On the module of effective relations of a standard algebra

By Francesc Planas-Vilanova

Dept. Matemàtica Aplicada I. ETSEIB-UPC. Diagonal 647, E-08028 Barcelona.

1 Introduction

Let A be a commutative ring. We denote by a standard A-algebra a commutative graded A-algebra $U = \oplus_{n\geq 0}U_n$ with $U_0 = A$ and such that U is generated as an A-algebra by the elements of U_1. Take $\underline{x}$ a (possibly infinite) set of generators of the A-module U_1. Let $V = A[\underline{t}]$ be the polynomial ring with as many variables $\underline{t}$ (of degree one) as $\underline{x}$ has elements and let $f : V \to U$ be the graded free presentation of U induced by the $\underline{x}$. For $n \geq 2$, we will call *module of effective n-relations* the A-module $E(U)_n = \ker f_n / V_1 \cdot \ker f_{n-1}$. The minimum positive integer $r \geq 1$ such that the effective n-relations are zero for all $n \geq r+1$ is known to be an invariant of U. It is called the relation type of U and is denoted by $\mathrm{rt}(U)$. For an ideal I of A, we define $E(I)_n = E(\mathcal{R}(I))_n$ and $\mathrm{rt}(I) = \mathrm{rt}(\mathcal{R}(I))$, where $\mathcal{R}(I) = \oplus_{n\geq 0} I^n t^n \subset A[t]$ is the Rees algebra of I.

In this paper, we give two descriptions of the A-module of effective n-relations. In terms of André-Quillen homology we have that $E(U)_n = H_1(A, U, A)_n$ (see 2.3). It turns out that this module does not depend on the chosen $\underline{x}$. In terms of Koszul homology we prove that $E(U)_n = H_1(\underline{x}; U)_n$ (see 2.4). Using these characterizations, we show later some properties on the module of effective n-relations and the relation type of a graded algebra. Meanwhile, our line of disquisition approaches us to several earlier works on the subject (see [2], [5], [6], [7], [9], [10], [13] and [14]).

Section 2 is devoted to state the above mentioned (co)homological characterizations of the A-module of effective n-relations and compare them with some already known results. In section 3, we give some applications. The interest is specially centered on the module of n-relations of powers of an ideal and the module of n-relations of Veronese subrings. In particular, one concludes that $\mathrm{rt}(U^{(p)}) \leq \mathrm{rt}(U_+^p)$ but, in general, $\mathrm{rt}(U^{(p)}) \neq \mathrm{rt}(U_+^p)$, where $U_+ = \oplus_{n>0}U_n$ is the irrelevant ideal of U and $U^{(p)} = \oplus_{n\geq 0}U_{np}$ is the p-th Veronese subring of U (see 3.12). Finally, in section 4 we characterize, in terms of a system of generators, which ideals have module of effective n-relations zero. In particular, a new characterization of sequences of linear type is obtained.

e-mail: planas@ma1.upc.es

2 Homological description of effective relations

Let $U = \oplus_{n\geq 0} U_n$ be a standard A-algebra. Put $U_+ = \oplus_{n>0} U_n$ its irrelevant ideal. If $E = \oplus_{n\geq 1} E_n$ is a graded U-module and $r \geq 1$, we denote by $F_r(E)$ the submodule of E generated by the elements of degree at most r. Put (possibly infinite)

$$\mathrm{s}(E) = \min\{r \geq 1 \mid E_n = 0 \,\text{for all}\, n \geq r+1\}\,.$$

Since $(E/U_+E)_n = E_n/U_1E_{n-1}$, then the following three conditions are equivalent: $F_r(E) = E$, $\mathrm{s}(E/U_+E) \leq r$, and, $E_n = U_1E_{n-1}$ for all $n \geq r+1$.

Given $h : W \to U$, a surjective graded morphism of standard A-algebras, we are interested in the graded A-module $E(h) = \ker h / W_+ \cdot \ker h$. The following is an elementary, but useful lemma:

Lemma 2.1 *Let $f : V \to U$ and $g : W \to V$ be two surjective graded morphisms of standard A-algebras. Then, there exists a graded exact sequence of A-modules:*

$$E(g) \to E(f \circ g) \xrightarrow{g} E(f) \to 0\,. \tag{1}$$

In particular, $\mathrm{s}(E(f)) \leq \mathrm{s}(E(f \circ g)) \leq \max(\mathrm{s}(E(f)), \mathrm{s}(E(g)))$. Moreover, if V and W are two symmetric algebras, then $E(g)_n = 0$ and $E(f \circ g)_n = E(f)_n$ for all $n \geq 2$.

Proof. Exact sequence (1) follows from the snake lemma applied to the commutative diagram:

$$\begin{array}{ccccccccc}
 & & W_1 \otimes \ker g_{n-1} & \longrightarrow & W_1 \otimes \ker(f\circ g)_{n-1} & \xrightarrow{1\otimes g_{n-1}} & W_1 \otimes \ker f_{n-1} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker g_n & \longrightarrow & \ker(f\circ g)_n & \xrightarrow[g_n]{} & \ker f_n & \longrightarrow & 0
\end{array}$$

Moreover, if $W = \mathbf{S}(W_1)$ and $V = \mathbf{S}(V_1)$, then $\ker g = F_1(\ker g)$. ∎

Definition 2.2 Let U be a standard A-algebra and let $\alpha : \mathbf{S}(U_1) \to U$ be the graded morphism of standard A-algebras induced by the identity on U_1. Given $n \geq 2$, the *module of effective n-relations* of U is defined to be $E(U)_n = \ker\alpha_n / U_1 \cdot \ker\alpha_{n-1}$. Put $E(U) = \oplus_{n\geq 2} E(U)_n = \ker\alpha / \mathbf{S}_+(U_1) \cdot \ker\alpha$. Then, the *relation type* of U is defined to be $\mathrm{rt}(U) = \mathrm{s}(E(U))$. Remark that if $h : W \to U$ is any *symmetric presentation of* U, that is, W is a symmetric algebra and h is a surjective graded morphism of standard A-algebras, then h can be factored into $h = f \circ g$, where $g : \mathbf{S}(W_1) \to \mathbf{S}(U_1)$ is the induced morphism by $h_1 : W_1 \to U_1$ and $f = \alpha$. Thus, applying Lemma 2.1, $E(U)_n = E(h)_n$ for all $n \geq 2$ and $\mathrm{s}(E(U)) = \mathrm{s}(E(h))$. If I is an ideal of A, the *module of effective n-relations* of I is $E(I)_n = E(\mathcal{R}(I))_n$ and the *relation type* of I is $\mathrm{rt}(I) = \mathrm{rt}(\mathcal{R}(I))$, where $\mathcal{R}(I) = \oplus_{n\geq 0} I^n t^n$ is the Rees algebra of I. An ideal with module of effective 2-relations zero is called *syzygetic.* An ideal of relation type 1 is called of *linear type* (see, e.g., [8]).

Remark 2.3 In fact, sequence (1) is part of a long exact sequence of André-Quillen homology. Indeed, the Jacobi-Zariski sequence associated to the morphisms $g : W \to V$ and $f : V \to U$, with respect to the U-module $A = U/U_+$, gives rise to

$$\ldots \to H_1(W, V, A) \to H_1(W, U, A) \to H_1(V, U, A) \to H_0(W, V, A) \to \ldots .$$

Using $H_1(A, A/I, M) = I/I^2 \otimes M$ and $H_0(A, A/I, M) = 0$ for any ideal I of A and any A/I-module M, we get (1) (see [1]).

On the other hand, the Jacobi-Zariski sequence associated to the morphisms $A \to \mathbf{S}(U_1)$ and $\alpha : \mathbf{S}(U_1) \to U$, with respect to the U-module A, is

$$\ldots \to H_1(A, \mathbf{S}(U_1), A) \to H_1(A, U, A) \to H_1(\mathbf{S}(U_1), U, A) \to H_0(A, \mathbf{S}(U_1), A) \to \ldots .$$

Using $H_1(A, \mathbf{S}(U_1), A) = 0$ and $H_0(A, \mathbf{S}(U_1), A) = H_0(A, U, A)$, we get the graded isomorphism of A-modules $H_1(A, U, A) = H_1(\mathbf{S}(U_1), U, A) = \ker\alpha / \mathbf{S}_+(U_1) \cdot \ker\alpha$. Thus, $H_1(A, U, A)_n = E(U)_n$ is the module of effective n-relations of U. In particular, $\mathrm{rt}(U) = \mathrm{s}(H_1(A, U, A))$.

There is also a description of the module of effective n-relations in terms of Koszul (co)homology. Let $f : V \to U$ be a surjective graded morphism of standard A-algebras. For each $p \geq 1$, consider the map $V_p \otimes U \to U$ sending $x \otimes y$ to $f_p(x)y$ and let $\mathcal{K}(f, p)$ be the Koszul complex associated to this U-linear form (see 1.6.1 of [3]). Since it is an homogeneous form of degree zero, $\mathcal{K}(f, p)$ is a complex of graded U-modules having differentials homogeneous morphisms of degree zero. Concretely, $\mathcal{K}(f, p) = \oplus_{n \geq 0} \mathcal{K}(f, p)_n$ where $\mathcal{K}(f, p)_n$ is the following subcomplex ($U_n = 0$ for $n < 0$):

$$\ldots \longrightarrow \mathbf{\Lambda}_2^A(V_p) \otimes_A U_{n-2p} \xrightarrow{\partial_2} V_p \otimes_A U_{n-p} \xrightarrow{\partial_1} U_n \longrightarrow 0 \,,$$

where $\partial_q((x_1 \wedge \ldots \wedge x_q) \otimes y) = \sum_{i=1}^{q} (-1)^{i-1} x_1 \wedge \ldots \wedge \widehat{x}_i \wedge \ldots \wedge x_q \otimes f_p(x_i)y$, for all $x_i \in V_p$ and $y \in U_{n-qp}$. In particular, for every $q \geq 0$, $H_q(\mathcal{K}(f, p))$ is a graded A-module with $H_q(\mathcal{K}(f, p))_n = H_q(\mathcal{K}(f, p)_n)$.

Theorem 2.4 *Let $f : V \to U$ and $g : W \to V$ be two surjective graded morphisms of standard A-algebras. Let $\alpha : \mathbf{S}(U_1) \to U$ be the canonical morphism and suppose W is a symmetric algebra. Given $(n \geq 2, p = 1)$ or $(n \geq 2p + 1, p \geq 2)$, there are isomorphisms of A-modules*

$$H_1(\mathcal{K}(f, p)_n) = \frac{\ker(f \circ g)_n}{W_p \cdot \ker(f \circ g)_{n-p}} = \frac{\ker\alpha_n}{\mathbf{S}_p(U_1) \cdot \ker\alpha_{n-p}} \,.$$

In particular, the module of effective n-relations of U is $E(U)_n = H_1(\mathcal{K}(f, 1)_n)$ and the relation type of U is $\mathrm{rt}(U) = \mathrm{s}(H_1(\mathcal{K}(f, 1)))$.

Proof. Put $h = f \circ g$. Since $n - p \geq p$, then $W_{n-p} \cdot \ker g_p \subset W_p \cdot \ker g_{n-p} \subset W_p \cdot \ker h_{n-p}$. Applying the snake lemma to the commutative diagram of exact rows

$$
\begin{array}{ccccccccc}
 & & \ker g_p \otimes W_{n-p} \oplus W_p \otimes \ker h_{n-p} & \longrightarrow & W_p \otimes W_{n-p} & \xrightarrow{g_p \otimes h_{n-p}} & V_p \otimes U_{n-p} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \ker h_n & \longrightarrow & W_n & \xrightarrow[h_n]{} & U_n & \longrightarrow & 0
\end{array}
$$

we get the exact sequence of A-modules

$$0 \to (g_p \otimes h_{n-p})(\mathcal{Z}_1(1_W, p)_n) \to \mathcal{Z}_1(f, p)_n \to \ker h_n / W_p \cdot \ker h_{n-p} \to 0\,,$$

where $\mathcal{Z}_1(1_W, p)_n$, $\mathcal{Z}_1(f, p)_n$ stand for the n-th component of the 1-cycles module of $\mathcal{K}(1_W, p)$, $\mathcal{K}(f, p)$. If $\mathcal{Z}_1(1_W, p)_n = \mathcal{B}_1(1_W, p)_n$ (the n-th component of the 1-boundaries module of $\mathcal{K}(1_W, p)$), then $(g_p \otimes h_{n-p})(\mathcal{Z}_1(1_W, p)_n) = \mathcal{B}_1(f, p)_n$ Thus, the first isomorphism is demonstrated provided we prove $H_1(\mathcal{K}(1_W, p))_n = 0$ for a symmetric algebra W (see next lemma). In particular, if we take $V = U$ and $f = 1_U$, then $h = f \circ g = g$ and one of the possible choices of h is the canonical morphism α. Hence, applying twice the first equality to α and to any $h : W \to U$ arising from a symmetric algebra W, we have

$$H_1(\mathcal{K}(1_U, p)_n) = \frac{\ker \alpha_n}{\mathbf{S}_p(U_1) \cdot \ker \alpha_{n-p}} = \frac{\ker h_n}{W_p \cdot \ker h_{n-p}}\,. \blacksquare$$

Lemma 2.5 *Let M be an A-module and $W = \mathbf{S}(M)$ the symmetric algebra of M. Then, for $(n \geq 1, p = 1)$ or $(n \geq 2p + 1, p \geq 2)$, $H_1(\mathcal{K}(1_W, p))_n = 0$.*

Proof. Put $\mathbf{T}(M)$ the tensorial algebra of M and $q = n - p$. Applying the snake lemma to the commutative diagram of exact rows

$$
\begin{array}{ccccccccc}
 & & & & \mathbf{T}_p(M) \otimes \mathbf{T}_q(M) & = & \mathbf{T}_n(M) & \longrightarrow & 0 \\
 & & & & \upsilon \downarrow & & \downarrow \varepsilon & & \\
0 & \longrightarrow & \ker \omega & \longrightarrow & W_p \otimes W_q & \xrightarrow[\omega]{} & W_n & \longrightarrow & 0
\end{array}
$$

we get the exact sequence $0 \to \ker \upsilon \to \ker \varepsilon \xrightarrow{\upsilon} \ker \omega \to 0$. Thus, $\mathcal{Z}_1(1_W, p)_n = \ker \omega = \upsilon(\ker \varepsilon)$ is the A-module generated by the elements

$$(x_1 \cdots x_{p-1} x_p) \otimes (y_1 y_2 \cdots y_q) - (x_1 \cdots x_{p-1} y_1) \otimes (x_p y_2 \cdots y_q)\,,$$

where $x_i, y_j \in M$ and $x_1 \cdots x_p$ stands for the product in $W = \mathbf{S}(M)$. Clearly, if $(n \geq 1$, $p = 1)$, then $\mathcal{Z}_1(1_W, p)_n = \mathcal{B}_1(1_W, p)_n$. Suppose $(n \geq 2p + 1$, $p \geq 2)$, i.e., $q > p$. Then, $H_1(\mathcal{K}(1_W, p)_n) = 0$ follows from the equality:

$$
\begin{aligned}
&(x_1 \cdots x_p) \otimes (y_1 \cdots y_q) - (x_1 \cdots x_{p-1} y_1) \otimes (x_p y_2 \cdots y_q) = \\
&(x_1 \cdots x_p) \otimes (y_1 \cdots y_q) - (y_2 \cdots y_{p+1}) \otimes (x_1 \cdots x_p y_1 y_{p+2} \cdots y_q) + \\
&(y_2 \cdots y_{p+1}) \otimes (x_1 \cdots x_{p-1} y_1 x_p y_{p+2} \cdots y_q) - (x_1 \cdots x_{p-1} y_1) \otimes (x_p y_2 \cdots y_q). \blacksquare
\end{aligned}
$$

Remark 2.6 Let $f:\mathbf{S}(F)\to\mathbf{S}(M)$ be the induced morphism on the symmetric algebras by an epimorphism $\pi: F\to M$ of A-modules. Then, the last three nonzero terms of $\mathcal{K}(f,p)_{p+q}$, $q\geq p\geq 1$, define the sequence:

$$\mathbf{\Lambda}_2^A(\mathbf{S}_p(F))\otimes_A \mathbf{S}_{q-p}(M)\stackrel{\partial_2}{\to}\mathbf{S}_p(F)\otimes_A\mathbf{S}_q(M)\stackrel{\partial_1}{\to}\mathbf{S}_{p+q}(M)\to 0\,, \tag{2}$$

with $\partial_2((x_1\cdots x_p)\wedge(y_1\cdots y_p)\otimes z)=(y_1\cdots y_p)\otimes f(x_1\cdots x_p)z-(x_1\cdots x_p)\otimes f(y_1\cdots y_p)z$ and $\partial_1((x_1\cdots x_p)\otimes t)=f(x_1\cdots x_p)t$, $x_i,y_j\in F$, $z\in\mathbf{S}_{q-p}(M)$ and $t\in\mathbf{S}_q(M)$.

On the other hand, Micali and Roby defined (in [10]) the sequence of A-modules

$$\mathbf{T}^A_{p+q}(F)\stackrel{\lambda}{\to}\mathbf{S}_p(F)\otimes_A\mathbf{S}_q(M)\stackrel{\mu}{\to}\mathbf{S}_{p+q}(M)\to 0\,, \tag{3}$$

with $\lambda(x_1\otimes\ldots\otimes x_{p+q})=(x_1\cdots x_p)\otimes f(x_{p+1}\cdots x_{p+q})-(x_1\cdots x_{p-1}x_{p+1})\otimes f(x_px_{p+2}\cdots x_{p+q})$ and $\mu=\partial_1$. By a similar argument to that one of the end of Lemma 2.5, one can prove that $\mathrm{Im}\partial_2$ is always contained in $\mathrm{Im}\lambda$ and that if $q>p$, then both modules are equal. Thus, the exactness of (2) (settled by Theorem 2.4 either for $q\geq p=1$ or either for $q>p\geq 2$) assures the exactness of (3). Nevertheless, if $q=p\geq 2$, then (2) might not be exact (see proof of Lemma 3.8) while (3) is always exact (see [10]).

Corollary 2.7 *Let U be a standard A-algebra and let $\underline{x}$ be a (possibly infinite) set of forms of degree one generating U_+. If $H_1(\underline{x};U)$ denotes the first Koszul homology group associated to $\underline{x}$, then $E(U)_n=H_1(\underline{x};U)_n$ for all $n\geq 2$. In particular,* $\mathrm{rt}(U)=\mathrm{s}(H_1(\underline{x};U))$.

Proof. Take in Theorem 2.4, $f:\mathbf{S}(F)\to U$ induced by a free presentation $F\to U_1$ associated to $\underline{x}$. Then, $\mathcal{K}(f,1)=\mathcal{K}(\underline{x};U)$ is the usual Koszul complex associated to the elements $\underline{x}$. ∎

Remark 2.8 Using duality between Koszul homology and cohomology (see 1.6.10 of [3]) we recover Schenzel's result $\mathrm{rt}(U)=\mathrm{s}(H^{d-1}(\underline{x};U))+d$, when $\underline{x}=x_1,\ldots,x_d$ has d elements (see [13]).

Remark 2.9 Let I be an ideal of A and $\mathcal{R}(I)=\oplus_{n\geq 0}I^nt^n$ its Rees algebra. Take $f=1_{\mathcal{R}}$, the identity on $\mathcal{R}(I)$, in Theorem 2.4. Then,

$$\mathcal{Z}_1(f,p)_n=\ker\left(I^p\otimes I^{n-p}\to I^n\right)=\mathrm{Tor}_1^A(A/I^p,I^{n-p})\,,$$

which is known to be isomorphic to $Z_1\cap I^{n-p}F/I^{n-p}Z_1$, where $0\to Z_1\to F\to I^p\to 0$ is a presentation of I^p with F free (see, e.g., 2.5 of [8]). Moreover, via the same isomorphism

$$\mathcal{B}_1(f,p)_n=\mathrm{Im}\left(\mathbf{\Lambda}_2^A(I^p)\otimes I^{n-2p}\to I^p\otimes I^{n-p}\right)=I^{n-2p}B_1/I^{n-p}Z_1\,.$$

Thus, by Theorem 2.4, we have

$$H_1(f,p)_n=\frac{\ker\alpha_n}{\mathbf{S}_p(I)\cdot\ker\alpha_{n-p}}=\frac{Z_1\cap I^{n-p}F}{I^{n-2p}B_1}\,,$$

which reproves an earlier result of Kühl (see 1.2 of [9]).

3 Some applications

The purpose of this section is to give some applications of Lemma 2.1 and Theorem 2.4.

Example 3.1 Cyclic standard algebras Let U be a cyclic standard A-algebra generated by a degree one form $x \in U_1$. Put $f : A[t] \to U$ with $f(t) = x$ in Theorem 2.4. Then, $E(U)_n = H_1(\mathcal{K}(f,1)_n) = (0 : x) \cap U_{n-1}$ and $\mathrm{rt}(U) = \min\{r \geq 1 \mid (0 : x^{r+1}) = (0 : x^r)\}$.

Example 3.2 Change of base ring Let U be a standard A-algebra and let $\varphi : A \to B$ be a homomorphism of rings. Take $f : V \to U$ any surjective graded morphism of standard A-algebras in Theorem 2.4. It induces $f \otimes 1 : V \otimes_A B \to U \otimes_A B$. Since $\mathcal{K}(f \otimes 1, p)_n = \mathcal{K}(f,p)_n \otimes_A B$, one can deduce $\mathrm{rt}(U \otimes_A B) \leq \mathrm{rt}(U)$. If φ is flat, then $H_1(\mathcal{K}(f \otimes 1, p)_n) = H_1(\mathcal{K}(f,p)_n) \otimes_A B$. In particular, $\mathrm{rt}(U) = \sup\{\mathrm{rt}(U_\mathfrak{p}) \mid \mathfrak{p} \in \mathrm{Spec}(A)\}$. If φ is faithfully flat, then $\mathrm{rt}(U \otimes_A B) = \mathrm{rt}(U)$. In particular, via the Nagata morphism $A \to A[t]_{\mathfrak{m}[t]} = B$, $\mathfrak{m}$ a maximal ideal of A, one can always suppose, when calculating the relation type of U, that A is a local ring of maximal $\mathfrak{m}$ and residual field $A/\mathfrak{m} = k$ infinite.

Let I be an ideal of A and $\mathcal{G}(I) = \oplus_{n \geq 0} I^n/I^{n+1}$ its associated graded ring. Since $\mathcal{G}(I) = \mathcal{R}(I) \otimes_A A/I$, then (by 3.2) $\mathrm{rt}(\mathcal{G}(I)) \leq \mathrm{rt}(\mathcal{R}(I)) = \mathrm{rt}(I)$. In [14], Valla showed that if $\mathrm{rt}(\mathcal{G}(I)) = 1$, then $\mathrm{rt}(I) = 1$ too. Next proposition is a generalization of that result.

Proposition 3.3 *If I is an ideal, there exists $E(I)_{n+1} \to E(I)_n \to E(\mathcal{G}(I))_n \to 0$, exact sequence of A-modules, for all $n \geq 2$. In particular, if $\mathrm{rt}(I) < \infty$, then $\mathrm{rt}(\mathcal{G}(I)) = \mathrm{rt}(I)$.*

Proof. If $1_\mathcal{R}$, $1_\mathcal{G}$, denote the identity on $\mathcal{R}(I)$, $\mathcal{G}(I)$, respectively, then for each $n \geq 1$, there is an exact sequence of complexes $\mathcal{K}(1_\mathcal{R},1)_{n+1} \to \mathcal{K}(1_\mathcal{R},1)_n \to \mathcal{K}(1_\mathcal{G},1)_n \to 0$. Since the 0-th component of the first morphism is injective and $H_0(\mathcal{K}(1_\mathcal{R},1)_{n+1}) = 0$, we have enough to deduce the exact sequence $E(I)_{n+1} \to E(I)_n \to E(\mathcal{G}(I))_n \to 0$. In particular, if $\mathrm{rt}(I) < \infty$, one can proceed by decreasing induction. ∎

Remark 3.4 If $\mathrm{rt}(I) = \infty$, then 3.3 might be false as Example 4.4 of [11] shows. Note that, as a consequence of next proposition, we will see that for the irrelevant ideal of a standard algebra hypothesis $\mathrm{rt}(I) < \infty$ can be removed.

Proposition 3.5 *Let U be a standard A-algebra and let $U_+ = \oplus_{n>0} U_n$ denote its irrelevant ideal. Take $f : W \to U$ a surjective graded morphism of standard A-algebras with W a symmetric algebra. Given $(n \geq 2, p = 1)$ or $(n \geq 3, p \geq 2)$, the module of effective n-relations of U_+^p is*

$$E(U_+^p)_n = \bigoplus_{q \geq np} \frac{\ker f_q}{W_p \cdot \ker f_{q-p}}.$$

In particular, $E(U_+^p)_n = 0$ if, and only if, $\mathrm{rt}(U_+^p) \leq n-1$. For $p = 1$, $\mathrm{rt}(U) = \mathrm{rt}(U_+)$. Moreover, U_+ is a syzygetic ideal if, and only if, U is a symmetric algebra.

Proof. Let $g : \mathbf{S}^U(U_p \otimes_A U) \to \mathcal{R}(U_+^p)$ be induced by the natural epimorphism of A-modules $U_p \otimes_A U \to U_+^p$. It is not hard to see $\mathcal{K}(g,1)_n = \oplus_{i\geq 0}\mathcal{K}(1_U,p)_{np+i}$. Moreover, if $(n \geq 2, p = 1)$, then $np+i \geq 2$ and if $(n \geq 3, p \geq 2)$, then $np+i \geq 2p+1$. Therefore, by Theorem 2.4,

$$E(U_+^p)_n = H_1(\mathcal{K}(g,1)_n) = \bigoplus_{i\geq 0} H_1(\mathcal{K}(1_U,p)_{np+i}) = \bigoplus_{i\geq 0} \frac{\ker f_{np+i}}{W_p \cdot \ker f_{(n-1)p+i}} = \bigoplus_{q\geq np} \frac{\ker f_q}{W_p \cdot \ker f_{q-p}}\,.$$

In particular, $E(U_+^p)_n \supset E(U_+^p)_{n+1}$. Thus, $E(U_+^p)_n = 0$ is equivalent to $\mathrm{rt}(U_+^p) \leq n-1$. For $p = 1$ and $n \geq 2$, $E(U_+)_n = \oplus_{i\geq 0}\ker f_{n+i}/W_1 \cdot \ker f_{n-1+i} = \oplus_{i\geq 0}E(U)_{n+i} = \oplus_{q\geq n}E(U)_q$. In particular, $\mathrm{rt}(U) = \mathrm{s}(E(U)) = \mathrm{s}(E(U_+)) = \mathrm{rt}(U_+)$. Moreover, $E(U_+)_2 = \oplus_{q\geq 2}E(U)_q = E(U)$. Thus, U_+ be syzygetic is equivalent to U be a symmetric algebra. ∎

Now, let us focus our attention into the relation type of Veronese subrings. Let U be a standard A-algebra. Recall that the p-th Veronese subring of U is defined to be the standard A-algebra $U^{(p)} = \oplus_{n\geq 0}U_{np}$. Clearly, if $f : V \to U$ is a (surjective) graded morphism of standard A-algebras, then it induces $f^{(p)} : V^{(p)} \to U^{(p)}$ another (surjective) graded morphism of standard A-algebras.

Lemma 3.6 *Let $f : V \to U$ be a surjective graded morphism of standard A-algebras. Then, for all $p \geq 1$, $\mathrm{s}(E(f^{(p)})) \leq 1 + [(\mathrm{s}(E(f)) - 1)/p]$ ($[a]$ is the integer part of a).*

Proof. Write $\mathrm{s}(E(f)) - 1 = pa + b$ with $0 \leq b < p$. So $[(\mathrm{s}(E(f)) - 1)/p] = a$. Take $n \geq 2 + a$. Then $(n-1)p \geq pa + p \geq \mathrm{s}(E(f))$. Thus, $\ker f_{np} = V_1 \cdot \ker f_{np-1} = \ldots = V_p \cdot \ker f_{(n-1)p}$ and hence $\mathrm{s}(E(f^{(p)})) \leq 1 + a$. ∎

Lemma 3.7 *Let U be a standard A-algebra and let $f : V \to U$ be a symmetric presentation of U. If $(n \geq 2, p = 1)$ or $(n \geq 3, p \geq 2)$, then the module of effective n-relations of $U^{(p)}$ is*

$$E(U^{(p)})_n = \frac{\ker f_{np}}{V_p \cdot \ker f_{(n-1)p}}\,.$$

Proof. Take $g : \mathbf{S}(V_p) \to U^{(p)}$ induced by $f_p : V_p \to U_p$ in degree one. We have $\mathcal{K}(g,1)_n = \mathcal{K}(f,p)_{np}$. Moreover, if $(n \geq 2, p = 1)$, then $np \geq 2$, and if $(n \geq 3, p \geq 2)$, then $np \geq 2p+1$. Thus, by Theorem 2.4, $E(U^{(p)})_n = H_1(\mathcal{K}(g,1)_n) = H_1(\mathcal{K}(f,p)_{np}) = (\ker f_{np})/(V_p \cdot \ker f_{(n-1)p})$. ∎

Lemma 3.8 *Let M be an A-module and $\mathbf{S}(M)$ its symmetric algebra. Then, for all $p \geq 1$, $\mathrm{rt}(\mathbf{S}(M)^{(p)}) \leq 2$. Moreover, if $p \geq 2$ and M is finitely generated, then $\mathrm{rt}(\mathbf{S}(M)^{(p)}) = 1$ if, and only if, M is locally cyclic.*

Proof. By Lemma 3.7, $E(\mathbf{S}(M)^{(p)})_n = 0$ for all $n \geq 3$. Thus, $\mathrm{rt}(\mathbf{S}(M)^{(p)}) \leq 2$. Suppose $p \geq 2$ and $(A, \mathfrak{m}, k)$ is local (see 3.2). If M is cyclic, then $\mathbf{S}(M)^{(p)} = \mathbf{S}(\mathbf{S}_p(M))$ and $\mathrm{rt}(\mathbf{S}(M)^{(p)}) = 1$. Conversely, suppose M finitely generated, but not cyclic. Take x, y part of a basis of $M \otimes k$ and $x^p, y^p, x^{p-1}y$ in $\mathbf{S}_p(M) \otimes k$. Then, $z = x^p \otimes y^p - x^{p-1}y \otimes$

$xy^{p-1} \in \mathcal{Z}_1(f \otimes 1_k, 1)_2$. Moreover, looking at the components of an element in a k-basis of $\mathcal{B}_1(f \otimes 1_k, 1)_2$, one sees that $z \notin \mathcal{B}_1(f \otimes 1_k, 1)_2$. Thus, $H_1(\mathcal{K}(f \otimes 1_k, 1)_2) \neq 0$, hence (by 3.2) $H_1(\mathcal{K}(f, 1)_2) \neq 0$ and $\mathrm{rt}(\mathbf{S}(M)^{(p)}) = 2$. ∎

Remark 3.9 Let I be an ideal of linear type finitely generated, but not locally principal. Then, by Lemma 3.8, $\mathrm{rt}(I^p) = 2$ for all $p \geq 2$, which reproves 2.6 of [7].

Theorem 3.10 *Let U be a standard A-algebra. Then,* $\mathrm{rt}(U^{(p)}) \leq \max(1+[(\mathrm{rt}(U)-1)/p], 2)$ *for all $p \geq 1$. Moroever, if U is finitely generated and $p \geq 2$, then* $\mathrm{rt}(U^{(p)}) = 1$ *if, and only if, U_p is locally generated by a d-sequence of length 1.*

Proof. Let $\alpha : \mathbf{S}(U_1) \to U$ be the canonical morphism. Put $g : \mathbf{S}(\mathbf{S}_p(U_1)) \to \mathbf{S}(U_1)^{(p)}$ and $f = \alpha^{(p)}$. Then, by Lemma 2.1, $\mathrm{rt}(U^{(p)}) \leq \max(\mathrm{s}(E(f)), \mathrm{s}(E(g)))$ and, by Lemmas 3.6 and 3.8, we prove the inequality. Suppose $p \geq 2$ and U finitely generated. By 3.2, one can suppose that $(A, \mathfrak{m}, k)$ is a local ring of infinite residual field k. If U_p is generated by a d-sequence of length 1, then (by 3.1) $\mathrm{rt}(U^{(p)}) = 1$. Conversely, suppose $\mathrm{rt}(U^{(p)}) = 1$. Take $V = U \otimes k$, so $V^{(p)} = U^{(p)} \otimes k$ and $\mathrm{rt}(V^{(p)}) \leq \mathrm{rt}(U^{(p)}) = 1$. Therefore, $V^{(p)}$ is a polynomial ring of Krull dimension $l = \mu(V_p) = \dim V^{(p)} = \dim V$ (since $V^{(p)} \subset V$ is an integral extension). Take $W \subset V$ a graded Noether normalization (it exists since k is infinite, see 1.5.17 of [3]). Thus, $\dim W = \dim V = l$ and so $\binom{l+p-1}{p} = \mu(W_p) \leq \mu(V_p) = l$, which forces $l = 1$. Hence, $\mu(U_p) = \mu(V_p) = 1$, $U_p = Ax$ is cyclic and, by 3.1 again, x is a d-sequence. ∎

Remark 3.11 The inequality of 3.10 was firstly proved by Backelin and Fröberg for finitely generated k-algebras (see [2]). Recently, Johnston and Katz showed a very similar statement to that of 3.10, but for $U = \mathcal{R}(I)$ the Rees algebra of an ideal I (see [7]). Since $\mathcal{G}(I)^{(p)} = \mathcal{R}(I)^{(p)} \otimes A/I = \mathcal{R}(I^p) \otimes A/I$, then (by 3.2) $\mathrm{rt}(\mathcal{G}(I)^{(p)}) \leq \mathrm{rt}(I^p)$. In particular, for $I = U_+$ the irrelevant ideal of a standard algebra U, $\mathcal{G}(I) = U$ and $\mathrm{rt}(U^{(p)}) \leq \mathrm{rt}(U_+^p)$. Thus, Johnston-Katz's result implies Backelin-Fröberg's result and the inequality of 3.10, when U is a Noetherian ring. Nevertheless, the whole Theorem 3.10 can not be deduced directly from earlier results since, in general, $\mathrm{rt}(U^{(p)}) \neq \mathrm{rt}(U_+^p)$ as next example shows.

Example 3.12 Put $U = k[x, y, z]/J$ with $J = (x^3y, xy^3, z^4, x^2y^2z^3)$. Then, $\mathrm{rt}(U) = 7$, $\mathrm{rt}(U^{(2)}) = 2$ and $\mathrm{rt}(U_+^2) = 3$ (remark that $\max(1 + [(\mathrm{rt}(U) - 1)/2], 2) = 4$). Indeed, since $E(U)_n = \ker\alpha_n/U_1 \cdot \ker\alpha_{n-1}$, $\alpha : \mathbf{S}(U_1) \to U$ the canonical morphism, then $E(U)_n = 0$ for all $n \geq 2$, $n \neq 4, 7$ and $E(U)_4 = k^{\oplus 3}$ and $E(U)_7 = k$. Thus, $\mathrm{rt}(U) = \mathrm{s}(E(U)) = 7$. Since $\ker\alpha_8 \subset F_4(\ker\alpha)$, then, by Lemma 3.7, $E(U^{(2)})_n = \ker\alpha_{2n}/\mathbf{S}_2(U_1) \cdot \ker\alpha_{2(n-1)} = 0$ for all $n \geq 3$. Thus, $\mathrm{rt}(U^{(2)}) \leq 2$. Moreover, $\mathrm{rt}(U^{(2)}) = 2$ since U_2 is not locally cyclic (see Theorem 3.10). Besides, using Proposition 3.5, $E(U_+^2)_4 = \oplus_{q \geq 8} (\ker\alpha_q/\mathbf{S}_2(U_1)\ker\alpha_{q-2}) = 0$, so $\mathrm{rt}(U_+^2) \leq 3$. But, since $\ker\alpha_7 \neq \mathbf{S}_2(U_1) \cdot \ker\alpha_5$, $E(U_+^2)_3 = \oplus_{q \geq 6}(\ker\alpha_q/\mathbf{S}_2(U_1)\ker\alpha_{q-2}) \neq 0$. Hence, $\mathrm{rt}(U_+^2) = 3$.

4 Conditions on the generators

In this section we characterize, in terms of a system of generators, which ideals have module of effective n-relations zero. Our work here is inspired in previous results by Costa, see [5] and [6]. Concretely, in [6], it was defined a *sequence of linear type* as a sequence of elements $x_1, \dots, x_d$ such that the ideals $(x_1, \dots, x_i)$ are of linear type for $i = 1, \dots, d$. As a consequence of the main result of this section (see 4.7), we get a new characterization of sequences of linear type involving annihilator ideals (see 4.9). For an ideal I generated by d elements $x_1, \dots, x_d$, we will denote by $I_{i_1,\dots,i_s}$ the ideal generated by the x_j, where $j \notin \{i_1, \dots, i_s\}$. For an A-module M, we will denote by $\mathcal{A}_d(M)$ the set of alternating $d \times d$ matrices with coefficients in M.

Lemma 4.1 *Let I be generated by d elements $x_1, \dots, x_d$ and take $n \geq 2$. Then, $E(I)_n = 0$ if, and only if, for all $(a_1, \dots, a_d) \in (I^{n-1})^{\oplus d}$ with $a_1x_1 + \dots + a_dx_d = 0$, there exists $(b_{i,j}) \in \mathcal{A}_d(I^{n-2})$ such that*

$$\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d} \\ -b_{1,2} & 0 & \dots & b_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1,d} & -b_{2,d} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}.$$

Proof. By Corollary 2.7, $E(I)_n = H_1(\underline{xt}; \mathcal{R}(I))_n$, where $\mathcal{K}(\underline{xt}; \mathcal{R}(I))_n$ is the n-th component of the Koszul complex associated to the elements $x_1t, \dots, x_dt$ in $\mathcal{R}(I) = \oplus_{n \geq 0} I^n t^n$. That is, $\cdots \to (I^{n-2})^{\oplus \binom{d}{2}} \xrightarrow{\partial_2} (I^{n-1})^{\oplus d} \xrightarrow{\partial_1} I^n \to 0$, with $\partial_2(b_{1,2}, \dots, b_{1,d}, b_{2,3}, \dots, b_{d-1,d}) = (a_1, \dots, a_d)$ defined by $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d} \\ -b_{1,2} & 0 & \dots & b_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1,d} & -b_{2,d} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$, and $\partial_1(a_1, \dots, a_d) = a_1x_1 + \dots + a_dx_d$. ∎

Lemma 4.2 *Let I be generated by d elements $x_1, \dots, x_d$ and take $n \geq 2$. If $E(I)_n = 0$, then $I_1I^{n-1} : x_1^n = I_1I^{n-2} : x_1^{n-1}$.*

Proof. If $a \in I_1I^{n-1} : x_1^n$, then $ax_1^n = a_2x_2 + \dots + a_dx_d$, $a_i \in I^{n-1}$. In particular, (by 4.1) $\begin{pmatrix} ax_1^{n-1} \\ -a_2 \\ \vdots \\ -a_d \end{pmatrix} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d} \\ -b_{1,2} & 0 & \dots & b_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ -b_{1,d} & -b_{2,d} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$, $b_{i,j} \in I^{n-2}$. Thus $ax_1^{n-1} \in I_1I^{n-2}$. ∎

Remark 4.3 If $d = 1$, then the necessary condition of Lemma 4.2 becomes $0 : x_1^n = 0 : x_1^{n-1}$, which is known to be sufficient to assure $E(I)_n = 0$ (see Example 3.1).

Lemma 4.4 *Let I be generated by d elements $x_1, \dots, x_d$ $(d \geq 2)$ and $n \geq 2$. If $E(I)_n = 0$, then*

$$(0 : x_1) \cap I^{n-1} = \left\{ \sum_{i=2}^{d} a_ix_i \mid a_i \in I^{n-2}, x_1 \begin{pmatrix} a_2 \\ \vdots \\ a_d \end{pmatrix} = (b_{i,j}) \begin{pmatrix} x_2 \\ \vdots \\ x_d \end{pmatrix} \text{ for } (b_{i,j}) \in \mathcal{A}_{d-1}(I_1^{n-2}) \right\}.$$

In particular, if $d=2$ *and* $E(I)_n=0$*, then* $(0:x_1)\cap I^{n-1}=x_2((0:x_1)\cap I^{n-2})$ *and* $(0:x_1x_2)\cap I^{n-2}=(0:x_1)\cap I^{n-2}+(0:x_2)\cap I^{n-2}$.

Proof. If $a\in(0:x_1)\cap I^{n-1}$, then (by 4.1) $\begin{pmatrix} a\\ 0\\ \vdots\\ 0\end{pmatrix}=\begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d}\\ -b_{1,2} & 0 & \dots & b_{2,d}\\ \vdots & \vdots & \ddots & \vdots\\ -b_{1,d} & -b_{2,d} & \dots & 0\end{pmatrix}\begin{pmatrix} x_1\\ x_2\\ \vdots\\ x_d\end{pmatrix}$ for some $(b_{i,j})\in\mathcal{A}_d(I^{n-2})$. Thus, $a=b_{1,2}x_2+\cdots+b_{1,d}x_d$ and

$$x_1\begin{pmatrix} b_{1,2}\\ b_{1,3}\\ \vdots\\ b_{1,d}\end{pmatrix}=\begin{pmatrix} 0 & b_{2,3} & \dots & b_{2,d}\\ -b_{2,3} & 0 & \dots & b_{3,d}\\ \vdots & \vdots & \ddots & \vdots\\ -b_{2,d} & -b_{3,d} & \dots & 0\end{pmatrix}=$$

$$=\begin{pmatrix} 0 & c_{2,3} & \dots & c_{2,d}\\ -c_{2,3} & 0 & \dots & c_{3,d}\\ \vdots & \vdots & \ddots & \vdots\\ -c_{2,d} & -c_{3,d} & \dots & 0\end{pmatrix}\begin{pmatrix} x_2\\ x_3\\ \vdots\\ x_d\end{pmatrix}+x_1\begin{pmatrix} 0 & e_{2,3} & \dots & e_{2,d}\\ -e_{2,3} & 0 & \dots & e_{3,d}\\ \vdots & \vdots & \ddots & \vdots\\ -e_{2,d} & -e_{3,d} & \dots & 0\end{pmatrix}\begin{pmatrix} x_2\\ x_3\\ \vdots\\ x_d\end{pmatrix}$$

with $c_{i,j}\in I_1^{n-2}$, $e_{i,j}\in I^{n-3}$ and $b_{i,j}=c_{i,j}+x_1e_{i,j}$ (if $n=2$, $I^{n-3}=0$). Put

$$\begin{pmatrix} a_2\\ a_3\\ \vdots\\ a_d\end{pmatrix}=\begin{pmatrix} b_{1,2}\\ b_{1,3}\\ \vdots\\ b_{1,d}\end{pmatrix}-\begin{pmatrix} 0 & e_{2,3} & \dots & e_{2,d}\\ -e_{2,3} & 0 & \dots & e_{3,d}\\ \vdots & \vdots & \ddots & \vdots\\ -e_{2,d} & -e_{3,d} & \dots & 0\end{pmatrix}\begin{pmatrix} x_2\\ x_3\\ \vdots\\ x_d\end{pmatrix}.$$

Then $x_1\begin{pmatrix} a_2\\ a_3\\ \vdots\\ a_d\end{pmatrix}=\begin{pmatrix} 0 & c_{2,3} & \dots & c_{2,d}\\ -c_{2,3} & 0 & \dots & c_{3,d}\\ \vdots & \vdots & \ddots & \vdots\\ -c_{2,d} & -c_{3,d} & \dots & 0\end{pmatrix}\begin{pmatrix} x_2\\ x_3\\ \vdots\\ x_d\end{pmatrix}$ and $a=a_2x_2+\cdots+a_dx_d$. Conversely, if $a=a_2x_2+\cdots+a_dx_d$, $a_i\in I^{n-2}$, with $x_1\begin{pmatrix} a_2\\ a_3\\ \vdots\\ a_d\end{pmatrix}=\begin{pmatrix} 0 & b_{2,3} & \dots & b_{2,d}\\ -b_{2,3} & 0 & \dots & b_{3,d}\\ \vdots & \vdots & \ddots & \vdots\\ -b_{2,d} & -b_{3,d} & \dots & 0\end{pmatrix}\begin{pmatrix} x_2\\ x_3\\ \vdots\\ x_d\end{pmatrix}$, $b_{i,j}\in I_1^{n-2}$, then clearly $ax_1=0$. In particular, for $d=2$, we have that $(0:x_1)\cap I^{n-1}=x_2((0:x_1)\cap I^{n-2})$. Moreover, if $a\in(0:x_1x_2)\cap I^{n-2}$, then $ax_2\in(0:x_1)\cap I^{n-1}$ and hence $ax_2=x_2b$ for some $b\in(0:x_1)\cap I^{n-2}$. Thus, $a=b+(a-b)$ where $b\in(0:x_1)\cap I^{n-2}$ and $(a-b)\in(0:x_2)\cap I^{n-2}$. ∎

Proposition 4.5 *Let* I *be generated by* x_1,x_2 *and* $n\geq 2$*. Then,* $E(I)_n=0$ *if, and only if, the following two conditions hold*

(*i*) $x_2I^{n-1}:x_1^n=x_2I^{n-2}:x_1^{n-1}$,

(*ii*) $(0:x_2)\cap I^{n-1}=x_1((0:x_2)\cap I^{n-2})$.

Proof. By Lemmas 4.2 and 4.4, $E(I)_n=0$ implies conditions (*i*) and (*ii*). Conversely, suppose (*i*) and (*ii*) are fulfilled and let us prove $E(I)_n=0$ via Lemma 4.1. Take $(a_1,a_2)\in$

$(I^{n-1})^{\oplus 2}$ with $a_1x_1 + a_2x_2 = 0$. Since $I^{n-1} = Ax_1^{n-1} + x_2I^{n-2}$, $a_1 = b_1x_1^{n-1} + b_2x_2$ with $b_1 \in A$ and $b_2 \in I^{n-2}$. Then, $0 = a_1x_1 + a_2x_2 = b_1x_1^n + (a_2 + b_2x_1)x_2$ and $b_1 \in x_2I^{n-1} : x_1^n = x_2I^{n-2} : x_1^{n-1}$ (by (i)). So $b_1x_1^{n-1} = c_2x_2$, $c_2 \in I^{n-2}$, and $a_1 = (b_2 + c_2)x_2$. Therefore, $0 = a_1x_1 + a_2x_2 = (a_2 + b_2x_1 + c_2x_1)x_2$. So, by (ii), $(a_2 + b_2x_1 + c_2x_1) \in (0 : x_2) \cap I^{n-1} = x_1((0 : x_2) \cap I^{n-2})$. We thus have $a_2 + b_2x_1 + c_2x_1 = c_1x_1$ with $c_1 \in I^{n-2}$ and $c_1x_2 = 0$. That is, $a_2 = (c_1 - b_2 - c_2)x_1$ and $a_1 = (b_2 + c_2)x_2 = (b_2 + c_2 - c_1)x_2$. ∎

Remark 4.6 If A is a domain and $I = (x_1, x_2)$, Costa showed (in Theorem 2 of [5]) that $\oplus_{n=2}^m E(I)_n = 0$ if, and only if, $x_2I^{n-1} : x_1^n = x_2I^{n-2} : x_1^{n-1}$ for all n, $2 \leq n \leq m$. Later, he improved his own result (see Theorem 4 of [6]) by avoiding the hypotesis A is a domain, but supposing the weaker condition $I = (x_1)$ is of linear type and adding a condition similar to (ii) of Proposition 4.5. However, for an ideal $I = (x_1, x_2)$, be of linear type does not imply $I_1 = (x_2)$ or $I_2 = (x_1)$ be of linear type (see Example 3.3 of [12] where $I = (x_1, x_2)$ of linear type is constructed with $0 : x^2 \neq 0 : x$ for all $x \in I$). Thus, Proposition 4.5 generalizes Costa's Theorem 2 of [5] and Theorem 4 of [6].

Theorem 4.7 *Let I be generated by d elements $x_1, \dots, x_d$ $(d \geq 3)$ and take $n \geq 2$. Then $E(I)_n = 0$ if, and only if, all the following conditions hold*

(i) $I_iI^{n-1} : x_i^n = I_iI^{n-2} : x_i^{n-1}$ *for all* $i = 1, \dots, d$,

(ii) $$\left(\left(\sum_{1 \leq i < j \leq d-1} x_ix_jI_d^{n-2} \right) : x_d \right) \cap I^{n-1} = \sum_{i=1}^{d-1} x_i \left((I_{i,d}I_d^{n-2} : x_d) \cap I^{n-2} \right),$$

(iii) If $$\begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \end{pmatrix} = \begin{pmatrix} 0 & \dots & b_{1,d-1} \\ \vdots & \ddots & \vdots \\ b_{d-1,1} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix}$$ *with* $\sum_{i=1}^{d-1} a_ix_i = 0$ *and* $b_{i,j} \in I_d^{n-2}$, *then*
$$\begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ 0 \end{pmatrix} = \begin{pmatrix} 0 & \dots & c_{1,d-1} & c_{1,d} \\ \vdots & \ddots & \vdots & \vdots \\ -c_{1,d-1} & \dots & 0 & c_{d-1,d} \\ -c_{1,d} & \dots & -c_{d-1,d} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \\ x_d \end{pmatrix}$$ *for some* $(c_{i,j}) \in \mathcal{A}_d(I^{n-2})$.

Proof. By Lemmas 4.2 and 4.1, $E(I)_n = 0$ clearly implies conditions (i) and (iii). Let us prove (ii) provided $E(I)_n = 0$. Take $a \in \sum_{i=1}^{d-1} x_i \left((I_{i,d}I_d^{n-2} : x_d) \cap I^{n-2} \right)$, so $a = a_1x_1 + \dots + a_{d-1}x_{d-1}$ with $a_i \in I^{n-2}$ and

$$x_d \begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{d-1} \end{pmatrix} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d-1} \\ b_{2,1} & 0 & \dots & b_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d-1,1} & b_{d-1,2} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \end{pmatrix},$$

$b_{i,j} \in I_d^{n-2}$. Therefore, $a \in I^{n-1}$ and $ax_d = \sum_{i,j \neq d}^{i \neq j} b_{i,j}x_ix_j \in \left(\sum_{1 \leq i < j \leq d-1} x_ix_jI_d^{n-2} \right)$. Conversely, take $a \in \left(\left(\sum_{1 \leq i < j \leq d-1} x_ix_jI_d^{n-2} \right) : x_d \right) \cap I^{n-1}$. So there exist $b_{i,j} \in I_d^{n-2}$ such that $ax_d = x_1c_1 + \cdots + x_{d-1}c_{d-1}$, where

$$\begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_{d-1} \end{pmatrix} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d-1} \\ b_{2,1} & 0 & \dots & b_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d-1,1} & b_{d-1,2} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \end{pmatrix}.$$

Since $E(I)_n = 0$, then (by Lemma 4.1)

$$\begin{pmatrix} -c_1 \\ -c_2 \\ \vdots \\ -c_{d-1} \\ a \end{pmatrix} = \left[\begin{pmatrix} 0 & 0 & \dots & 0 & e_{1,d} \\ 0 & 0 & \dots & 0 & e_{2,d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \dots & 0 & e_{d-1,d} \\ -e_{1,d} & -e_{2,d} & \dots & -e_{d-1,d} & 0 \end{pmatrix} + \begin{pmatrix} 0 & f_{1,2} & \dots & f_{1,d-1} & 0 \\ -f_{1,2} & 0 & \dots & f_{2,d-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -f_{1,d-1} & -f_{2,d-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \right.$$

$$\left. +x_d \begin{pmatrix} 0 & g_{1,2} & \dots & g_{1,d-1} & 0 \\ -g_{1,2} & 0 & \dots & g_{2,d-1} & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -g_{1,d-1} & -g_{2,d-1} & \dots & 0 & 0 \\ 0 & 0 & \dots & 0 & 0 \end{pmatrix} \right] \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \\ x_d \end{pmatrix}$$

where $e_{i,d} \in I^{n-2}$, $f_{i,j} \in I_d^{n-2}$ and $g_{i,j} \in I^{n-3}$. Put

$$\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_{d-1} \end{pmatrix} = \begin{pmatrix} -e_{1,d} \\ -e_{2,d} \\ \vdots \\ -e_{d-1,d} \end{pmatrix} - \begin{pmatrix} 0 & g_{1,2} & \dots & g_{1,d-1} \\ -g_{1,2} & 0 & \dots & g_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ -g_{1,d-1} & -g_{2,d-1} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \end{pmatrix}.$$

Then, $x_1h_1 + \cdots + x_{d-1}h_{d-1} = a$ and $x_d \begin{pmatrix} h_1 \\ \vdots \\ h_{d-1} \end{pmatrix} = x_d \begin{pmatrix} -e_{1,d} \\ \vdots \\ -e_{d-1,d} \end{pmatrix} - x_d(g_{i,j}) \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} =$ $\begin{pmatrix} c_1 \\ \vdots \\ c_{d-1} \end{pmatrix} + (f_{i,j}) \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix} = (b_{i,j}+f_{i,j}) \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \end{pmatrix}$. Thus, $a \in \sum_{i=1}^{d-1} x_i \left((I_{i,d} I_d^{n-2} : x_d) \cap I^{n-2} \right)$.

Now, suppose that (i), (ii) and (iii) hold and let us prove $E(I)_n = 0$ by using Lemma 4.1. Take $(a_1, \dots, a_d) \in (I^{n-1})^{\oplus d}$ such that $a_1x_1 + \cdots a_dx_d = 0$. Fix $i \in \{1, \dots, d\}$. Since $a_i \in I^{n-1} = Ax_i^{n-1} + I_iI^{n-2}$, then $a_i = b_ix_i^{n-1} + \sum_{j \neq i} b_jx_j$ with $b_i \in A$ and $b_j \in I^{n-2}$. Since $0 = \sum_{j=1}^d a_jx_j = b_ix_i^n + \sum_{j \neq i}(a_j + b_jx_i)x_j$, then $b_i \in I_iI^{n-1} : x_i^n = I_iI^{n-2} : x_i^{n-1}$ (by (i)). Thus $b_ix_i^{n-1} = \sum_{j \neq i} c_jx_j$, $c_j \in I^{n-2}$ and $a_i = \sum_{j \neq i}(b_j + c_j)x_j \in I_iI^{n-2}$. Hence, we can write $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_d \end{pmatrix} = \begin{pmatrix} 0 & b_{1,2} & \dots & b_{1,d} \\ b_{2,1} & 0 & \dots & b_{2,d} \\ \vdots & \vdots & \ddots & \vdots \\ b_{d,1} & b_{d,2} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_d \end{pmatrix}$ where $b_{i,j} \in I^{n-2}$. For $i, j \neq d$, $i \neq j$, put $b_{i,j} = e_{i,j} + x_dh_{i,j}$ with $e_{i,j} \in I_d^{n-2}$ and $h_{i,j} \in I^{n-3}$. For $i \neq d$, put $c_{i,d} = b_{i,d} + \sum_{j \neq i,d} h_{i,j}x_j$. Then,

$$\begin{pmatrix} a_1 \\ \vdots \\ a_{d-1} \\ a_d \end{pmatrix} = \begin{pmatrix} 0 & \dots & 0 & c_{1,d} \\ \vdots & \ddots & \vdots & \vdots \\ 0 & \dots & 0 & c_{d-1,d} \\ b_{d,1} & \dots & b_{d,d-1} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \\ x_d \end{pmatrix} + \begin{pmatrix} 0 & \dots & e_{1,d-1} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ e_{d-1,d} & \dots & 0 & 0 \\ 0 & \dots & 0 & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ \vdots \\ x_{d-1} \\ x_d \end{pmatrix}$$

Since $\sum_{i=1}^{d} a_i x_i = 0$, then $x_d(\sum_{i=1}^{d-1}(b_{d,i}+c_{i,d})x_i) = -\sum_{i,j\neq d}^{i\neq j} x_i x_j e_{i,j} \in \sum_{1\leq i<j\leq d-1} x_i x_j I_d^{n-2}$ and, by hypothesis (ii), $(b_{d,1}+c_{1,d})x_1 + \cdots (b_{d,d-1}+c_{d-1,d})x_{d-1} = f_1 x_1 + \cdots + f_{d-1}x_{d-1}$ where $f_i \in I^{n-2}$ and $x_d \begin{pmatrix} f_1 \\ f_2 \\ \vdots \\ f_{d-1} \end{pmatrix} = \begin{pmatrix} 0 & g_{1,2} & \dots & g_{1,d-1} \\ g_{2,1} & 0 & \dots & g_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ g_{d-1,1} & g_{d-1,2} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \end{pmatrix}$, $g_{i,j} \in I_d^{n-2}$.

Therefore,

$$\begin{cases} a_d &= b_{d,1}x_1 + \cdots + b_{d,d-1}x_{d-1} = (f_1 - c_{1,d})x_1 + \cdots + (f_{d-1} - c_{d-1,d})x_{d-1} \\ a_{d-1} &= e_{d-1,1}x_1 + \cdots + e_{d-1,d-2}x_{d-2} + c_{d-1,d}x_d = \\ &\quad (e_{d-1,1} + g_{d-1,1})x_1 + \cdots + (e_{d-1,d-2} + g_{d-1,d-2})x_{d-2} + (c_{d-1,d} - f_{d-1})x_d \\ \vdots & \quad \vdots \\ a_1 &= e_{1,2}x_2 + \cdots + e_{1,d-1}x_{d-1} + c_{1,d}x_d = \\ &\quad (e_{1,2} + g_{1,2})x_2 + \cdots + (e_{1,d-1} + g_{1,d-1})x_{d-1} + (c_{1,d} - f_1)x_d. \end{cases}$$

We thus can write $\begin{pmatrix} a_1 \\ a_2 \\ \vdots \\ a_{d-1} \\ a_d \end{pmatrix} = \begin{pmatrix} 0 & \tilde{c}_{1,2} & \dots & \tilde{c}_{1,d-1} & \tilde{b}_{1,d} \\ \tilde{c}_{2,1} & 0 & \dots & \tilde{c}_{2,d-1} & \tilde{b}_{2,d} \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ \tilde{c}_{d-1,1} & \tilde{c}_{d-1,2} & \dots & 0 & \tilde{b}_{d-1,d} \\ -\tilde{b}_{1,d} & -\tilde{b}_{2,d} & \dots & -\tilde{b}_{d-1,d} & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \\ x_d \end{pmatrix}$, with $\tilde{b}_{i,d} \in I^{n-2}$, but $\tilde{c}_{i,j} \in I_d^{n-2}$. Applying hypothesis (iii) to $\begin{pmatrix} 0 & \tilde{c}_{1,2} & \dots & \tilde{c}_{1,d-1} \\ \tilde{c}_{2,1} & 0 & \dots & \tilde{c}_{2,d-1} \\ \vdots & \vdots & \ddots & \vdots \\ \tilde{c}_{d-1,1} & \tilde{c}_{d-1,2} & \dots & 0 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_{d-1} \end{pmatrix}$, we finish. ∎

Corollary 4.8 INDUCTION THEOREM *Let I be generated by d elements $x_1, \dots, x_d$ $(d \geq 3)$ and take $n \geq 2$. Suppose that $E(I_d)_n = 0$. Then $E(I)_n = 0$ if, and only if, the following two conditions hold*

- *(i)* $I_i I^{n-1} : x_i^n = I_i I^{n-2} : x_i^{n-1}$ *for all* $i = 1, \dots, d$,
- *(ii)* $\left((\sum_{1\leq i<j\leq d-1} x_i x_j I_d^{n-2}) : x_d \right) \cap I^{n-1} = \sum_{i=1}^{d-1} x_i \left((I_{i,d} I_d^{n-2} : x_d) \cap I^{n-2} \right)$.

Proof. By lemma 4.1, $E(I_d)_n = 0$ assures that condition (iii) of Theorem 4.7 is fulfilled. ∎

Corollary 4.9 *Let $\underline{x} = x_1, \dots, x_d$ be d elements of A. Then, $\underline{x}$ is a sequence of linear type if, and only if, the following two conditions hold for all $n \geq 2$*

- *(i)* $(x_1, \dots, \widehat{x}_i, \dots, x_k)(x_1, \dots, x_k)^{n-1} : x_i^n = (x_1, \dots, \widehat{x}_i, \dots, x_k)(x_1, \dots, x_k)^{n-2} : x_i^{n-1}$ *for all* $1 \leq i \leq k \leq d$,
- *(ii) For all* $1 \leq i < j < k \leq d$,

$$\left((\sum_{1\leq i<j\leq k-1} x_i x_j (x_1, \dots, x_{k-1})^{n-2}) : x_k \right) \cap (x_1, \dots, x_k)^{n-1} =$$
$$\sum_{i=1}^{k-1} x_i \left(((x_1, \dots, \widehat{x}_i, \dots, x_{k-1})(x_1, \dots, x_{k-1})^{n-2} : x_k) \cap (x_1, \dots, x_k)^{n-2} \right),$$

(understanding $\sum_{1\le i<j\le k-1}(\ldots)=0$ *for* $k\le 2$ *and* $\sum_{i=1}^{k-1}(\ldots)=0$ *for* $k=1$*).*

Remark 4.10 With the hypothesis $E(I_d)_n=0$ of Corollary 4.8, it is not hard to prove that $E(I)_n=0$ if, and only if, the following two conditions hold

(i) $I_dI^{n-1}:x_d^n=I_dI^{n-2}:x_d^{n-1}$,

(ii) If $(a_1,\ldots,a_{d-1})\in(I^{n-1})^{\oplus(d-1)}$ with $a_1x_1+\ldots+a_{d-1}x_{d-1}=0$, then there exists $(b_1,\ldots,b_{d-1})\in(I^{n-2})^{\oplus(d-1)}$ and $(c_1,\ldots,c_{d-1})\in(I_d^{n-1})^{\oplus(d-1)}$ such that $b_1x_1+\ldots+b_{d-1}x_{d-1}=0$ and $a_i=x_db_i+c_i$ for all $i=1,\ldots,d-1$.

In fact, this is the expected generalization of Costa's Induction Theorem (see 4 of [6]).

Corollary 4.11 *Let* I *be generated by* x_1,x_2,x_3 *and take* $n\ge 2$*. Then,* $E(I)_n=0$ *if, and only if, the following three conditions hold*

(i) $I_iI^{n-1}:x_i^n=I_iI^{n-2}:x_i^{n-1}$ *for all* $i=1,2,3$,

(ii) $(x_1x_2I_3^{n-2}:x_3)\cap I^{n-1}=x_1\left((x_2I_3^{n-2}:x_3)\cap I^{n-2}\right)+x_2\left((x_1I_3^{n-2}:x_3)\cap I^{n-2}\right)$,

(iii) $(0:x_1)\cap I^{n-1}=\{a_2x_2+a_3x_3\mid a_i\in I^{n-2}\,,a_2x_1=bx_3,a_3x_1=-bx_2\,\text{for}\,b\in I_1^{n-2}\}$.

Moreover, if $(0:x_1x_2)\cap I_3^{n-2}=(0:x_1)\cap I_3^{n-2}+(0:x_2)\cap I_3^{n-2}$ *(for instance, if* $E(I_3)_n=0$*) then condition (iii) can be skipped.*

Proof. Suppose $E(I)_n=0$. Then, Lemma 4.2 assures (i), Lemma 4.4 assures (iii) and Theorem 4.7 assures (ii). Conversely, suppose (i), (ii) and (iii) hold and let us prove $E(I)_n=0$ by proving (iii) of Theorem 4.7. So take $\begin{pmatrix}a_1\\a_2\end{pmatrix}=\begin{pmatrix}0&b\\c&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix}$ with $a_1x_1+a_2x_2=0$ and $b,c\in I_3^{n-2}$. Since $(b+c)x_1x_2=0$, then $(b+c)x_2\in(0:x_1)\cap I^{n-1}$ and, by hypothesis (iii), $(b+c)x_2=ex_2+fx_3$ with $e,f\in I^{n-2}$ and $ex_1=gx_3$, $fx_1=-gx_2$ for some $g\in I_1^{n-2}$. Thus, $\begin{pmatrix}a_1\\a_2\\0\end{pmatrix}=\begin{pmatrix}0&e-c&-f\\c-e&0&g\\f&-g&0\end{pmatrix}\begin{pmatrix}x_1\\x_2\\x_3\end{pmatrix}$. Analogously, one could prove that $(0:x_1x_2)\cap I_3^{n-2}=(0:x_1)\cap I_3^{n-2}+(0:x_2)\cap I_3^{n-2}$ implies (iii) of 4.7. ∎

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