New Large Graphs with Given Degree and Diameter Six

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Abstract: In this paper, a method for obtaining large diameter 6 graphs by replacing some vertices of a Moore bipartite diameter 6 graph with complete K_h graphs is proposed. These complete graphs are joined to each other and to the remaining nonmodified graphs by means of new edges and by using a special diameter 2 graph. The degree of the graph so constructed coincides with the original one. © 1999 John Wiley & Sons, Inc. Networks 34: 154–161, 1999

1. INTRODUCTION

A question of special interest in graph theory is the construction of graphs with an order as large as possible for a given degree and diameter or (Δ, D)-problem. This problem deserves much attention due to its implications in the design of topologies for interconnection networks and other questions such as the data alignment problem and the description of some cryptographic protocols.

The (Δ, D)-problem for undirected graphs has been approached in different ways. It is possible to give bounds on the order of the graphs for a given degree and diameter (see [7]). As the theoretical bounds are difficult to attain, most of the work dealt with the construction of graphs, which for this given diameter and degree have a number of vertices as close as possible to the theoretical bounds (see [6] for a review).

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Various techniques which depend on the way graphs are generated and their parameters are calculated have been developed. Many (Δ, D)-graphs correspond to Cayley graphs [8, 9, 21, 14] and have been found by computer search. However, the use of computers is only efficient when the degree and the diameter are not too large. Compounding is another technique that has proved useful and consists of replacing one or more vertices of a given graph with another graph or copies of a graph and rearranging the edges suitably (see, for instance, [15, 16]). Compound graphs and Cayley graphs make up many of the large (Δ, D)-graphs described in the literature for a small diameter.

Other large graphs have been found as graph products or special methods. For instance, a graph on an alphabet may be constructed as follows: The vertices are labeled with words of a given alphabet and a rule that relates two different words provides the edges.

In this paper, compounding is used to construct new families of large (Δ, D)-graphs that improve some known

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results for diameter 6. The technique is a generalization of a method used by Quisquater [20], based on the replacement by a complete graph of a single vertex from a bipartite Moore graph. In [10, 16], several authors modified the technique in order to replace several vertices. This paper presents a new technique for working out a general rule for the replacement of a large number of vertices using a special diameter 2 graph.

Section 2 is devoted to introduce some notation and some known results concerning Moore bipartite graphs. In the following section, a type of diameter 2 graph is defined. In Section 4, we describe a general technique for the construction of large diameter 6 graphs using the graphs introduced in the previous section. Finally, in Section 5, new large graphs are proposed.

2. NOTATION AND KNOWN RESULTS

A graph, $G = (V, E)$, consists of a nonempty finite set V of elements called *vertices* and a set E of unordered pairs of elements of V called *edges*. The number of vertices $N = N(G) = |V|$ is the *order* of the graph. If $\{x, y\}$ is an edge of E, we say that x and y (or y and x) are *adjacent* and this is usually written $x \sim y$. It is also said that x and y are the *endvertices* of the edge $\{x, y\}$ or x and y are *incident* to $\{x, y\}$. The graph G is *bipartite* if $V = V_1 \cup V_2$ and $V_1 \cap V_2 = \emptyset$ and any edge $\{x, y\} \in E$ has one endvertex in V_1 and the other in V_2. The *degree* of a vertex $\delta(x)$ is the number of vertices adjacent to x. The (maximum) *degree* of G is $\Delta = \max_{x\in V}\delta(x)$. A graph is regular of degree Δ or Δ-*regular* if the degree of all vertices is Δ. The *distance* between two vertices x and y, $d(x, y)$, is the number of edges of a shortest path between x and y, and its maximum value over all pairs of vertices, $D = \max_{x,y\in V} d(x, y)$, is the *diameter* of the graph. A (Δ, D)-*graph* is a graph with degree Δ and diameter, at most, D.

The order of a graph with degree Δ ($\Delta > 2$) and diameter D is easily seen to be upper-bounded by

$$1 + \Delta + \Delta(\Delta - 1) + \cdots + \Delta(\Delta - 1)^{D-1} = \frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2} = N(\Delta, D).$$

This value is called the *Moore bound*, and it is known that, for $D \geq 2$ and $\Delta \geq 3$, this bound is only attained if $D = 2$ and $\Delta = 3$, 7, and (perhaps) 57, (see [7]). In this context, it is of great interest to find graphs which, for a given diameter and degree, have a number of vertices as close as possible to the Moore bound.

A way of modifying some known large bipartite graphs in order to obtain new large graphs for some values of the degree and the diameter was shown in [10]. In this paper, this method is improved in order to obtain better results.

By counting arguments, it is easy to obtain the following upper bound (see [7]) for the maximum order of a (Δ, D)-bipartite graph:

$$b(\Delta, D) = 2\,\frac{(\Delta - 1)^D - 1}{\Delta - 2},\ \Delta > 2.$$

The bipartite graphs that reach this bound are called *bipartite Moore graphs*. They exist only for $D = 2$ (complete bipartite graphs $K_{\Delta,\Delta}$) or $D = 3, 4, 6$. For these values of D, bipartite Moore graphs have been constructed when $\Delta - 1 = q$ is a prime power (see [5, 7]). For $D = 3$ and $\Delta = q + 1$, the bipartite Moore graph, P_q, is the incident graph of the projective plane $PG(2, q)$. The points of $PG(2, q)$ are the 1-dimensional vector subspaces of the 3-dimensional vector space K^3 over a finite field K with q elements. This fact enables us to define the so-called, graph P'_q as follows: The vertex set is $PG(2, q)$ and the adjacency rule is, for each $a, b \in P(2, q)$, a and b are adjacent if and only if they are orthogonal. It is easy to see that P'_q is a regular graph with order $N = q^2 + q + 1$, degree $\Delta = q + 1$, and diameter $D = 2$. Some of these graphs will be used in this paper in order to expand some graphs. The bipartite Moore graphs with $D = 6$, H_q, are called *generalized hexagons* (see [3]). They are graphs whose vertices are the points and some lines of a nondegenerate quadric surface in the 6-dimensional projective space $PG(6, q)$, with two vertices being adjacent, if and only if they correspond to an incident point-line pair on the surface. They have order $N = 2[(q^6 - 1)/(q - 1)]$ and degree $\Delta = q + 1$.

Here, we have some known results that will be used in the next sections:

- If $G = (V_1 \cup V_2, E)$ is any bipartite graph of even (odd) diameter D, the distance between $x \in V_1$ and any $y \in V_2$ ($y \in V_1$) is at most $D - 1$.
- If G is a (Δ, D)-bipartite Moore graph, for any $x, y \in V(G)$ with $d(x, y) = D$, there exist Δ disjoint paths between x and y of length D.
- The girth of any bipartite Moore graph with diameter D is $g = 2D$.

The construction proposed in [10] consists of expanding the bipartite Moore graph H_q by replacing several vertices by complete graphs and creating some new adjacencies. In it, some vertices x_{ijk} (see Fig. 1) are replaced by copies of the complete graph K_h ($h \leq \Delta$) whose vertex set is denoted by K_{ijk}. These replacements must verify some conditions. In this paper, we will use the same notation for the replaced vertices but our conditions are changed. To be more precise, these conditions have become only one, which is simpler and easier to verify. The next section discusses a type of

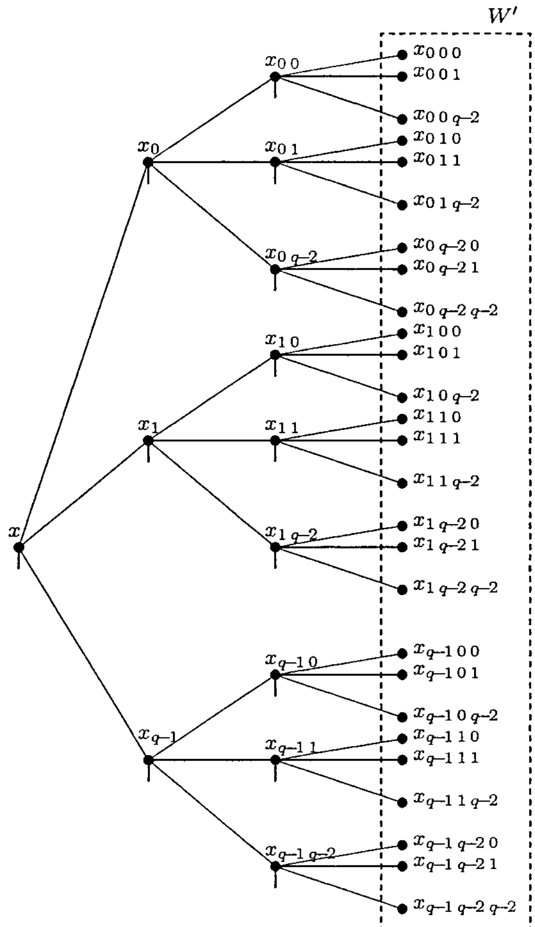

Fig. 1. The subgraph of H_q to be modified.

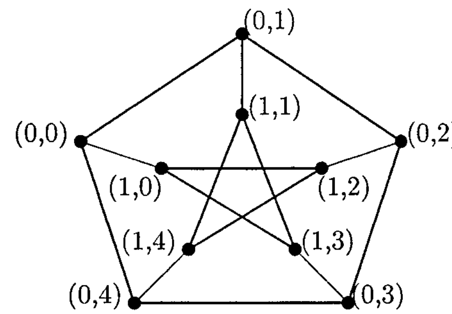

Fig. 2. Petersen graph.

graph with diameter 2 which will enable this condition to be defined.

3. [l, λ]-CLIQUES

Definition 1. *A graph $G = (V, E)$ is an $[l, \lambda]$-clique if it is possible to partition V into λ partite sets $V_1, \ldots, V_\lambda$, whose induced graphs are cliques and such that $|V_1| \leq \cdots \leq |V_\lambda| = l$.*

Observe that if $\tilde{G} = (\tilde{W}, \tilde{E})$ is an $[l, \lambda]$-clique, then $|\tilde{W}| \leq l.\lambda$, and it is also a $[k, \mu]$-clique, for each $k \leq l$ and for some $\mu \geq \lambda$. Next, some examples of $[l, \lambda]$-cliques are given. Most of them will be used to construct new large graphs in Section 5.

Example 1.

1. Every graph is a $[2, \lambda]$-clique, but for the graphs without edges.
2. The complete graph K_h is a $[h, 1]$-clique and it is also a $[k, 2]$-clique for every $k \in \{\lceil \frac{h}{2} \rceil, \ldots, h - 1\}$.
3. Every graph G that contains a clique K_l is an $[l, \lambda]$-clique with λ at most, $N(G) - l + 1$.
4. The Petersen graph P is a $[2, 5]$-clique. A partition is $\{\{(0, i), (1, i)\}, i = 0, 1, 2, 3, 4\}$ (see Fig. 2).
5. The largest known graph for degree 6 and diameter 2 is $K_4 * X_8$. It is constructed by means of joining four copies of X_8 (see [4]). Figure 3 shows two copies of an X_8 graph. The second one shows a partition of it. So, it is a $[2, 4]$-clique. Thus, $K_4 * X_8$ is a $[2, 16]$-clique.
6. The graph P'_q has degree $\Delta = q + 1$, diameter $D = 2$, and order $N = q^2 + q + 1$. For $q = 9$, 11, and 13, the graph P'_q is the largest known graph and, moreover, it is a $[3, \lambda]$-clique. In fact, according to Corollary 3 of Theorem 4.3.6 of [19], P'_9 can be partitioned into 7 P'_3. Besides, P'_3 is a $[3, 6]$-clique for a partition with 1 K_3 and 5 K_2 (see [17]). As a consequence, P'_9 is a $[3, 42]$-clique with 7 K_3 and 35 K_2. The same theorem cannot be applied to the graphs P'_{11} and P'_{13}. However, a study by computer shows that P'_{11} is a $[3, 55]$-clique (see [17]). The partition obtained has 32 K_3, 14 K_2, and 9 K_1. Another analogous study shows that P'_{13} is a $[3, 72]$-clique with 44 K_3, 23 K_2, and 5 K_1 (see [17]).
7. Let l be a positive integer less than or equal to $q^2 + q + 1$. Let us choose a partition into λ parts of the vertex set of P'_q so that
    - Each part has, at most, l vertices.
    - There exists a part with l vertices.

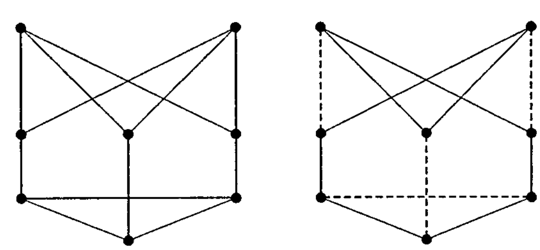

Fig. 3. X_8 and its partition.

We construct a new family of graphs, denoted by $P'_q l$ from a copy of P'_q, by adding new edges between nonadjacent vertices of the same part so that in each part all the vertices are then adjacent among themselves. So, any graph $P'_q l$ is composed by the union of λ complete graphs. As a consequence, the degree of $P'_q l$ is, at most, $q + l$. Furthermore, it is an $[l, \lambda]$-clique for the previous partition.

4. $H_q(K_h)$ GRAPHS

Let us consider the subgraph of the bipartite Moore graph of degree $\Delta = q + 1$, $H_q = (V \cup W, E)$ (see Fig. 1.) We consider a vertex $x \in V$. According to this figure,

$$\Gamma(x) = \{x_0, x_1, \ldots, x_q\} \subset W,$$

$$\Gamma(x_i) = \{x_{i0}, x_{i1}, \ldots, x_{iq-1}\} \cup \{x\} \subset V, \quad \forall\, i \in \{0, \ldots, q-1\}$$

$$\Gamma(x_{ij}) = \{x_{ij0}, x_{ij1}, \ldots, x_{ijq-1}\} \cup \{x_i\} \subset W, \quad \forall\, j \in \{0, \ldots, q-2\} \tag{1}$$

Hence, the subset W' of W that we call *set of replaceable vertices* has the following expression:

$$W' = \bigcup_{i=0}^{q-1} \bigcup_{j=0}^{q-2} \Gamma(x_{ij}) \backslash \{x_i, x_{ijq-1}\}.$$

The set of incident edges to vertex x_{ijk} is denoted by E_{ijk}.

With the notation $W_{ij} = \Gamma(x_{ij}) \backslash \{x_i, x_{ijq-1}\}$, the set $\{W_{ij}, i = 1, 2, \ldots, q-1, j = 1, 2, \ldots, q-2\}$ is a partition of W' that we call *standard partition.* It consists of $q(q-1)$ parts with $q-1$ elements in each one.

Definition 2. *Let q be a prime power and let h be an integer so that $1 \le h \le q+1$. Let us denote $H_q(K_h)$ any graph obtained from H_q carrying out the following steps:*

1. *Let us choose a subset $\tilde{W}$ of W'.*
2. *Each vertex $x_{ijk} \in \tilde{W}$ is replaced by a complete graph K_h, whose vertex set is denoted by $K_{ijk} = \{y^1_{ijk}, \ldots, y^h_{ijk}\}$. The set of added vertices is called $\tilde{W}(K_h)$. Thus,*

$$\tilde{W}(K_h) = \bigcup_{x_{ijk} \in \tilde{W}} K_{ijk}.$$

3. *The incident edges to each $x_{ijk} \in \tilde{W}$ are joined to the vertices of K_{ijk} so that each vertex y^l_{ijk}, $l = 1, 2, \ldots, h$ is incident, at least, to one of them.*

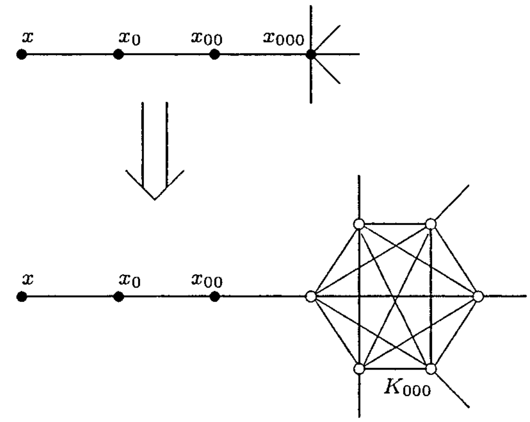

Fig. 4. An expansion of H_5.

4. *Some new edges may be added between vertices of $\tilde{W}(K_h)$ so that the constructed graph has degree $\Delta = q + 1$. The set of these additional edges is denoted by $\tilde{E}_{ijk}$.*

Example 2. Consider a bipartite Moore graph H_5 with diameter 6, degree 6, and order 7812. The subset chosen is $\tilde{W} = \{x_{000}\}$. This vertex is replaced by a complete graph K_6. The edges of E_{000} are now incident to vertices of K_{000} according to Figure 4.

The graph so constructed is not bipartite anymore. However, it is still 6-regular and its order is greater than the original one (five vertices). This example, put forward by Quisquater in [20], corresponds to a particular case of our construction. This author showed that this graph still has diameter 6.

Observe that any $H_q(K_h)$ graph verifies the following properties:

1. After making the first three steps, each vertex of $\tilde{W}(K_h)$ has degree not greater than $q + 1$.
2. If $h > 1$, $H_q(K_h)$ is not bipartite.
3. $|\tilde{W}| \le (\Delta - 1)(\Delta - 2)^2 = q \cdot (q-1)^2$.
4. $N(H_q(K_h)) = N(H_q) + |\tilde{W}|(h-1)$.

The next proposition provides an upper bound to the diameter of any $H_q(K_h)$ graph.

Proposition 1. *The diameter of any $H_q(K_h)$ graph is bounded by 7.*

Proof. After the replacement of the vertices x_{ijk} of H_q, we observe the following:

i. A path of maximum length 6 in H_q, without x_{ijk} vertices, is unaltered in $H_q(K_h)$.
ii. The shortest path between any two vertices will increase its length by at most two (the maximum number of x_{ijk}

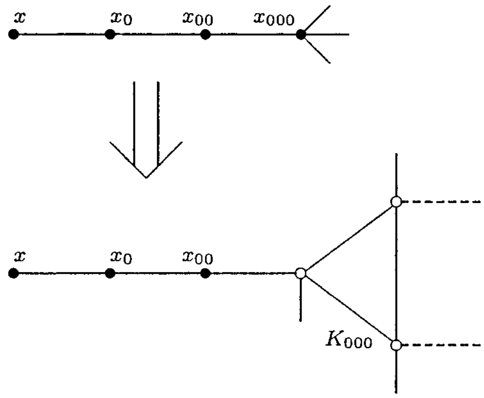

Fig. 5. Margin of $H_3(K_3)$ is 2.

vertices that might be contained in this path); see Figure 1 and remember that $g = 2D$.

iii. Because of condition 3, the maximum distance between two vertices of type $y = y^l_{ijk}$ and $y' = y^0_{rst}$ is 7. In fact, since H_q is bipartite, the distance between x_{ijk} and x_{rst} has to be 0, 2, 4, or 6. If it is less than or equal to 4, then by the previous observation, the distance between y and y' is at most 6. Otherwise, since H_q contains Δ disjoint paths of length 6 between x_{ijk} and x_{rst}, at least a path of the same length runs between y and y' or between y and a vertex of K_{rst} in $H_q(K_h)$, y^m_{rst}, which is adjacent to vertex y'.

Thus, we have only to examine the following case: Let two vertices be at distance 6 in such a way that at least one of them is not of the form y^l_{ijk}. Then, as Δ disjoint paths of length 6 run between these vertices, from condition (3) of Definition 2 and Figure 1, it follows that there exists one path among these vertices, which is unaffected by the replacements. ■

To put forward more complicated expansions of the H_q graphs in an easy way than the ones presented by Quisquater in [20] and Comellas and Gómez in [10], we need to introduce the following definition:

Definition 3. *The margin M of an $H_q(K_h)$ graph is the number of edges needed to add in step 4 of its construction to vertices of K_{ijk} so that each vertex has degree $\Delta = q + 1$.*

Example 3. The vertex x_{000} is replaced by a copy of K_3 in H_3 (see Fig. 5). It is easy to see that it is necessary to add two edges (drawn in broken line) so that each vertex has degree 4. Thus, $M = 2$ in this case.

Proposition 2. *Given an $H_q(K_h)$ graph, if $h \le \Delta = q + 1$, then*

$$M = (\Delta - h)(h - 1). \tag{2}$$

Proof. After making the first three steps in order to construct the $H_q(K_h)$ graph, we have

$$h\Delta = \Delta + h(h - 1) + M,$$

and the desired result is obtained by isolating M from this equation. ■

When a subset $\tilde{W}$ of W' is chosen, the standard partition of W' leads to a new partition that we call *standard partition* of $\tilde{W}$:

$$\tilde{W} = \bigcup_{i=1}^{\lambda} \tilde{W}_i, \tag{3}$$

where λ is the number of nonempty parts. Note that $\lambda \le q(q - 1)$ and for each $i \in \{1, 2, \ldots, \lambda\}$, $|\tilde{W}_i| \le q - 1$.

As we said above, to construct an $H_q(K_h)$ graph, each vertex $x_{ijk} \in \tilde{W}$ is replaced by a complete graph K_h. Then, the edges of E_{ijk} are joined to the vertices of K_{ijk} so that each vertex of K_{ijk} is adjacent, at least, to some not-replaced vertex of H_q. Finally, some new edges are added, joining vertices of the copies K_{ijk} in such a way that each new vertex has degree less than or equal to $\Delta = q + 1$. Observe that this last new set $\tilde{E}_{ijk}$ of added edges has, at most, M elements.

As a consequence of Proposition 1, the diameter of any $H_q(K_h)$ graph is 6 or 7. As a matter of fact, it is easy to check that there are many of them with diameter 7. However, not all of them have this diameter (remember the construction proposed by Quisquater). To present a family

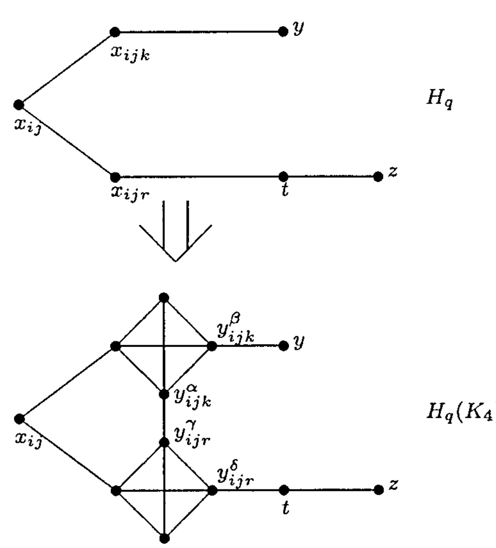

Fig. 6. A path of length 6 between y and z in $H_q(K_4)$.

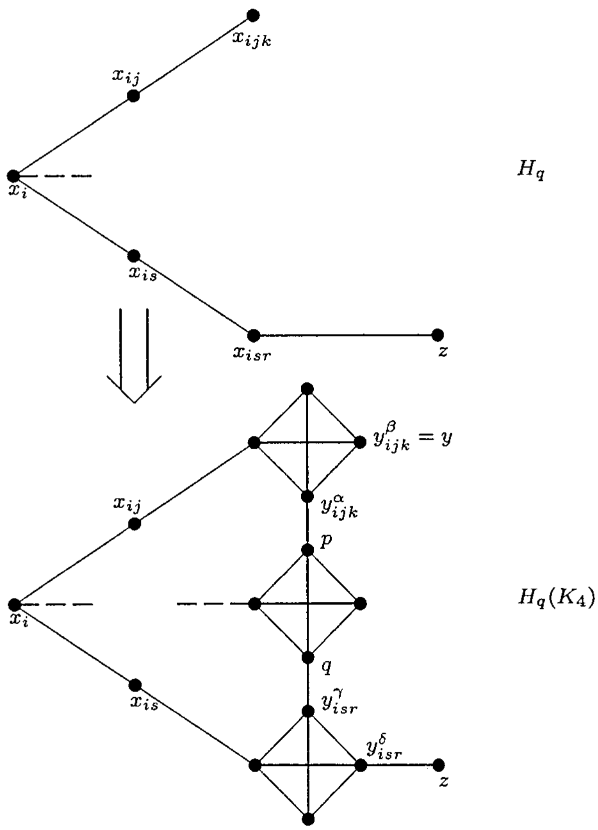

Fig. 7. A path of length 6 between y and z.

of large $H_q(K_h)$ graphs with diameter 6, a new graph is required. This is denoted by $\tilde{G} = (\tilde{W}, \tilde{E})$, where $\tilde{W}$ is presented in (3) and $\tilde{E}$ is defined as follows:

$$(x_{ijk}, x_{rst}) \in \tilde{E} \leftrightarrow \exists\, \alpha, \gamma \in \{1, 2, \ldots, h\} | \quad (y^{\alpha}_{ijk}, y^{\gamma}_{rst}) \in \tilde{E}_{ijk} \cap \tilde{E}_{rst}. \quad (4)$$

The graph so constructed has degree not greater than M. Additionally, the parameters of the standard partition of $\tilde{W}$ verify that

$$l = \max_{i=1,2,\ldots,\lambda} |\tilde{W}_i|$$

$$\lambda \leq q(q-1)$$

$$l \leq q - 1.$$

Theorem 1. *If the graph $\tilde{G} = (\tilde{W}, \tilde{E})$ has diameter 2 and it is an $[l, \lambda]$-clique for the standard partition of $\tilde{W}$, then $H_q(K_h)$ is a graph of (maximum) degree $\Delta = q + 1$, diameter $D = 6$, and order $N(H_q(K_h)) = N(H_q) + |\tilde{W}| \cdot (h - 1)$*

Proof. According to Proposition 1, it is sufficient to consider the three following cases:

i. Let us consider two vertices in H_q, say y and z, at distance 5 joined by the path: $y, x_{ijk}, x_{ij}, x_{ijr}, t, z$. Since x_{ijk} and x_{ijr} belong to the same part of the standard partition of $\tilde{W}$, they are adjacent in G. Therefore, according to (4), there exist $\alpha, \gamma \in \{1, 2, \ldots, h\}$ so that vertices y^{α}_{ijk} and y^{β}_{ijr} are adjacent in $H_q(K_h)$. Thus, a path of length at most, 6 between y and z in $H_q(K_h)$ is $y, y^{\beta}_{ijk}, y^{\alpha}_{ijk}, y^{\gamma}_{ijr}, y^{\delta}_{ijr}, t, z$. See Figure 6, where this is illustrated for $h = 4$.

ii. Let us consider these two vertices, $y = y^{\beta}_{ijk}$ and z, in $H_q(K_h)$, where x_{ijk} and z are at distance 5 in H_q and the shortest path between them is: $x_{ijk}, x_{ij}, x_i, x_{is}, x_{isr}, z$. Since x_{ijk} is at distance less than or equal to 2 of x_{isr} in G, there exist $\alpha, \gamma \in \{1, 2, \ldots, h\}$ such that the vertices y^{α}_{ijk} and y^{γ}_{isr} are, at most, at distance 3 in $H_q(K_h)$. See Figure 7, where this is shown for $h = 4$. So, a path of length at most 6 between $y = y^{\beta}_{ijk}$ and z is $y^{\beta}_{ijk}, y^{\alpha}_{ijk}, p, q, y^{\gamma}_{isr}, y^{\delta}_{isr}, z$.

iii. Let us consider these two vertices, y^{α}_{ijk} and y^{β}_{rst}, with $i \neq r$. Since x_{ijk} and x_{rst} are at distance, at most, 2 in $\tilde{G}$, by using the same reasoning as in the previous case, we find that the distance between y^{α}_{ijk} and y^{β}_{rst} is, at most, 5.

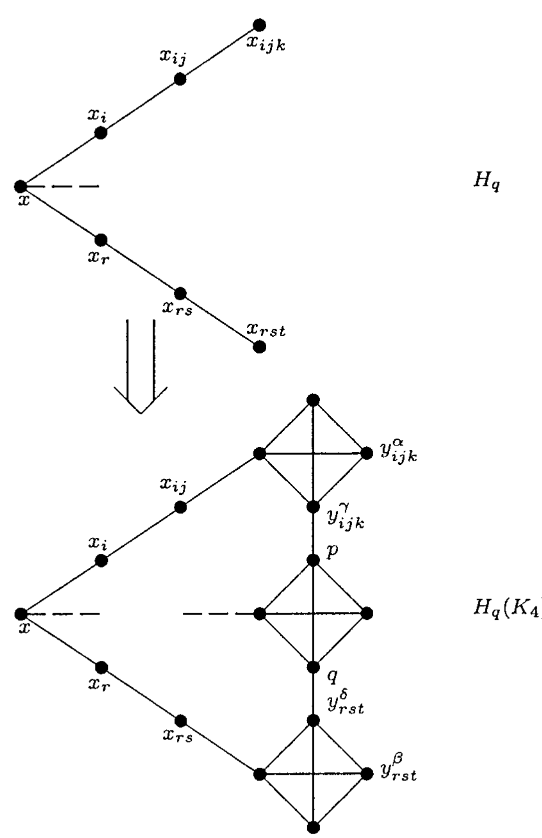

Fig. 8. A path of length 5 between y^{α}_{ijk} and y^{β}_{rst}.

TABLE I. New values of large graphs from γ-graphs

Δ	h	M	$\tilde{G}$	$N(\tilde{G})$	l	λ	N
5	4	3	P	10	2	5	**2760**
6	4	6	$K_4 * X_8$	32	2	16	**7908**
8	6	10	P'_9	91	3	42	**39671**
9	7	12	P'_{11}	133	3	55	**75696**
10	8	14	P'_{13}	183	3	72	**134141**

(Fig. 8 shows the path for $h = 4$.) ■

5. NEW LARGE GRAPHS OF DIAMETER 6

Theorem 1 gives a method to construct some $H_q(K_h)$ graphs of diameter 6. To be more precise, we take a copy of a graph H_q and we choose a complete K_h graph where

$$h \leq q + 1. \tag{5}$$

Next, we take a $[l, \lambda]$-clique $\tilde{G} = (\tilde{W}, \tilde{E})$ with diameter $D = 2$ and degree $\Delta_{\tilde{G}}$, where

$$\lambda \leq q(q - 1) \tag{6}$$

$$l \leq q - 1 \tag{7}$$

$$\Delta_{\tilde{G}} \leq (q + 1 - h)(h - 1). \tag{8}$$

In the following subsections, some of the examples of $[l, \lambda]$-clique with diameter $D = 2$ are used in order to obtain new large graphs of diameter 6.

5.1. Large Graphs from γ-Graphs

All the largest known graphs with diameter 2 are called in this paper *γ-graphs*. Five of them are used in this subsection in order to obtain new large graphs. To be more precise, we use the graphs Petersen, $K_4{*}X_8$, P'_9, P'_{11}, and P'_{13}, which correspond to the ones already mentioned in Examples 1.4, 1.5, and 1.6, to obtain five new large graphs (see Table I).

TABLE II. New values of large graphs from δ-graphs

Δ	h	M	$\tilde{G}$	$N(\tilde{G})$	l	λ	N
10	6	20	$P'_{17}5$	307	5	72	**134395**
12	8	28	$P'_{23}6$	553	6	110	**358183**
14	9	40	$P'_{32}7$	1057	7	151	**812924**

TABLE III. New large graphs

Δ	$x-graph$	G	$N(G)$
5	γ	$H_4(K_4)$	**2 760**
6	γ	$H_5(K_4)$	**7 908**
8	γ	$H_7(K_6)$	**39 671**
9	γ	$H_8(K_7)$	**75 696**
10	δ	$H_9(K_6)$	**134 395**
12	δ	$H_{11}(K_8)$	**358 183**
14	δ	$H_{13}(K_9)$	**812 924**

5.2. Large Graphs from δ-Graphs

From the family presented in Example 1.7, which we call *δ-graphs*, three new large graphs are obtained (see Table II).

The δ-graphs used are constructed as follows:

Graph $H_9(K_6)$ is constructed using a graph $P'_{17}5$. This is done by means of a partition of P'_{17} in $\lambda = 72$ parts (see [17]). It consists of 50 C_5, 8 $K_{1,3}$, 1 K_3, 9 K_2, and 4 K_1. Observe that to obtain complete graphs from them it is enough to increase their degree by, at most, two units. So, $P'_{17}5$ has degree 20 which coincides with the margin in this case.

Graph $H_{11}(K_8)$ is constructed using a graph $P'_{23}6$. This is done by means of a partition of P'_{23} in $\lambda = 110$ parts. It consists of 107 sets with order 5 each and 3 sets containing 2 K_3 each (see [17]). Observe that to obtain complete graphs from them it is enough to increase their degree by, at most, four units. So, $P'_{23}6$ has degree 28 which coincides again with the margin in this case.

Graph $H_{13}(K_9)$ is constructed using a graph $P'_{32}7$. The last one is made by means of a partition of P'_{32} in $\lambda = 151$ parts. It consists of 151 sets with 7 arbitrary vertices. In this case, to obtain 151 complete graphs K_7, the degree of each vertex is increased by, at most, 6 units. So, $P'_{32}7$ has degree 39 which is less than the margin in this case.

Table III shows the best values obtained in this work for degree less than or equal to 14 that improve previous values (see [23]).

The method put forward in this paper enables one to improve many values for diameter 6 and degree Δ greater than 14, by means of the corresponding family of $[l, \lambda]$-cliques. For instance, using δ-graphs for degree $\Delta = q + 1$ (with q a prime power), this method gives graphs on order about

$$N = N(H_q) + \frac{\Delta^5}{2^6}.$$

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