MLD-set for short.

[4] NLD-set for short.

▷ A set $S \subset V$ is a *neighbor-locating-dominating number* [4] if for every two vertices $u, v \in V \setminus S$,

$$\boxed{\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset}$$

[4] NLD-set for short.

▷ A set $S \subset V$ is a *neighbor-locating-dominating number* [4] if for every two vertices $u, v \in V \setminus S$,

$$\boxed{\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset}$$

▷ The *NLD number* $\lambda(G)$ is the minimum cardinality of an NLD-set.

[4] NLD-set for short.

▷ A set $S \subset V$ is a *neighbor-locating-dominating number* [4] if for every two vertices $u, v \in V \setminus S$,

$$\boxed{\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset}$$

▷ The *NLD number* $\lambda(G)$ is the minimum cardinality of an NLD-set.

▷ An NLD-set of cardinality $\lambda(G)$ is called a λ-*code*.

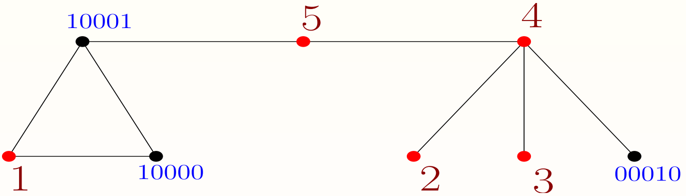
[4] NLD-set for short.

- A set $S \subset V$ is a *neighbor-locating-dominating number*[4] if for every two vertices $u, v \in V \setminus S$,

$$\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset$$

- The *NLD number* $\lambda(G)$ is the minimum cardinality of an NLD-set.

- An NLD-set of cardinality $\lambda(G)$ is called a λ-*code*.



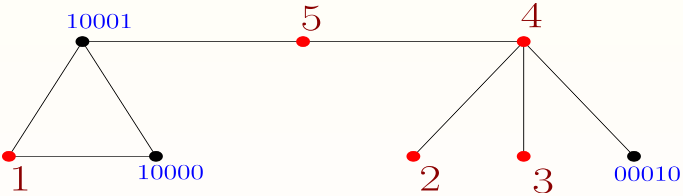
[4] NLD-set for short.

▷ A set $S \subset V$ is a *neighbor-locating-dominating number* [4] if for every two vertices $u, v \in V \setminus S$,

$$\boxed{\emptyset \neq N(u) \cap S \neq N(v) \cap S \neq \emptyset}$$

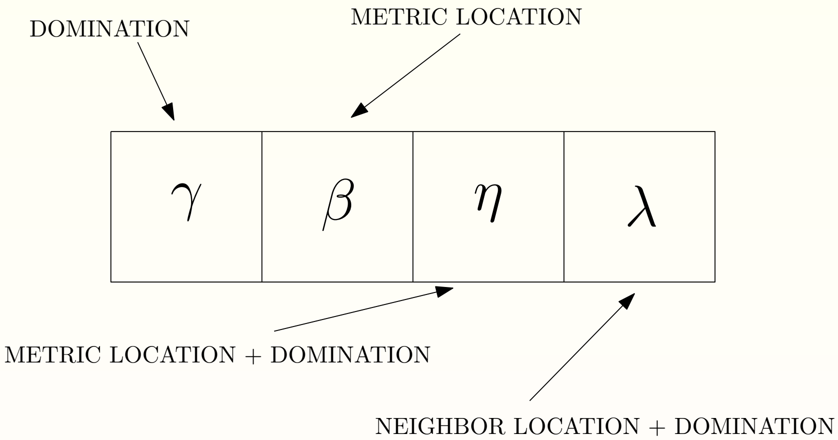
▷ The *NLD number* $\lambda(G)$ is the minimum cardinality of an NLD-set.

▷ An NLD-set of cardinality $\lambda(G)$ is called a λ-*code*.



- $\boxed{\lambda(G) = 5}$ ($S = \{1, 2, 3, 4, 5\}$ is a λ-code).

[4] NLD-set for short.



$$\max\{\gamma, \beta\} \leq \eta \leq \min\{\gamma + \beta, \lambda\}$$

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V *dominates* G if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$,

$$\boxed{d(v, S_j) = 1}$$

- ▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ of V *dominates* G if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$,

$$\boxed{d(v, S_j) = 1}$$

- ▷ The *partition domination number* $\gamma_p(G)$ is the minimum cardinality of a dominating partition of G.

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V *dominates* G if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$,

$$\boxed{d(v, S_j) = 1}$$

- The *partition domination number* $\gamma_p(G)$ is the minimum cardinality of a dominating partition of G.

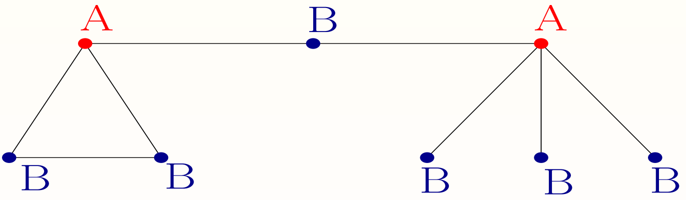
- A dominating partition of cardinality $\gamma_p(G)$ is called a γ_p*-partition*.

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V *dominates* G if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$,

$$\boxed{d(v, S_j) = 1}$$

- The *partition domination number* $\gamma_p(G)$ is the minimum cardinality of a dominating partition of G.

- A dominating partition of cardinality $\gamma_p(G)$ is called a γ_p*-partition*.

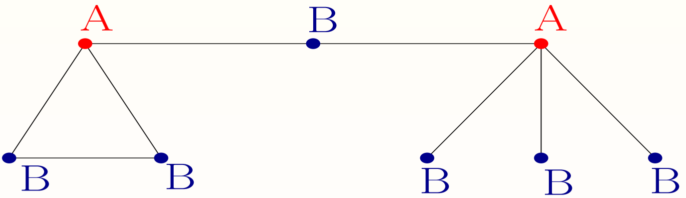


- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V *dominates* G if, for every $i \in \{1, \ldots, k\}$, for every vertex $v \in S_i$ and for some $j \in \{1, \ldots, k\}$,

$$d(v, S_j) = 1$$

- The *partition domination number* $\gamma_p(G)$ is the minimum cardinality of a dominating partition of G.

- A dominating partition of cardinality $\gamma_p(G)$ is called a γ_p*-partition*.



- $\gamma_p(G) = 2$ (TRUE for every graph).

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is an *ML-partition* of G if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$\boxed{d(u, S_j) \neq d(v, S_j)}$$

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is an *ML-partition* of G if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) \neq d(v, S_j)$$

- The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of G.

- ▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is an *ML-partition* of G if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) \neq d(v, S_j)$$

- ▷ The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of G.

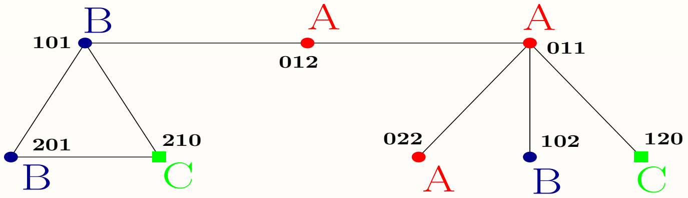
- $\tau(G) \leq \beta_p(G) \leq \beta(G) + 1$ ($\tau(G)$ is the twin number of G).

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is an *ML-partition* of G if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) \neq d(v, S_j)$$

- The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of G.

- $\tau(G) \leq \beta_p(G) \leq \beta(G) + 1$ ($\tau(G)$ is the twin number of G).

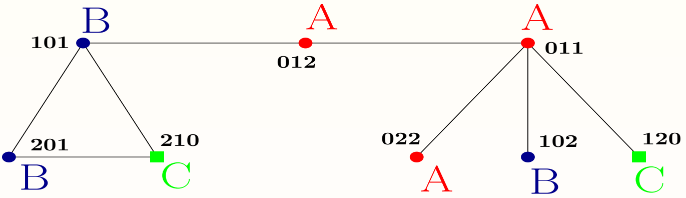


- A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is an *ML-partition* of G if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) \neq d(v, S_j)$$

- The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of G.

- $\tau(G) \leq \beta_p(G) \leq \beta(G) + 1$ ($\tau(G)$ is the twin number of G).



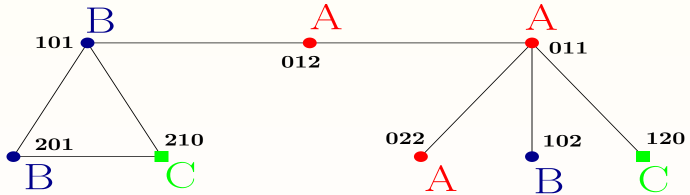
- $\beta_p(G) = \tau(G) = 3$ ($\Pi = \{A, B, C\}$ is a β_p-partition).

▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ of V is an *ML-partition* of G if, for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) \neq d(v, S_j)$$

▷ The *partition dimension* $\beta_p(G)$ is the minimum cardinality of a metric-locating partition of G.

- $\tau(G) \leq \beta_p(G) \leq \beta(G) + 1$ ($\tau(G)$ is the twin number of G).



- $\beta_p(G) = \tau(G) = 3$ ($\Pi = \{A, B, C\}$ is a β_p-partition).

→ Π is not dominating (**022** is an <u>internal</u> vertex of part A).

- A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *MLD-partition* of G if it is both a **dominating** and a **metric-locating** partition of G.

- ▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *MLD-partition* of G if it is both a **dominating** and a **metric-locating** partition of G.

- ▷ The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of an MLD-partition of G.

- A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *MLD-partition* of G if it is both a **dominating** and a **metric-locating** partition of G.

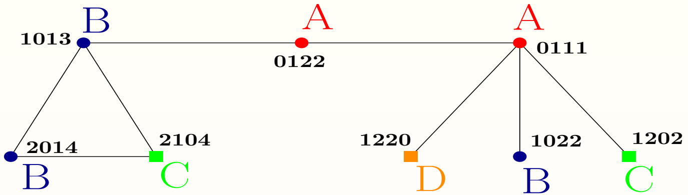
- The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of an MLD-partition of G.

- $\boxed{\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1}$

▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *MLD-partition* of G if it is both a **dominating** and a **metric-locating** partition of G.

▷ The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of an MLD-partition of G.

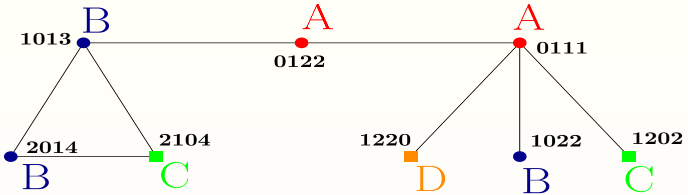
- $\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1$



- A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *MLD-partition* of G if it is both a **dominating** and a **metric-locating** partition of G.

- The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of an MLD-partition of G.

- $\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1$

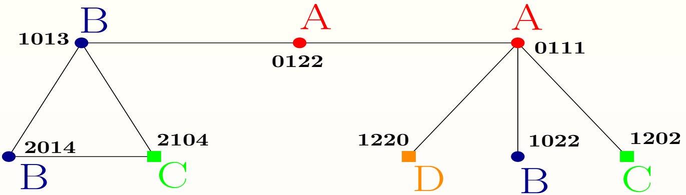


- $\eta_p(G) = 4$ ($\{A, B, C, D\}$ is an η_p-partition).

- ▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *MLD-partition* of G if it is both a **dominating** and a **metric-locating** partition of G.

- ▷ The *partition metric-location-domination number* $\eta_p(G)$ is the minimum cardinality of an MLD-partition of G.

- $\beta_p(G) \leq \eta_p(G) \leq \eta(G) + 1$



- $\eta_p(G) = 4$ ($\{A, B, C, D\}$ is an η_p-partition).

- $\eta_p(G) \leq \beta_p(G) + 1$

- A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *NLD-partition* of G if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) = 1 \text{ and } d(v, S_j) > 1$$

- A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *NLD-partition* of G if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) = 1 \text{ and } d(v, S_j) > 1$$

- The *partition neighbor-location-domination number* $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of G.

▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *NLD-partition* of G if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) = 1 \text{ and } d(v, S_j) > 1$$

▷ The *partition neighbor-location-domination number* $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of G.

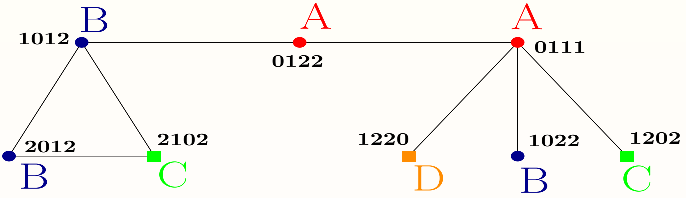
- $\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$

- A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *NLD-partition* of G if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) = 1 \text{ and } d(v, S_j) > 1$$

- The *partition neighbor-location-domination number* $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of G.

- $\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$

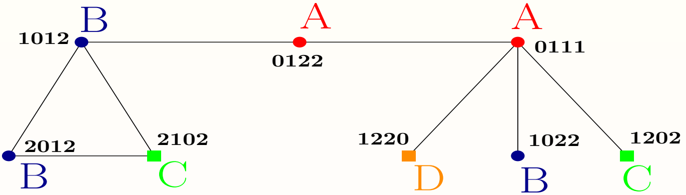


- ▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *NLD-partition* of G if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) = 1 \text{ and } d(v, S_j) > 1$$

- ▷ The *partition neighbor-location-domination number* $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of G.

- $\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$



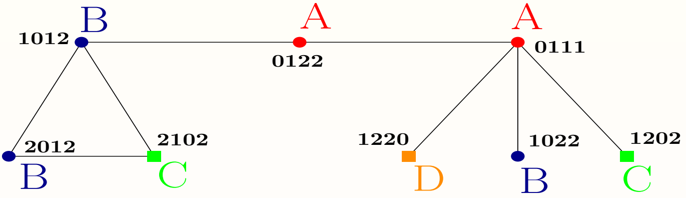
- $\lambda_p(G) = 4$, ($\{A, B, C, D\}$ is a λ_p-partition).

- ▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ is an *NLD-partition* of G if for every $i \in \{1, \ldots, k\}$, every pair $u, v \in S_i$ and some $j \in \{1, \ldots, k\}$,

$$d(u, S_j) = 1 \text{ and } d(v, S_j) > 1$$

- ▷ The *partition neighbor-location-domination number* $\lambda_p(G)$ is the minimum cardinality of an NLD-partition of G.

- $\beta_p(G) \leq \eta_p(G) \leq \lambda_p(G) \leq \lambda(G) + 1$



- $\lambda_p(G) = 4$, ($\{A, B, C, D\}$ is a λ_p-partition).
- $\beta_p(P_{10}) = 2$, $\eta_p(P_{10}) = 3$, $\lambda_p(P_{10}) = 4$

$\gamma \leq \eta$

$\gamma_p = 2$

$$\begin{array}{ccccc}
 & & \beta_p + 1 & & \\
 & & \rotatebox{90}{$\geq$} & & \\
\beta_p & \leq & \eta_p & \leq & \lambda_p \\
\rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} & & \rotatebox{90}{$\leq$} \\
\beta + 1 & \leq & \eta + 1 & \leq & \lambda + 1 \\
 & & \rotatebox{90}{$\leq$} & & \\
 & & \gamma + \beta + 1 & &
\end{array}$$

- $\gamma_p = 2$

- $\gamma_p = 2$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\boxed{\gamma_p = 2}$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\boxed{\beta_p \leq \beta + 1}$

- $\gamma_p = 2$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\beta_p \leq \beta + 1$

▷ $S = \{v_1, \ldots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \ldots \{v_k\}, V \setminus S\}$ ML-partition.

- $\boxed{\gamma_p = 2}$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\boxed{\beta_p \leq \beta + 1}$

▷ $S = \{v_1, \dots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \dots \{v_k\}, V \setminus S\}$ ML-partition.

- $\boxed{\eta_p \leq \beta_p + 1}$

- $\gamma_p = 2$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\beta_p \leq \beta + 1$

▷ $S = \{v_1, \ldots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \ldots \{v_k\}, V \setminus S\}$ ML-partition.

- $\eta_p \leq \beta_p + 1$

▷ $\Pi = \{S_1, \ldots, S_k\}$ ML-partition.

- $\gamma_p = 2$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\beta_p \leq \beta + 1$

▷ $S = \{v_1, \ldots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \ldots \{v_k\}, V \setminus S\}$ ML-partition.

- $\eta_p \leq \beta_p + 1$

▷ $\Pi = \{S_1, \ldots, S_k\}$ ML-partition.

▷ $1 \leq i \leq k$: $B_i = \{w \in S_i : N(w) \subseteq S_i\}$ (internal vertices of S_i).

- $\boxed{\gamma_p = 2}$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\boxed{\beta_p \leq \beta + 1}$

▷ $S = \{v_1, \ldots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \ldots \{v_k\}, V \setminus S\}$ ML-partition.

- $\boxed{\eta_p \leq \beta_p + 1}$

▷ $\Pi = \{S_1, \ldots, S_k\}$ ML-partition.

▷ $1 \leq i \leq k$: $B_i = \{w \in S_i : N(w) \subseteq S_i\}$ (internal vertices of S_i).

▷ $1 \leq i \leq k$: $C_i = \{w \in B_i : d(w, V \setminus S)$ is even$\}$.

- $\boxed{\gamma_p = 2}$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\boxed{\beta_p \leq \beta + 1}$

▷ $S = \{v_1, \ldots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \ldots \{v_k\}, V \setminus S\}$ ML-partition.

- $\boxed{\eta_p \leq \beta_p + 1}$

▷ $\Pi = \{S_1, \ldots, S_k\}$ ML-partition.

▷ $1 \leq i \leq k$: $B_i = \{w \in S_i : N(w) \subseteq S_i\}$ (internal vertices of S_i).

▷ $1 \leq i \leq k$: $C_i = \{w \in B_i : d(w, V \setminus S)$ is even$\}$.

▷ $D = C_1 \cup \ldots \cup C_k$.

- $\gamma_p = 2$

▷ S γ-code $\Rightarrow \{S, V \setminus S\}$ γ_p-partition.

- $\beta_p \leq \beta + 1$

▷ $S = \{v_1, \ldots, v_k\}$ ML-set $\Rightarrow \{\{v_1\}, \ldots \{v_k\}, V \setminus S\}$ ML-partition.

- $\eta_p \leq \beta_p + 1$

▷ $\Pi = \{S_1, \ldots, S_k\}$ ML-partition.

▷ $1 \leq i \leq k$: $B_i = \{w \in S_i : N(w) \subseteq S_i\}$ (internal vertices of S_i).

▷ $1 \leq i \leq k$: $C_i = \{w \in B_i : d(w, V \setminus S)$ is even$\}$.

▷ $D = C_1 \cup \ldots \cup C_k$.

▷ $\Pi' = \{S_1, \ldots, S_k, D\}$ MLD-partition.

- $\beta_p = n$: K_n.

- $\beta_p = n$: K_n.
- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

- $\beta_p = n$: K_n.
- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

- $\beta_p = n-1$: $K_{1,n-1}$, $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.

- $\beta_p = n$: K_n.
- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

- $\beta_p = n-1$: $K_{1,n-1}$, $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.
- $\eta_p = n-1 \Leftrightarrow \lambda_p = n-1$: $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.

- $\beta_p = n$: K_n.
- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

- $\beta_p = n-1$: $K_{1,n-1}$, $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.
- $\eta_p = n-1 \Leftrightarrow \lambda_p = n-1$: $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.

- $\beta_p = n-2$: $\{H_i\}_{i=1}^{15}$.

- $\beta_p = n$: K_n.
- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

- $\beta_p = n-1$: $K_{1,n-1}$, $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.
- $\eta_p = n-1 \Leftrightarrow \lambda_p = n-1$: $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.

- $\beta_p = n-2$: $\{H_i\}_{i=1}^{15}$.

▷ **Discrete Math., 308 (2008), 5026–5031.** (wrong result: 22 families)

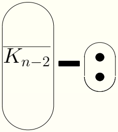
- $\beta_p = n$: K_n.
- $\eta_p = n \Leftrightarrow \lambda_p = n$: K_n, $K_{1,n-1}$.

- $\beta_p = n-1$: $K_{1,n-1}$, $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.
- $\eta_p = n-1 \Leftrightarrow \lambda_p = n-1$: $K_{n-2} \vee \overline{K_2}$, $K_1 \vee (K_1 + K_{n-2})$.

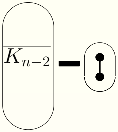
- $\beta_p = n-2$: $\{H_i\}_{i=1}^{15}$.

▷ **Discrete Math., 308 (2008), 5026–5031.** (wrong result: 22 families)

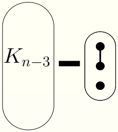
- $\eta_p = n-2 \Leftrightarrow \lambda_p = n-2$: $\{H_i\}_{i=1}^{17}$.



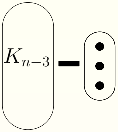
$H_1 \cong K_{2,n-2}$



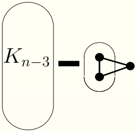
$H_2 \cong \overline{K_{n-2}} \vee K_2$



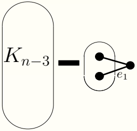
$H_3 \cong K_{n-3} \vee (K_2 + K_1)$



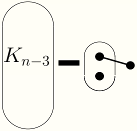
$H_4 \cong K_{n-3} \vee \overline{K_3}$



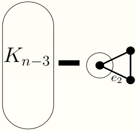
$H_5 \cong (K_{n-3} + K_1) \vee K_2$



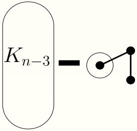
$H_6 \cong (K_{n-3} + K_1) \vee \overline{K_2}$



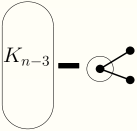
$H_7 \cong H_6 - e_1$



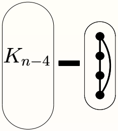
$H_8 \cong (K_{n-3} + K_2) \vee K_1$



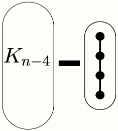
$H_9 \cong H_8 - e_2$



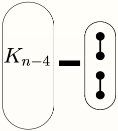
$H_{10} \cong (K_{n-3} + \overline{K_2}) \vee K_1$



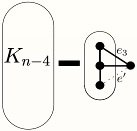
$H_{11} \cong K_{n-4} \vee C_4$



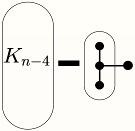
$H_{12} \cong K_{n-4} \vee P_4$



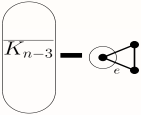
$H_{13} \cong K_{n-4} \vee 2K_2$



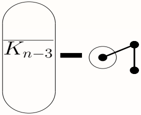
$H_{14} \cong (K_{n-4} + K_1) \vee P_3 - e'$



$H_{15} \cong H_{14} - e_3$



$H_{16} \cong K_1 \vee (\overline{K_{n-3}} + K_2)$



$H_{17} \cong H_{16} - e$

[5] also called *stable partition*.

▷ A partition $\Pi = \{S_1, \ldots, S_k\}$ of a graph G is a *coloring partition* [5] of G, if for every $i \in \{1, \ldots, k\}$, $\boxed{G[S_i] = \overline{K_{|S_i|}}}$, i.e., if S_i is an independent set.

[5] also called *stable partition*.

- A partition $\Pi = \{S_1, \ldots, S_k\}$ of a graph G is a *coloring partition* [5] of G, if for every $i \in \{1, \ldots, k\}$, $\boxed{G[S_i] = \overline{K_{|S_i|}}}$, i.e., if S_i is an independent set.

- Every part is said to be a *color*.

[5] also called *stable partition*.

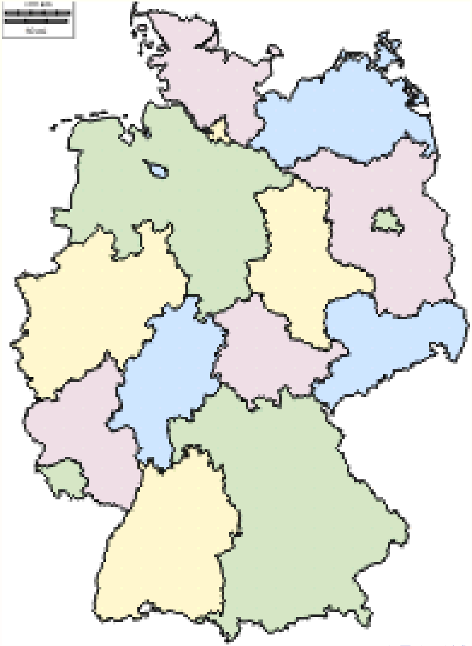
- A partition $\Pi = \{S_1, \ldots, S_k\}$ of a graph G is a *coloring partition* [5] of G, if for every $i \in \{1, \ldots, k\}$, $\boxed{G[S_i] = \overline{K_{|S_i|}}}$, i.e., if S_i is an independent set.

- Every part is said to be a *color*.

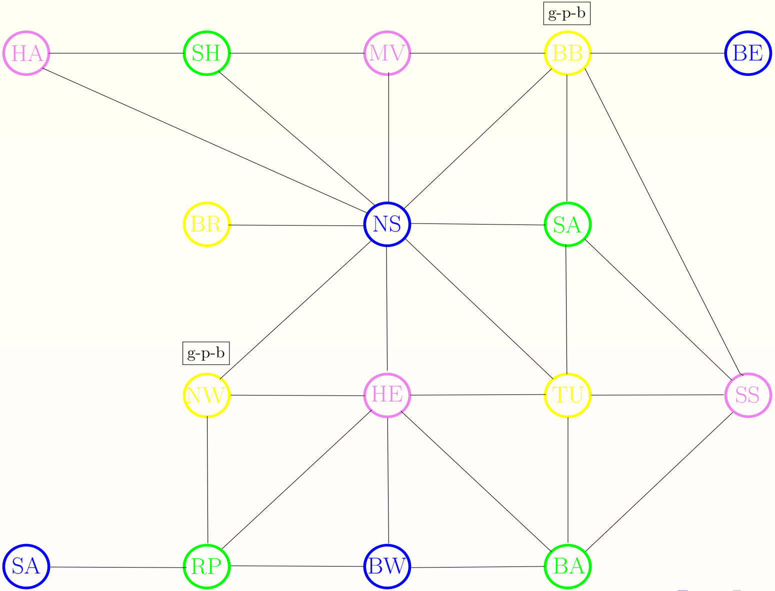
γ_p	$\longmapsto$	γ_p^i	=	χ	chromatic number
β_p	$\longmapsto$	β_p^i	=		
		$\Vert$			
η_p	$\longmapsto$	η_p^i	=	χ_{ML}	ML-chromatic number
λ_p	$\longmapsto$	λ_p^i	=	$\boxed{\chi_{NL}}$	NL-chromatic number

[5] also called *stable partition*.

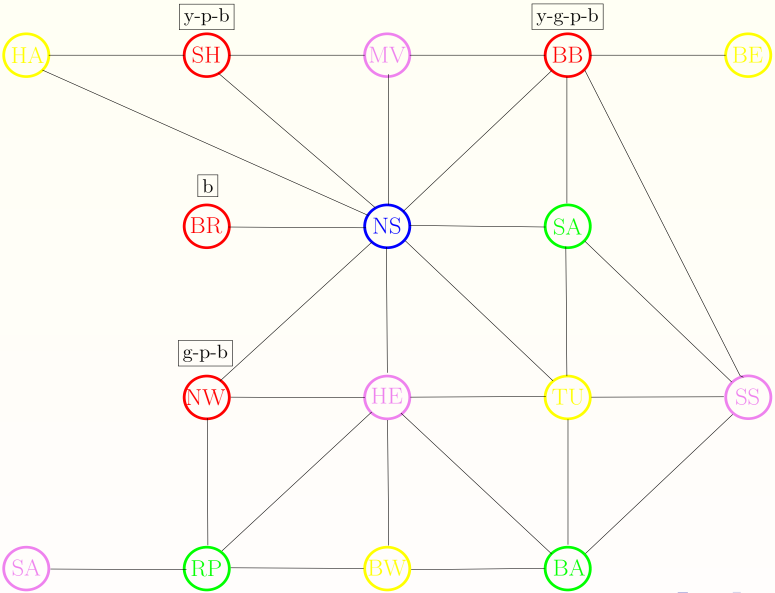
$n = 13 + 3 = 16$
$m = 29$
$D = 5$
$\delta = 1$
$\Delta = 9$



$\chi = 4$



$$\chi_{NL} = 5$$



Google, Yahoo: arxiv pelayo locating partition

Google, Yahoo: arxiv pelayo locating partition