Vanishing of the André-Quillen homology module $H_2(A, B, \mathbf{G}(I))$

Francesc Planas-Vilanova

Dept. Matemàtica Aplicada I. ETSEIB, Universitat Politècnica de Catalunya.
Diagonal 647, E-08028 Barcelona. Spain. E-mail: planas@ma1.upc.es

Abstract

Let I be an ideal of a commutative Noetherian ring A, $A \supset \mathbb{Q}$, $B = A/I$ and $\mathbf{G}(I)$ the associated graded ring to I. It is known that $H_2(A, B, B) = 0$ is equivalent to I being syzygetic. We prove that the vanishing of $H_2(A, B, \mathbf{G}(I))$ is equivalent to I being of linear type and $\sigma_{3,q} : \mathbf{\Lambda}_3^B(I/I^2) \otimes_B I^q/I^{q+1} \to \mathrm{Tor}_3^A(B, A/I^{q+1})$, the $(3, q)$-antisymmetrization morphism, being surjective for all $q \geq 0$. Using this and a theorem of Ulrich on a conjecture of Herzog, we deduce that, in a regular local ring A, a Gorenstein, licci ideal I verifies $H_2(A, B, \mathbf{G}(I)) = 0$ if and only if I is a complete intersection. Thus, we characterize perfect (respectively, Gorenstein) ideals of grade two (respectively, three) with $H_2(A, B, \mathbf{G}(I)) = 0$ as those ideals which are of linear type (respectively, complete intersection). With any grade, but small deviation, we show that a licci ideal, generically a complete intersection and of deviation one, verifies $H_2(A, B, \mathbf{G}(I)) = 0$. This is not true for licci ideals of linear type and of deviation two.

1 Introduction

Let I be an ideal of a commutative ring A and $B = A/I$. Let $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ denote the canonical graded morphism from the symmetric algebra of I, $\mathbf{S}(I)$, onto its Rees algebra, $\mathbf{R}(I) = \oplus_{q \geq 0} I^q$. The ideal I is said to be of linear type if α is an isomorphism, and it is said to be syzygetic if α_2 is an isomorphism. Clearly, if I is of linear type, then the canonical morphism $\beta : \mathbf{S}^B(I/I^2) \to \mathbf{G}(I)$ from the symmetric algebra of the conormal module of I onto its associated graded ring, $\mathbf{G}(I) = \oplus_{q \geq 0} I^q/I^{q+1}$, is bijective. It is well known (see [19]) that the converse is true whenever A is Noetherian.

As we see next, there is a relationship between the linear type condition and the André-Quillen homology theory for commutative rings. We will denote by $H_p(A, B, W)$ the p-th André-Quillen homology module of the A-algebra B with coefficients in a B-module W. We describe now the relationship between the linear type condition and the André-Quillen homology. In Theorem 10.3 of [16], Quillen proved that $H_p(A, B, W) = 0$ for all $p \geq 2$ and all B-module W is equivalent to the canonical morphism $\gamma : \mathbf{\Lambda}^B(I/I^2) \to \mathrm{Tor}^A(B, B)$

being an isomorphism and I/I^2 a flat B-module. Moreover (see 8.5, [16]), if one (and hence, both) of the above condition holds, then β is an isomorphism. To prove it, Quillen used a spectral sequence (see 6, [16]) whose five term exact sequence is:

$$\mathrm{Tor}_3^A(B,B) \to H_3(A,B,B) \to \mathbf{\Lambda}_2^B(I/I^2) \stackrel{\gamma_2}{\to} \mathrm{Tor}_2^A(B,B) \to H_2(A,B,B) \to 0\,. \qquad (1)$$

It can be shown (see for more details Corollary 2.6) that $\gamma_2 : \mathbf{\Lambda}_2^B(I/I^2) \to \mathrm{Tor}_2^A(B,B)$ is defined by $\gamma_2(\overline{u} \wedge \overline{v}) = v \otimes u - u \otimes v \in \mathrm{Tor}_2^A(B,B) = \mathrm{Ker}(I \otimes I \to I^2)$. In particular, using the exact sequence (1):

$$H_2(A,B,B) = \mathrm{Coker}\gamma_2 = \frac{\mathrm{Ker}(I \otimes I \to I^2)}{< v \otimes u - u \otimes v >} = \mathrm{Ker}(\alpha_2 : \mathbf{S}_2^A(I) \to I^2)\,.$$

Thus, I is syzygetic if and only if $H_2(A,B,B) = 0$. This was firstly remarked by André in 11 of [3]. As a corollary, using $\mathbf{S}_+(I) = \oplus_{q>0}\, \mathbf{S}_q(I)$ is a syzygetic ideal of $\mathbf{S}(I)$ (see, for instance, 2.3, [8]) and taking $\mathbf{S}(I) \supset \mathbf{S}_+(I) \supset \mathrm{Ker}\alpha$ in 25.3 of [1], one can deduce that

$$H_2(\mathbf{R}(I), A, A) = \frac{\mathrm{Ker}\alpha}{\mathbf{S}_+(I) \cdot \mathrm{Ker}\alpha}\,.$$

That is, I is of linear type if and only if $H_2(\mathbf{R}(I), A, A) = 0$.

A similar result to Quillen's Theorem 10.3, [16], was proved by André in [3], where he showed that $H_p(A,B,W) = 0$ for all $p \geq 2$ and all B-module W is equivalent to β being an isomorphism, I/I^2 a flat B-module and $\tau_{p,q} : \mathrm{Tor}_p^A(B, A/I^q) \to \mathrm{Tor}_p^A(B, A/I^{q-1})$ being zero for all $p, q \geq 2$. In fact, André proved that $H_2(A,B,\cdot) = 0$ is enough to imply β bijective.

In [14], it was proved that $H_2(A,B,W) = 0$ for all B-module W is equivalent to α being an isomorphism and I/I^2 a flat B-module. To show it, the author used that the vanishing of $H_2(A,B,\mathbf{G}(I))$ (under a mild hypothesis, fulfilled if A contains the field of rational numbers $\mathbb{Q}$) suffices to assure that I is of linear type (see 3.3 of [14]). In connection with the above mentioned result of André, it was also proved that α being an isomorphism is equivalent to β being an isomorphism and $\tau_{2,q} = 0$ for all $q \geq 2$. To show it, in 2.4 of [14] some morphisms $\sigma_{2,q} : \mathbf{\Lambda}_2^B(I/I^2) \otimes_B I^q/I^{q+1} \to \mathrm{Tor}_2^A(B, A/I^{q+1})$ with $\mathrm{Coker}\sigma_{2,q} = \mathrm{Ker}\alpha_{q+2}/I \cdot \mathrm{Ker}\alpha_{q+1}$ were introduced. These morphisms can be thought, in some sense, as a generalization of the morphism $\gamma_2 : \mathbf{\Lambda}_2^B(I/I^2) \to \mathrm{Tor}_2^A(B,B)$ whose $\mathrm{Coker}\gamma_2 = \mathrm{Ker}\alpha_2$.

Motivated by its naturality, the purpose of this paper is to clarify the meaning of $H_2(A,B,\mathbf{G}(I)) = 0$ and its relationship with some other already known conditions on the ideal I, like being a complete intersection, being generated by a d-sequence or being of linear type. Our first result (see Theorem 3.4) characterizes ideals I with $H_2(A,B,\mathbf{G}(I)) = 0$ as those ideals with $\sigma_{p,q} : \mathbf{\Lambda}_p^B(I/I^2) \otimes_B I^q/I^{q+1} \to \mathrm{Tor}_p^A(B, A/I^{q+1})$ being surjective for $p = 2,3$ and all $q \geq 0$, where $\sigma_{p,q}$ is the (p,q)-*antisymmetrization morphism*, a straightforward extension of $\sigma_{2,q}$ introduced in 2.4 of [14]. In particular, if $H_2(A,B,\mathbf{G}(I)) = 0$, then I is an ideal of linear type. Moreover, the converse is true, at least, for perfect ideals of grade 2. In Theorem 3.4 we also prove that if $H_2(A,B,\mathbf{G}(I)) = 0$, then $H_p(A,B,B) = 0$ for $p = 2,3$. This result calls our attention to the following result of Ulrich on a conjecture of Herzog (see 2.20 of [17] and [7]): if I is a licci ideal of a regular local ring A, $A \supset \mathbb{Q}$ and $B = A/I$, then $H_p(A,B,B) = 0$ for $p = 3,4$ is equivalent to I being a complete intersection. In this way, using our Theorem 3.4 and this result of Ulrich, we deduce that in a regular local ring A, a licci, Gorenstein ideal I verifies $H_2(A,B,\mathbf{G}(I)) = 0$ if and only if I is a complete intersection (see Theorem 4.2). As a corollary, we characterize perfect (respectively, Gorenstein) ideals of grade two (respectively, three) with $H_2(A,B,\mathbf{G}(I)) = 0$ as those ideals which are of linear type (respectively, complete intersection) (see 4.3). Finally, we show that in a Gorenstein local ring A, a licci ideal I of any grade, but deviation one and generically a complete intersection verifies $H_2(A,B,\mathbf{G}(I)) = 0$ (see Theorem 4.4). We remark that this result is not true for deviation two (see Example 4.7).

Some technical devices we need to prove Theorem 3.4 are developed in section 2. Concretely, we give an expression for the images of the canonical morphism γ and, more in general, for the images of $\sigma_{p,q} : \mathbf{\Lambda}^B_p(I/I^2) \otimes_B I^q/I^{q+1} \to \mathrm{Tor}^A_p(B, A/I^{q+1})$ in terms of a projective A-resolution of I (see Proposition 2.2). For the cases $p = 2,3$, those involved in Theorem 3.4, this expression becomes even more explicit (see 2.6 and 2.9).

2 Antisymmetrization morphisms

Let I be an ideal of a commutative ring A, $B = A/I$. Let C be a commutative B-algebra. Consider $\gamma^C : \mathbf{\Lambda}^C(I/I^2 \otimes_B C) \to \mathrm{Tor}^A(B,C)$, the only graded morphism of alternated C-algebras which extends the natural isomorphism $I/I^2 \otimes_B C \xrightarrow{\simeq} \mathrm{Tor}^A_1(B,C)$. Clearly, the following is a commutative diagram of canonical morphisms:

$$\begin{array}{ccc}
\mathbf{\Lambda}^B(I/I^2) \underset{B}{\otimes} C & \xrightarrow{\gamma^B \otimes 1} & \mathrm{Tor}^A(B,B) \underset{B}{\otimes} C \\
\xi \downarrow \simeq & & \downarrow \rho \\
\mathbf{\Lambda}^C(I/I^2 \underset{B}{\otimes} C) & \xrightarrow[\gamma^C]{} & \mathrm{Tor}^A(B,C)
\end{array}$$

Let $\psi : \mathbf{\Lambda}^B(I/I^2) \otimes_B C \longrightarrow \mathrm{Tor}^A(B,C)$ denote the composition $\psi = \rho \circ (\gamma^B \otimes 1) = \gamma^C \circ \xi$. If C is a graded B-algebra, then ψ is bigraded. Let $\psi_{p,q} : \mathbf{\Lambda}^B_p(I/I^2) \otimes_B C_q \longrightarrow \mathrm{Tor}^A_p(B, C_q)$ stand for its (p,q)-component.

Consider $C = \mathbf{G}(I) = \oplus_{q \geq 0} I^q/I^{q+1}$ the associated graded ring to I. Then, for each $q \geq 1$, the short exact sequence $0 \to I^q/I^{q+1} \to A/I^{q+1} \to A/I^q \to 0$ induces the long exact sequence of $\mathrm{Tor}(,)$:

$$\ldots \to \mathrm{Tor}^A_p(B, I^q/I^{q+1}) \xrightarrow{i_{p,q}} \mathrm{Tor}^A_p(B, A/I^{q+1}) \xrightarrow{\tau_{p,q+1}} \mathrm{Tor}^A_p(B, A/I^q) \xrightarrow{c_{p,q}} \ldots$$

We will denote by $\sigma_{p,q}$ the following composition:

$$\sigma_{p,q} = i_{p,q} \circ \psi_{p,q} : \mathbf{\Lambda}^B_p(I/I^2) \underset{B}{\otimes} I^q/I^{q+1} \to \mathrm{Tor}^A_p(B, I^q/I^{q+1}) \to \mathrm{Tor}^A_p(B, A/I^{q+1})$$

and we will call $\sigma_{p,q}$ the (p,q)-*antisymmetrization morphism.* Remark that when $q=0$, then $\sigma_{p,0}=\gamma_p^B:\mathbf{\Lambda}_p^B(I/I^2)\to \mathrm{Tor}_p^A(B,B)$, which we will simply denote by γ_p.

Remark 2.1 If I is an ideal locally generated by a regular sequence, then γ_p is an isomorphism and $\tau_{p,q}=0$ for all $p,q\geq 0$. Therefore, $\sigma_{p,q}$ is surjective for all $p,q\geq 0$.

Proposition 2.2 *Let I be an ideal of A, $B=A/I$ and $\sigma_{p,q}$ the (p,q)-antisymmetrization morphism. Let $(X,\partial)\stackrel{\varepsilon_X}{\to} B$ be the Koszul complex of $\lambda: X_1\to A$, a projective A-presentation of I. Let $(Y,\mathrm{d})\stackrel{\varepsilon_Y}{\to} B$ be a projective A-resolution of B and consider $f:X\to Y$ a morphism of complexes such that $\varepsilon_X=\varepsilon_Y\circ f$. Denote by $\overline{x}_i$ the class of $x_i\in I$ in I/I^2, $\overline{y}$ (respectively, $\tilde{y}$) the class of $y\in I^q$ in I^q/I^{q+1}, (respectively, in A/I^{q+1}) and take $u_i\in X_1$ such that $\lambda(u_i)=x_i$. Then,*

$$\sigma_{p,q}\left((\overline{x}_1\wedge\ldots\wedge\overline{x}_p)\otimes\overline{y}\right)=[f_p(u_1\wedge\ldots\wedge u_p)\otimes\tilde{y}]\ ,$$

where $[-]$ stands for the homology class of a p-cycle in $H_p(Y\otimes A/I^{q+1})=\mathrm{Tor}_p^A(B,A/I^{q+1})$.

Proof. Since $\sigma_{p,q}=i_{p,q}\circ\psi_{p,q}=i_{p,q}\circ\rho_{p,q}\circ(\gamma_p\otimes 1_{I^q/I^{q+1}})$, with $\gamma_p:\mathbf{\Lambda}_p^B(I/I^2)\to\mathrm{Tor}_p^A(B,B)$ and $\rho_{p,q}:\mathrm{Tor}_p^A(B,B)\otimes_B I^q/I^{q+1}\to\mathrm{Tor}_p^A(B,I^q/I^{q+1})$, it is enough to prove that

$$\gamma_p(\overline{x}_1\wedge\ldots\wedge\overline{x}_p)=[f_p(u_1\wedge\ldots\wedge u_p)\otimes 1_B]\in H_p(Y\underset{A}{\otimes}B)=\mathrm{Tor}_p^A(B,B)\,.$$

Let $g:X_1\to I$ be defined by $g(u)=\lambda(u)$. Denote by $\overline{g}:X_1/IX_1\to I/I^2$ the induced epimorphism of B-modules and $\mathbf{\Lambda}(\overline{g}):\mathbf{\Lambda}^B(X_1/IX_1)\to\mathbf{\Lambda}^B(I/I^2)$ the surjective graded morphism of B-algebras induced by $\overline{g}$. Through the natural isomorphism $H(X\otimes_A B)\simeq X\otimes_A B\simeq\mathbf{\Lambda}^B(X_1/IX_1)$, we can consider the morphism

$$\Phi: H(X\underset{A}{\otimes}B)\stackrel{\simeq}{\longrightarrow}\mathbf{\Lambda}^B(X_1/IX_1)\stackrel{\mathbf{\Lambda}(\overline{g})}{\longrightarrow}\mathbf{\Lambda}^B(I/I^2)\stackrel{\gamma}{\longrightarrow}\mathrm{Tor}^A(B,B)\,,$$

which is a graded morphism of B-algebras with $\Phi_1=\gamma_1\circ\overline{g}$. On the other hand, f induces the homology morphism $[f\otimes 1_B]:H(X\otimes_A B)\to H(Y\otimes_A B)=\mathrm{Tor}^A(B,B)$ with $[f\otimes 1_B]_1=\Phi_1$. If we prove that $[f\otimes 1_B]$ is a morphism of algebras, then $\Phi=[f\otimes 1_B]$ and hence

$$\gamma_p(\overline{x}_1\wedge\ldots\wedge\overline{x}_p)=\Phi_p([(u_1\wedge\ldots\wedge u_p)\otimes 1_B])=[f_p(u_1\wedge\ldots\wedge u_p)\otimes 1_B]\,.$$

Thus, it remains to prove next lemma. ∎

Lemma 2.3 *Let I be an ideal of A, $B=A/I$. Let $(X,\partial)\stackrel{\varepsilon_X}{\to}B$ be an augmented projective differential graded A-algebra. Let $(Y,\mathrm{d})\stackrel{\varepsilon_Y}{\to}B$ be a projective A-resolution of B and consider $f:X\to Y$ a morphism of complexes such that $\varepsilon_X=\varepsilon_Y\circ f$ (it does exist and it is uniquely determinate up to homotopy). Then, $f\otimes 1_B$ induces in homology a graded morphism of B-algebras*

$$[f\otimes 1_B]:H(X\underset{A}{\otimes}B)\longrightarrow H(Y\underset{A}{\otimes}B)=\mathrm{Tor}^A(B,B)\,.$$

Proof. By the comparison theorem (3.1 of [4]), there exists a morphism of A-complexes $u : Y \otimes_A Y \to Y$, defined uniquely up to homotopy, such that

$$\begin{array}{ccc} Y \underset{A}{\otimes} Y & \xrightarrow{\varepsilon_Y \otimes \varepsilon_Y} & B \\ {\scriptstyle u}\downarrow & & \parallel \\ Y & \xrightarrow[\varepsilon_Y]{} & B \end{array}$$

is a commutative diagram. Recall (see, for instance, 7, ex. 7, [4]) the product in $\mathrm{Tor}^A(B,B)$ is defined via the composition:

$$H(Y \underset{A}{\otimes} B) \underset{A}{\otimes} H(Y \underset{A}{\otimes} B) \longrightarrow H(Y \underset{A}{\otimes} B \underset{A}{\otimes} Y \underset{A}{\otimes} B) \stackrel{\simeq}{\longrightarrow} H(Y \underset{A}{\otimes} Y \underset{A}{\otimes} B) \stackrel{H(u \otimes 1_B)}{\longrightarrow} H(Y \underset{A}{\otimes} B)\,.$$

Thus, to prove that $[f \otimes 1_B]$ is a morphism of A-algebras it is enough to see that the following diagram is commutative up to homotopy:

$$\begin{array}{ccc} X \underset{A}{\otimes} X & \xrightarrow{f \otimes f} & Y \underset{A}{\otimes} Y \\ {\scriptstyle p}\downarrow & & \downarrow{\scriptstyle u} \\ X & \xrightarrow[f]{} & Y \end{array}$$

where $p : X \otimes_A X \to X$ stands for the product in X. Consider now

$$\begin{array}{ccc} X \underset{A}{\otimes} X & \xrightarrow{\varepsilon_X \otimes \varepsilon_X} & B \\ {\scriptstyle u \circ (f \otimes f)}\downarrow\downarrow{\scriptstyle f \circ p} & & \parallel \\ Y & \xrightarrow[\varepsilon_Y]{} & B \end{array}$$

where $\varepsilon_Y \circ f \circ p = \varepsilon_X \otimes \varepsilon_X = \varepsilon_Y \circ u \circ (f \otimes f)$. Again, by the comparison theorem, we conclude that $u \circ (f \otimes f)$ and $f \circ p$ are homotopic. ∎

To take some advantage of Proposition 2.2, we need to give a more detailed description of $\mathrm{Tor}_p^A(M,P)$ for any two A-modules M and P.

Remark 2.4 Let M, P be two A-modules. Let $(Y, \mathrm{d}) \stackrel{\varepsilon_Y}{\to} M$ be a flat A-resolution of M and, for each $p \geq 1$, denote by $N_p = \mathrm{Ker}\,(\mathrm{d}_p : Y_p \to Y_{p-1})$ and by $N_0 = \mathrm{Ker}(\varepsilon_Y : Y_0 \to M)$. Then, for each $p \geq 2$, the A-module $\mathrm{Tor}_p^A(M,P) = H_p(Y \otimes_A P)$ is canonically isomorphic (by "décalage") to $\mathrm{Tor}_1^A(N_{p-2}, P)$ which can be displayed either

$$\mathrm{Ker}\,(N_{p-1} \otimes P \to Y_{p-1} \otimes P)\ , \tag{2}$$

via the flat A-presentation $0 \to N_{p-1} \stackrel{i_{p-1}}{\to} Y_{p-1} \stackrel{\mathrm{d}_{p-1}}{\to} N_{p-2} \to 0$ of N_{p-2}, or either

$$\mathrm{Ker}\,(N_{p-2} \otimes W \to N_{p-2} \otimes L)\ , \tag{3}$$

via a given flat A-presentation $0 \to W \overset{j}{\to} L \overset{\pi}{\to} P \to 0$ of P.

Moreover, the homology class of a p-cycle $\sum x_i \otimes y_i$ of $Y \otimes P$ leads to the element $\sum \mathrm{d}_p(x_i) \otimes y_i$ in (2). In the other hand, using the snake lemma applied to the diagram

$$\begin{array}{ccccccccc}
 & & N_{p-1}\otimes W & \longrightarrow & N_{p-1}\otimes L & \xrightarrow[1\otimes\pi]{} & N_{p-1}\otimes P & \longrightarrow & 0\\
 & & \downarrow & & {\scriptstyle i_{p-1}\otimes 1_L}\downarrow & & \downarrow & & \\
0 & \longrightarrow & Y_{p-1}\otimes W & \xrightarrow[1\otimes j]{} & Y_{p-1}\otimes L & \longrightarrow & Y_{p-1}\otimes P & \longrightarrow & 0\\
 & & {\scriptstyle \mathrm{d}_{p-1}\otimes 1}\downarrow & & \downarrow & & \downarrow & & \\
 & & N_{p-2}\otimes W & \longrightarrow & N_{p-2}\otimes L & \longrightarrow & N_{p-2}\otimes P & \longrightarrow & 0
\end{array}$$

the homology class of the p-cycle $\sum x_i \otimes y_i$ leads to the element

$$(\mathrm{d}_{p-1} \otimes 1_W) \circ (1_{Y_{p-1}} \otimes j)^{-1} \circ (i_{p-1} \otimes 1_L) \circ (1_{N_{p-1}} \otimes \pi)^{-1}(\sum \mathrm{d}_p(x_i) \otimes y_i)$$

in (3), where if g is a map (not necessarily bijective), then $g^{-1}(y) = x$ whenever $g(x) = y$.

For $p = 2$ and M and P two cyclic A-modules, we have:

Example 2.5 Let I, J be two ideals of A. Let $(X, \partial) \overset{\varepsilon_X}{\to} A/I$ be the Koszul complex of $\lambda : X_1 \to A$, a flat A-presentation of I and, for each $p \geq 1$, denote by $Z_p = \mathrm{Ker}\partial_p$. Then, $\mathrm{Tor}_2^A(A/I, A/J)$ can be displayed either $Z_1 \cap JX_1/JZ_1$ or either $\mathrm{Ker}\,(I \otimes J \to IJ)$. Moreover, if $\sum a_i u_i \in Z_1$, $a_i \in J$, $u_i \in X_1$, then $(\sum a_i u_i) + JZ_1$ in $Z_1 \cap JX_1/JZ_1$ corresponds to $\sum \lambda(u_i) \otimes a_i$ in $\mathrm{Ker}\,(I \otimes J \to IJ)$.

As a corollary of Proposition 2.2 and Example 2.5, we give a more explicit expression for the images of $\sigma_{2,q}$. In particular, this expression shows that our $\sigma_{2,q}$ defined in this section coincide with those $\sigma_{2,q}$ defined in Corollary 2.4 of [14]. Moreover, we obtain a new proof of Proposition 26.2 of [1].

Corollary 2.6 *Let $\alpha : \mathbf{S}(I) \to \mathbf{R}(I)$ and $\gamma : \mathbf{\Lambda}^B(I/I^2) \to \mathrm{Tor}^A(B, B)$ be the canonical morphisms for an ideal I of A, $B = A/I$. Let $\sigma_{p,q} : \mathbf{\Lambda}_p^B(I/I^2) \otimes_B I^q/I^{q+1} \to \mathrm{Tor}_p^A(B, A/I^{q+1})$ be the (p,q)-antisymmetrization morphism. If we denote by $\overline{x}_i$ the class of the element $x_i \in I$ in I/I^2 and by $\overline{y}$ the class of the element $y \in I^q$ in I^q/I^{q+1}, then*

$$\sigma_{2,q}\left((\overline{x}_1 \wedge \overline{x}_2) \otimes \overline{y}\right) = x_2 \otimes (yx_1) - x_1 \otimes (yx_2) \in \mathrm{Tor}_2^A(B, A/I^{q+1}) = \mathrm{Ker}\left(I \otimes I^{q+1} \to I^{q+2}\right) .$$

In particular, there is a canonical isomorphism $H_2(A, B, B) = \mathrm{Coker}\gamma_2 = \mathrm{Ker}\alpha_2$. Moreover, if $(X, \partial) \overset{\varepsilon_X}{\to} B$ denotes the Koszul complex of $\lambda : X_1 \to A$, a flat A-presentation of I and, for each $p \geq 1$, $Z_p = \mathrm{Ker}\partial_p$ and $B_p = \mathrm{Im}\partial_{p+1}$, then

$$\mathrm{Coker}\sigma_{2,q} = \frac{Z_1 \cap I^{q+1}X_1}{I^q B_1} .$$

Remark 2.7 Let I be an ideal of A, $B = A/I$. Let $(X, \partial) \stackrel{\varepsilon_X}{\to} B$ be the Koszul complex of $\lambda : X_1 \to A$, a flat A-presentation of I and, for each $p \geq 1$, denote by $Z_p = \mathrm{Ker}\partial_p$ and by $B_p = \mathrm{Im}\partial_{p+1}$. It is known (see 1.2 of [11], for the finitely generated case, or 2.6 of [14], for the general case) that, for each $q \geq 0$,

$$\left(\frac{\mathrm{Ker}\alpha}{\mathbf{S}_+(I) \cdot \mathrm{Ker}\alpha}\right)_{q+2} = \frac{\mathrm{Ker}\alpha_{q+2}}{I \cdot \mathrm{Ker}\alpha_{q+1}} = \frac{Z_1 \cap I^{q+1} X_1}{I^q B_1} .$$

In particular, I is an ideal of linear type if and only if $\sigma_{2,q}$ is an epimorphism for all $q \geq 0$.

For any $p \geq 2$, and as a corollary of Proposition 2.2 and Remark 2.4, we have a generalized version of Corollary 2.6:

Corollary 2.8 *Let I be an ideal of A, $B = A/I$. Let $(X, \partial) \stackrel{\varepsilon_X}{\to} B$ be the Koszul complex of $\lambda : X_1 \to A$, a projective A-presentation of I. Let $(Y, \mathrm{d}) \stackrel{\varepsilon_Y}{\to} B$ be a projective A-resolution of B and consider $f : X \to Y$ a morphism of complexes such that $\varepsilon_X = \varepsilon_Y \circ f$. For each $p \geq 1$, denote by $B_p = \mathrm{Im}\partial_{p+1}$ and by $N_p = \mathrm{Ker}\mathrm{d}_p$. Then, for all $p \geq 2$ and $q \geq 0$,*

$$\mathrm{Coker}\sigma_{p,q} = \frac{N_{p-1} \cap I^{q+1} Y_{p-1}}{I^q f(B_{p-1}) + I^{q+1} N_{p-1}} .$$

This corollary will be helpful later to show Proposition 3.10. To finish this section, we give the following example which will be used again in Example 3.8.

Example 2.9 Let $x_1, \ldots, x_n$ be a regular sequence of elements of a commutative ring A. Let y_1, y_2, y_3 be three monomials in the x_i and denote by I the ideal they generate. Then, $\sigma_{3,q}$ is surjective if and only if

$$I^{q+1} : \left(\frac{|\, y_1, y_2, y_3 \,|}{|\, y_1, y_2 \,|}, \frac{|\, y_1, y_2, y_3 \,|}{|\, y_1, y_3 \,|}, \frac{|\, y_1, y_2, y_3 \,|}{|\, y_2, y_3 \,|} \right) = I^q \cdot \frac{y_1 y_2 y_3}{|\, y_1, y_2, y_3 \,|} + I^{q+1} ,$$

where $|\, y_{i_1}, \ldots, y_{i_s} \,|$ denotes the least common multiple of the $y_{i_1}, \ldots, y_{i_s}$. In particular, if $y_1 = x_1 x_2$, $y_2 = x_2 x_3$ and $y_3 = x_3 x_1$, then $\sigma_{3,q}$ is surjective for all $q \geq 0$. On the other hand, if $y_1 = x_1 x_2$, $y_2 = x_1 x_3$ and $y_3 = x_1 x_4$, then $\sigma_{3,q}$ is not surjective for any $q \geq 0$.

Proof. Take as Y, in Corollary 2.8, the Taylor's resolution (see [12]) associated to the three monomials y_1, y_2, y_3 and take $f_2 : X_2 \to Y_2$ defined by $f_2(e_{i_1} \wedge e_{i_2}) = a_{i_1,i_2}(e_{i_1} \wedge e_{i_2})$ with $a_{i_1,i_2} = (y_{i_1} y_{i_2} / \,|\, y_{i_1}, y_{i_2} \,|)$. ∎

Remark 2.10 While $\sigma_{2,q}$ is surjective for large $q \geq 0$ when A is Noetherian, Example 2.9 shows that this is not true for $\sigma_{3,q}$.

3 Ideals of strong linear type

All along the section I continues being an ideal of a commutative ring A, $B = A/I$, and $\mathbf{G}(I)$ its associated graded ring of I. The purpose of this section is to characterize $H_2(A, B, \mathbf{G}(I)) = 0$ in terms of the antisymmetrization morphisms introduced in section 2. To do this, let us begin by recovering a diagram similar to one build by Quillen in [16] (see also [15]).

It is proved in 8.2 of [16], that the morphism $\mathrm{d} = (\mathrm{d}_{p,q})_{p,q\geq 0}$ defined, for each $p, q \geq 0$, as the composition

$$\mathrm{d}_{p,q} : \mathrm{Tor}_p^A(B, I^q/I^{q+1}) \xrightarrow{i_{p,q}} \mathrm{Tor}_p^A(B, A/I^{q+1}) \xrightarrow{c_{p,q+1}} \mathrm{Tor}_{p-1}^A(B, I^{q+1}/I^{q+2})$$

gives $\mathrm{Tor}^A(B, \mathbf{G}(I))$ a structure of graded differential B-algebra.

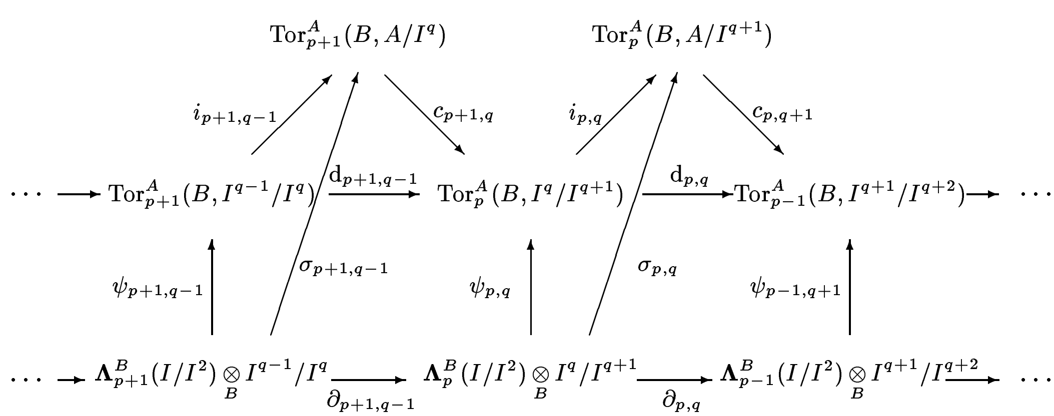
On the other hand, in $\Lambda^{\mathbf{G}(I)}(I/I^2 \otimes_B \mathbf{G}(I))$, one has defined the graded Koszul differential induced by the canonical homogeneous $\mathbf{G}(I)$-linear form $I/I^2 \otimes_B \mathbf{G}(I) \to \mathbf{G}(I)$. Consider in $\mathbf{\Lambda}^B(I/I^2) \otimes_B \mathbf{G}(I)$, the differential transported via the isomorphism

$$\xi : \mathbf{\Lambda}^B(I/I^2) \underset{B}{\otimes} \mathbf{G}(I) \xrightarrow{\simeq} \Lambda^{\mathbf{G}(I)}(I/I^2 \underset{B}{\otimes} \mathbf{G}(I)) \,.$$

By the universal property of the Koszul complex, one can assure that the morphism

$$\psi : \mathbf{\Lambda}^B(I/I^2) \underset{B}{\otimes} \mathbf{G}(I) \to \mathrm{Tor}^A(B, \mathbf{G}(I))$$

is a graded morphism of differential algebras. In other words, for each $p, q \geq 0$, the following diagram (which will be refereed as Quillen's diagram QD_{p+q}) is a commutative diagram:



The rows are complexes of B-modules and $\mathrm{Im}c_{p+1,q} = \mathrm{Ker}i_{p,q}$.

Lemma 3.1 *If for a $p \geq 1$ and a $q \geq 1$, $\sigma_{p,q}$ and $\sigma_{p+1,q-1}$ are surjective, then $\psi_{p,q}$ is surjective too.*

Proof. Given $x \in \mathrm{Tor}_p^A(B, I^q/I^{q+1})$, take $y \in \mathbf{\Lambda}_p^B(I/I^2) \otimes I^q/I^{q+1}$ with $i_{p,q}(x) = \sigma_{p,q}(y)$ (it does exist since $\sigma_{p,q}$ is surjective). Consider $z \in \mathrm{Tor}_{p+1}^A(B, A/I^q)$ such that $c_{p+1,q}(z) =$

$x - \psi_{p,q}(y)$. Since $\sigma_{p+1,q-1}$ is surjective, there exists $t \in \mathbf{\Lambda}^B_{p+1}(I/I^2) \otimes I^{q-1}/I^q$ such that $z = \sigma_{p+1,q-1}(t)$. Thus, $x = \psi_{p,q}(\partial_{p+1,q-1}(t) + y)$. ∎

Lemma 3.2 *Assume that $A \supset \mathbb{Q}$ and take $n \geq 0$. If for all q, $0 \leq q \leq n$, $\psi_{2,q}$ is surjective, then $\sigma_{2,q}$ is surjective and $\tau_{2,q+2} = 0$ for all q, $0 \leq q \leq n$.*

Proof. Let us proceed by induction on $n \geq 0$. If $n = 0$, $i_{2,0} = 1$ and $\sigma_{2,0} = i_{2,0} \circ \psi_{2,0} = \psi_{2,0}$. Moreover, since $A \supset \mathbb{Q}$, $\partial_{2,0}$ is injective. Thus, $\psi_{2,0}$ surjective and $\psi_{1,1}$ and $\partial_{2,0}$ both injective imply $\mathrm{d}_{2,0} = c_{2,1}$ is injective. In other words, $\tau_{2,2} = 0$.

Suppose $n \geq 1$ and true the lemma for $n - 1$. In particular, $\tau_{2,n+1} = 0$, i.e. $i_{2,n}$ is surjective and, hence, $\sigma_{2,n} = i_{2,n} \circ \psi_{2,n}$ is surjective too. It remains to prove $\tau_{2,n+2} = 0$. By corollary 2.6 and Remark 2.7, $0 = \mathrm{Coker}\sigma_{2,q} = \mathrm{Ker}\alpha_{q+2}/I \cdot \mathrm{Ker}\alpha_{q+1}$ for all q, $0 \leq q \leq n$. Since $\mathrm{Ker}\alpha_1 = 0$, then α_q is an isomorphism, and so is $\beta_q = \alpha_q \otimes 1_B$, for all q, $0 \leq q \leq n+2$. Therefore, the bottom row of diagram QD_{n+2} becomes isomorphic to the graded component of degree $n+2$ of the Koszul complex $\mathbf{\Lambda}^B(I/I^2) \otimes \mathbf{S}^B(I/I^2)$ (see, e.g. 9, ex. 1, [4]). The hypothesis $A \supset \mathbb{Q}$, assures its exactness. In particular, $\mathrm{Ker}\partial_{2,n} = \mathrm{Im}\partial_{3,n-1}$. Take now $x \in \mathrm{Tor}^A_2(B, A/I^{n+1})$ with $c_{2,n+1}(x) = 0$. Since $\sigma_{2,n}$ is surjective, there exists $y \in \mathbf{\Lambda}^B_2(I/I^2) \otimes I^n/I^{n+1}$ such that $\sigma_{2,n}(y) = x$. Then, $0 = c_{2,n+1}(x) = c_{2,n+1}(\sigma_{2,n}(y)) = \psi_{1,n+1}(\partial_{2,n}(y))$ and, since $\psi_{1,n+1}$ is an isomorphism, then $y \in \mathrm{Ker}\partial_{2,n} = \mathrm{Im}\partial_{3,n-1}$. Thus, there exists $z \in \mathbf{\Lambda}^B_3(I/I^2) \otimes I^{n-1}/I^n$ with $\partial_{3,n-1}(z) = y$. Then, $x = \sigma_{2,n}(\partial_{3,n-1}(z)) = (i_{2,n} \circ c_{3,n})(\sigma_{3,n-1}(z)) = 0$ because $i_{2,n} \circ c_{3,n} = 0$. That is, $c_{2,n+1}$ is injective and $\tau_{2,n+2} = 0$. ∎

Lemma 3.3 *Assume that $A \supset \mathbb{Q}$ and that A is Noetherian. If for all $q \geq 0$, $\psi_{2,q}$ is surjective, then $\sigma_{3,q}$ is surjective for all $q \geq 0$.*

Proof. Consider diagram QD_{q+3} and $x \in \mathrm{Tor}^A_3(B, A/I^{q+1})$. Since $\psi_{2,q+1}$ is surjective, there exists $y \in \mathbf{\Lambda}^B_2(I/I^2) \otimes I^{q+1}/I^{q+2}$ such that $\psi_{2,q+1}(y) = c_{3,q+1}(x)$. Since the bottom row is exact (see the proof of lemma 3.2), there exists $z \in \mathbf{\Lambda}^B_3(I/I^2) \otimes I^q/I^{q+1}$ such that $\partial_{3,q}(z) = y$. Then, $x - \sigma_{3,q}(z) \in \mathrm{Ker}c_{3,q+1} = \mathrm{Im}\tau_{3,q+2}$. So, there exists $x_2 \in \mathrm{Tor}^A_3(B, A/I^{q+2})$ such that $x = \sigma_{3,q}(z) + \tau_{3,q+2}(x_2)$.

Repeating the same argument with diagram QD_{q+4} and $x_2 \in \mathrm{Tor}^A_3(B, A/I^{q+2})$, we get $z_2 \in \mathbf{\Lambda}^B_3(I/I^2) \otimes I^{q+1}/I^{q+2}$ and $x_3 \in \mathrm{Tor}^A_3(B, A/I^{q+3})$ such that $x_2 = \sigma_{3,q+1}(z_2) + \tau_{3,q+3}(x_3)$. Since $\tau_{3,q+2} \circ \sigma_{3,q+1} = \tau_{3,q+2} \circ i_{3,q+1} \circ \psi_{3,q+1} = 0$, then

$$x = \sigma_{3,q}(z) + \tau_{3,q+2}(\sigma_{3,q+1}(z_2) + \tau_{3,q+3}(x_3)) = \sigma_{3,q}(z) + (\tau_{3,q+2} \circ \tau_{3,q+3})(x_3)\,.$$

Recursively, there exist $z_{k-1} \in \mathbf{\Lambda}^B_3(I/I^2) \otimes I^{q+k-2}/I^{q+k-1}$ and $x_k \in \mathrm{Tor}^A_3(B, A/I^{q+k})$ such that $x_{k-1} = \sigma_{3,q+k-2}(z_{k-1}) + \tau_{3,q+k}(x_k)$, for each $k \geq 3$. Since $\tau_{3,q+k-1} \circ \sigma_{3,q+k-2} = 0$, then $x = \sigma_{3,q}(z) + (\tau_{3,q+2} \circ \ldots \circ \tau_{3,q+k})(x_k)$. Finally, since A is Noetherian, I is an ideal of Artin-Rees, that is, for each $p \geq 0$ and $q \geq 0$, there exists $k \geq 0$ such that $(\tau_{p,q+k} \circ \ldots \circ \tau_{p,q}) = 0$ (see 10.11 of [2] or 9.9 of [16]). Therefore, $x \in \mathrm{Im}\sigma_{3,q}$. ∎

Theorem 3.4 *Let A be a Noetherian ring, $A \supset \mathbb{Q}$. Let I be an ideal of A, $B = A/I$ and $\mathbf{G}(I)$ its associated graded ring. Then, the following conditions are equivalent:*

(i) $H_2(A, B, \mathbf{G}(I)) = 0$.

(ii) $\sigma_{p,q} : \mathbf{\Lambda}_p^B(I/I^2) \underset{B}{\otimes} I^q/I^{q+1} \to \mathrm{Tor}_p^A(B, A/I^{q+1})$ *is surjective for* $p = 2, 3$ *and* $q \geq 0$.

In particular, if (i) holds, then I is of linear type, $H_p(A, B, B) = 0$ and $\tau_{p,q} = 0$ for all $p = 2, 3$ and $q \geq 0$.

Proof. Consider the five term exact sequence associated to the fundamental spectral sequence (see 6, [16]) for the B-module $\mathbf{G}_q = I^q/I^{q+1}$:

$$\mathrm{Tor}_3^A(B, \mathbf{G}_q) \overset{h_{3,q}}{\to} H_3(A, B, \mathbf{G}_q) \to \mathbf{\Lambda}_2^B(I/I^2) \underset{B}{\otimes} \mathbf{G}_q \overset{\psi_{2,q}}{\to} \mathrm{Tor}_2^A(B, \mathbf{G}_q) \to H_2(A, B, \mathbf{G}_q) \to 0 \qquad (4)$$

We have $\mathrm{Coker}\psi_{2,q} = H_2(A, B, I^q/I^{q+1})$. Thus, the proof of the equivalence between (*i*) and (*ii*) follows from lemmas 3.1, 3.2 and 3.3. In particular, if $\sigma_{2,q}$ is surjective for all $q \geq 0$, then (by Remark 2.7) I is an ideal of linear type. Moreover, if $\sigma_{p,q} = i_{p,q} \circ \psi_{p,q}$ is surjective (for $p = 2, 3$ and all $q \geq 0$), then $i_{p,q}$ is surjective too and, hence, $\tau_{p,q+1} = 0$ (for $p = 2, 3$ and all $q \geq 0$). It remains to prove $H_3(A, B, B) = 0$ whenever (*i*) or (*ii*) hold. But if (*ii*) holds, then $\gamma_3 = \sigma_{3,0}$ is surjective. In particular, $\mathrm{Tor}_3^A(B, B)$ is made up of decomposable elements and, hence, the morphism $h_{3,0} = 0$ in the exact sequence (4). On the other hand, $\psi_{2,0} = \gamma_2$ is injective since $A \supset \mathbb{Q}$. ∎

Remark 3.5 If $H_2(A, B, \mathbf{G}(I)) = 0$, but A is not Noetherian, then $\sigma_{3,0}$ might not be surjective (see Example 3.3 of [15]).

Definition 3.6 Let A be a commutative ring, $A \supset \mathbb{Q}$. An ideal I of A will be called of *strong linear type* if $H_2(A, B, \mathbf{G}(I)) = 0$.

Remark that I of linear type does not imply I of strong linear type, as we see next for principal ideals.

Example 3.7 Let $I = (x)$ be a nonzero principal ideal of a commutative ring A, $B = A/I$. Then $H_2(A, B, B) = (0 : x) \cap I$ and $H_2(A, B, I^q/I^{q+1}) = (0 : x)/(0 : x) \cdot (0 : x^q)$ for all $q \geq 1$. In particular, if A is a Noetherian local ring, then $I = (x)$ is of strong linear type if and only if x is a nonzero divisor of A. On the other hand, I is of linear type if and only if x is a d-sequence.

Proof. By exact sequence (4) and via $0 \to (0 : x) \to A \to I \to 0$, free A-presentation of I, we have: $H_2(A, B, \mathbf{G}_q) = \mathrm{Tor}_2^A(B, \mathbf{G}_q) = \mathrm{Tor}_1^A(I, \mathbf{G}_q) = \mathrm{Ker}\,((0 : x) \otimes \mathbf{G}_q \to A \otimes \mathbf{G}_q)$. Thus, if $q = 0$, then $H_2(A, B, B) = \mathrm{Ker}\,((0 : x)/I \cdot (0 : x) \to A/I) = (0 : x) \cap I$. On the other hand, if $q \geq 1$, then $H_2(A, B, \mathbf{G}_q) = (0 : x) \otimes \mathbf{G}_q = (0 : x) \otimes I^q = \mathrm{Tor}_1^A(I, I^q) = \mathrm{Tor}_1^A(A/(0 : x), I^q)$ and, via the free A-presentation $0 \to (0 : x^q) \to A \to I^q \to 0$ of I^q, we finish. ∎

Except for principal ideals, in general, to be generated by a d-sequence is strictly stronger than to be of linear type (see, e.g., [8] or [19]). Even so, in general, to be generated by a d-sequence does not imply to be of strong linear type.

Example 3.8 Let $y_1 = x_1x_2$, $y_2 = x_1x_3$ and $y_3 = x_1x_4$ be three monomials on a regular sequence x_1, x_2, x_3, x_4 of a commutative ring A and let I be the ideal generated by the y_i. By Example 2.9 and Theorem 3.4, I is not of strong linear type although (it is easy to check) y_1, y_2, y_3 is a d-sequence.

Example 3.9 Let R be a regular local ring and let M be a finitely generated R-module of projective dimension $\mathrm{pd}(M) = 2$. Let $A = \mathbf{S}(M)$ be the symmetric algebra of M, $I = \mathbf{S}_+(M)$ the irrelevant ideal of A, $B = A/I = R$. Then, I is an ideal of linear type, but not of strong linear type.

Proof. By 2.3 of [8], I is of linear type. In particular, and using Lemma 4.1 of [14],

$$H_2(A, B, \mathbf{G}(I)) = \mathrm{Tor}_1^B(I/I^2, \mathbf{G}(I)) = \mathrm{Tor}_1^R(M, \mathbf{S}(M)) .$$

By 4.2.2 of [20] and since $\mathrm{pd}(M) > 1$, $\mathrm{Tor}_1^R(M, \mathbf{S}(M))$ can not be zero. ∎

As a corollary of Theorem 3.4 we have the following:

Proposition 3.10 *Let I be an ideal of a commutative ring A. If I is an ideal of linear type and its flat dimension is $\mathrm{fd}(I) \leq 1$, then I is an ideal of strong linear type. Moreover, the converse is true when A is regular local, $A \supset \mathbb{Q}$, and $\mathrm{Tor}_1^A(I, I) = 0$.*

Proof. Since I is of linear type, $\sigma_{2,q}$ is surjective for all $q \geq 0$ (see Remark 2.7). Since $\mathrm{fd}(I) \leq 1$, then $\mathrm{Tor}_3^A(B, \cdot) = 0$ and, in particular , $\sigma_{3,q}$ is surjective for all $q \geq 0$. Thus, by Theorem 3.4, $H_2(A, B, \mathbf{G}(I)) = 0$. Conversely, if I is of strong linear type, then, by Corollary 2.8 and with its notations, $N_2 \cap I^{q+1}Y_2 = I^q f(B_2) + I^{q+1}N_2$ for all $q \geq 0$. Since $\mathrm{Tor}_3^A(B, B) = 0$, then $f(B_2) \subset N_2 \cap IY_2 = IN_2$ and, therefore, $I^q f(B_2) \subset I^{q+1}N_2$. Thus, $N_2 \cap I^{q+1}Y_2 = I^{q+1}N_2$ and hence, $\mathrm{Tor}_3^A(B, A/I^{q+1}) = 0$ for all $q \geq 0$. We finish by using again 4.2.2 of [20]. ∎

Corollary 3.11 *Let A be a Noetherian ring, $A \supset \mathbb{Q}$, of Krull dimension $\dim(A) \leq 3$. Let I be a perfect ideal of A. Then, I is of strong linear type if and only if I is of linear type.*

Proof. Since I is perfect, $\mathrm{pd}(B) = \mathrm{grade}(I) \leq \mathrm{ht}(I) \leq 3$, where $\mathrm{ht}(I)$ stands for the height of I. If $\mathrm{grade}(I) \leq 2$, then $\mathrm{pd}(I) \leq 1$ and (by Proposition 3.10) I is of strong linear type. If $\mathrm{grade}(I) = \mathrm{ht}(I) = 3$, then I is generated by a regular sequence since I of linear type implies the minimal number of generators of I is $\mu(I) \leq 3$ (see, e.g., 2.4 of [8]). ∎

4 Licci ideals of strong linear type

In this section A will be a Cohen-Macaulay local ring, I will be a proper ideal of A and $B = A/I$. The difference $\mathrm{d}(I) = \mu(I) - \mathrm{grade}(I)$ between the minimal number of generators of I and its grade is called the deviation of I, and I is said to be a complete intersection if $\mathrm{d}(I) = 0$.

Definition 4.1 ([13],[17]) Two proper ideals I and J are said to be *linked* (write $I \sim J$ or $I \underset{x}{\sim} J$) if there is a regular sequence $x = x_1, \dots x_g$ (allowing $\{x\} = \emptyset$) contained in $I \cap J$ such that $J = (x) : I$ and $I = (x) : J$.

It is said that I is *licci* (in the linkage class of a complete intersection) if there is a sequence of links $I = I_0 \sim I_1 \sim \dots \sim I_n$ where I_n is a complete intersection.

As a corollary of the methods used in Section 3 and the work done by Ulrich in [17] on a conjecture of Herzog (see [7]), we are able to prove the following:

Theorem 4.2 *Let A be a regular local ring, $A \supset \mathbb{Q}$. Let I be a licci ideal of A, $B = A/I$. Assume that B is Gorenstein. Then, the following conditions are equivalent:*

(i) I is an ideal of strong linear type.

(ii) $H_2(A, B, B) = 0$, $H_2(A, B, I/I^2) = 0$ and $\tau_{3,2} = 0$.

(iii) I is complete intersection.

Proof. By Theorem 2.20 of [17], we have only to prove that (ii) implies $H_3(A, B, B) = 0$. Since $\mathrm{Coker}\psi_{2,q} = H_2(A, B, I^q/I^{q+1})$, by hypothesis (ii), $\psi_{2,0}$ and $\psi_{2,1}$ are surjective. Thus, like in proof of Lemma 3.2, we can assure that the bootom row of Quillen's diagram QD_3 is exact. Since $\tau_{3,2} = 0$, then $\mathrm{d}_{3,0}$ is injective. Moreover, $\psi_{1,2}$ is always injective. With all that hypotheses on QD_3, one can easily check that γ_3 is surjective. We finish like at the end of proof of Theorem 3.4. ∎

For perfect ideals of small grade, we have:

Corollary 4.3 *Let A be a regular local ring, $A \supset \mathbb{Q}$, and I a perfect ideal of* $\mathrm{grade}(I) = g$.

(a) If $g = 2$, then I is an ideal of strong linear type if and only if I is of linear type.

(b) If $g = 3$ and I is Gorenstein, then I is an ideal of strong linear type if and only if I is a complete intersection. Moreover, if $I^2 : I = I$ and I is generically (i.e., locally at each of its associated primes) a complete intersection, then I being of strong linear type, I being a complete intersection and I being such that $H_2(A, B, I/I^2) = 0$, are three equivalent conditions.

Proof. (a) follows from Proposition 3.10. If I is a perfect ideal of grade 3 and A/I is Gorenstein, then (by a theorem of Watanabe [23]) I is licci. Thus, the first part of (b) follows from Theorem 4.2. For the proof of the second part of (b), the Buchsbaum-Eisenbud free resolution of B (see [6]) assures that there exists an exact sequence of A-modules of the form

$$0 \to A \xrightarrow{f_3} A^n \xrightarrow{f_2} A^n \xrightarrow{f_1} A \to B \to 0\,,$$

where $f_3(1) = (y_1, \ldots, y_n)$ with $y_1, \ldots, y_n$ a system of generators of the ideal I. This allows us to prove that $\mathrm{Im}\tau_{3,2} = (I^2 : I)/I$. Moreover, a licci ideal generically a complete intersection is syzygetic, thus $H_2(A, B, B) = 0$. We finish by applying again Theorem 4.2. ∎

For licci ideals of any grade, but deviation 1 we have:

Theorem 4.4 *Let A be a Gorenstein local ring, $A \supset \mathbb{Q}$, and I a licci, generically a complete intersection ideal of A of deviation* $\mathrm{d}(I) = 1$. *Then, I is an ideal of strong linear type.*

Proof. Since $\mathrm{d}(I) = 1$ and I is generically a complete intersection, then I is of linear type and the approximation complex $\mathcal{M}(I, A)$ associated to a free presentation of I, $f : F \to A$, gives rise to $0 \to H_1(I) \otimes_A \mathbf{S}(F) \to \mathbf{S}^B(F/IF) \to \mathbf{G}(I) \to 0$, exact sequence of B-modules, where $H_1(I)$ stands for the first Koszul homology group associated to a generating set of I and $B = A/I$ (see page 108 of [8]). Applying $H_*(A, B, \cdot)$, we get

$$\ldots \to H_2(A, B, \mathbf{G}(I)) \to H_1(A, B, H_1(I) \underset{A}{\otimes} \mathbf{S}(F)) \to H_1(A, B, \mathbf{S}^B(F/IF)) \to \ldots$$

Since I is syzygetic, $H_2(A, B, \mathbf{S}^B(F/IF)) = 0$. Since $B = A/I$, then $H_1(A, B, \cdot) = I/I^2 \otimes \cdot$. Thus, $0 \to H_2(A, B, \mathbf{G}(I)) \to I \otimes H_1(I) \otimes \mathbf{S}(F) \to I \otimes \mathbf{S}^B(F/IF)$ is exact. In particular, any associated prime to $H_2(A, B, \mathbf{G}(I))$ is an associated prime to $I \otimes H_1(I)$. Since A is Gorenstein local and $\mathrm{d}(I) = 1$, then the canonical module of B exists and is $\omega_B \simeq H_1(I)$ (see 3.1 of [18]). Moreover, since I is licci and by 5.1, [18], then $I \otimes H_1(I) = I \otimes \omega_B$ is a torsion free B-module. In particular, and using B is Cohen-Macaulay, we have

$$Ass\,(H_2(A, B, \mathbf{G}(I))) \subseteq Ass\,(I \otimes H_1(I)) \subseteq Min(B)\,.$$

Using I is generically a complete intersection we deduce $H_2(A, B, \mathbf{G}(I)) = 0$. ∎

Next three examples show that we can not avoid any of the three hypotheses on the ideal I in Theorem 4.4 ($B = A/I$ as always).

Example 4.5 Let $(A, \mathfrak{m})$ be a regular local ring of $\dim(A) = 2$ and $I = \mathfrak{m}^2$. Then, I is licci, $\mathrm{d}(I) = 1$, but not generically a complete intersection and $H_2(A, B, B) \neq 0$.

Example 4.6 There exists a regular local ring $(A, \mathfrak{m})$ such that it contains a perfect ideal I, generically a complete intersection with $\mathrm{d}(I) = 1$ and generated by a d-sequence, but

$H_2(A,B,I/I^2) \neq 0$. Indeed, take k an infinite field of characteristic zero, $X = (X_{i,j})$ a generic 3×3 matrix and $A = k[\![X]\!]$ the formal power series ring in the variables $X_{i,j}$. Let $\mathfrak{m}$ denote the maximal ideal of A and $J = I_2(X)$ the ideal of A generated by the 2×2 minors of X. It is known (see, e.g. 2.5, [10]) that J is a prime, Gorenstein ideal of grade 4. In particular, $J \otimes \omega_{A/J} = J \otimes A/J = J/J^2$ which is known to have nonzero torsion (see 10.13, [5]). Moreover, $J_{\mathfrak{p}}$ is a complete intersection for all prime $\mathfrak{p} \supseteq J$, $\mathfrak{p} \neq \mathfrak{m}$ (see 2.A (ii) of [5]). Choose now a regular sequence $y = y_1, y_2, y_3, y_4$ contained in J and take $I = (y) : J$. In particular, I is linked with J and, hence, I is a perfect ideal of grade 4 and $\mathrm{d}(I) = 1$ (see [18]). Moreover, y can be chosen in such a way that $I_{\mathfrak{p}}$ is a complete intersection for all prime $\mathfrak{p} \supseteq I$, $\mathrm{ht}(\mathfrak{p}/I) \leq 1$ (see, e.g. 5 of [21]). In particular, since I is generically a complete intersection and $\mathrm{d}(I) = 1$, then I can be generated by a d-sequence (see [8]). Moreover, $\mu(I_{\mathfrak{p}}) = \mathrm{ht}(I_{\mathfrak{p}}) = 4 < 5$ and the analytic spread of $I_{\mathfrak{p}}$ is $l(I_{\mathfrak{p}}) < \mathrm{ht}(\mathfrak{p})$ for all prime $\mathfrak{p}$ with $\mathrm{ht}(\mathfrak{p}/I) = 1$. Then (by 2.5 of [9]) $I^n = I^{(n)}$ for all $n \geq 1$. Since I is generically a complete intersection and I/I^2 is torsion free, then (by 2.2 of [17]) $H_2(A,B,I/I^2) = T(H_1(I) \otimes I) = T(I \otimes \omega_{A/I})$, where $T(\cdot)$ stands for the torsion. Since $I \sim J$ and $T(J \otimes \omega_{A/J}) = T(J/J^2) \neq 0$, then (by 5.1 of [18]) $T(I \otimes \omega_{A/I}) \neq 0$. Therefore, $H_2(A,B,I/I^2) \neq 0$.

Example 4.7 Let X and Y be two generic matrices, 2×3 and 3×1, respectively. Let $A = k[X,Y]_{(X,Y)}$ and $I = I_2(X) + I_1(XY)$, where $I_t(Z)$ denotes the ideal generated by the $t \times t$ minors of a matrix Z. It is proved (2.3, [10]) that I is a prime, Gorenstein ideal of grade 3, of linear type and not a complete intersection (in fact $\mu(I) = 5$ and $\mathrm{d}(I) = 2$). In particular (by Corollary 4.3) $H_2(A,B,I/I^2) \neq 0$ although I is licci of linear type.

We finish by generalizing Corollary 3.11 to rings of Krull dimension 4.

Proposition 4.8 *Let A be a Gorenstein local ring, $A \supset \mathbb{Q}$, with $\dim(A) = 4$. Let I be a perfect ideal of A. Then, I is an ideal of strong linear type if and only if I is of linear type.*

Proof. If I is a perfect ideal of linear type, then $\mathrm{pd}(B) = \mathrm{grade}(I) \leq \mu(I) \leq 4$. If $\mathrm{grade}(I) \leq 2$, then (by Proposition 3.10) I is of strong linear type. If $\mathrm{grade}(I) = \mu(I) = 3$ or $\mathrm{grade}(I) = \mu(I) = 4$, then I is a complete intersection. Thus, it remains the case $\mathrm{grade}(I) = 3$ and $\mu(I) = 4$. Since I is of linear type and B is Cohen-Macaulay, I is generically a complete intersection. Since $\mathrm{d}(I) = 1$ and the residual field is infinite, we can choose $x = x_1, x_2, x_3$, a regular sequence, part of a minimal system of generators of I. Then (by 2.4 and 2.7 of [18]) $J = (x) : I$ is a perfect ideal of grade 3 and A/J is Gorenstein. In particular, (by [23]) J is licci and hence, I is licci too. We finish by applying Theorem 4.4 to I. ∎

Remark that Proposition 4.8 is not true in dimension 5.

Example 4.9 Let $A = k[\![x_1, \ldots, x_5]\!]$ and $I = (x_1x_2, x_2x_3, x_3x_4, x_4x_5, x_5x_1)$. Then, I is a grade 3, Gorenstein ideal of linear type with $\mathrm{d}(I) = 2$ and $I^2 : I = I$ (see [22]). In particular, by Corollary 4.3, $H_2(A,B,I/I^2) \neq 0$.

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