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Optical routing of uniform instances in tori ⋆

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Abstract. We consider the problem of routing *uniform* communication instances in switched optical tori that use the *wavelength division multiplexing* (or WDM) approach. A communication instance is called uniform if it consists exactly of all pairs of nodes in the graph whose distance is equal to one from a specified set $S = \{d_1, d_2, \ldots, d_k\}$. We give bounds on the optimal load induced on an edge for any uniform instance in a torus $T_{n\times n}$. When $k = 1$, we prove necessary and sufficient conditions on the value in S relative to n for the wavelength index to be equal to the load. When $k \geq 2$, we show that for any set S, there exists an n_0, such that for all $n > n_0$, there is an optimal wavelength assignment for the communication instance specified by S on the torus $T_{n\times n}$. We also show an approximation for the wavelength index for any S and n. Finally, we give some results for rectangular tori.

1 Introduction

Optical networks, in which data are transmitted in optical form and where the optical form is maintained for switching, provide transmission rates that are orders of magnitude higher than traditional electronic networks. A single optical fiber can support simultaneous transmission of multiple channels of data, voice and video.

Wavelength-division multiplexing is the most common approach to realize such high-capacity networks [4, 5]. A switched optical network using the WDM approach consists of nodes connected by point-to-point fiber-optic links, each of which can support a fixed number of channels or wavelengths. Incoming data streams can be redirected at switches along different outgoing links based on wavelengths. Different data streams can use the same link at the same time as long as they are assigned distinct wavelengths.

Two point x and y that are connected usually have one fiber-optic line for the transmission of signals from x to y and another one for signals from y to x. Thus,

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optical networks are generally modeled by *symmetric digraphs*, that is, a directed graph G with vertex set $V(G)$ and edge set $E(G)$ such that the edge $[x, y]$ is in $E(G)$ if and only if the edge $[y, x]$ is also in $E(G)$. In the following, whenever we talk about a graph, we always assume that we consider the associated *symmetric digraph* where any edge between x and y is replaced by two directed edges $[x, y]$ and $[y, x]$.

In a network, a *request* is an ordered pair of nodes (x, y) which corresponds to a message to be sent from x to y. An *instance* I is a collection of requests. Given an instance I in the network, an *optical routing problem* is to determine for each request (x, y) in I a dipath from x to y in the network, and assign it a wavelength, so that any two requests whose dipaths share a link are assigned different wavelengths. Thus, an optical routing problem contains the related tasks of *route assignment* and *wavelength assignment.* A *routing* R for a given instance I is a set of dipaths $\{P(x, y) \mid (x, y) \in I\}$, where $P(x, y)$ is a dipath from x to y in the network. By representing a wavelength by a color, the wavelength assignment can be seen as a coloring problem where one color is assigned to all the edges of a path given by the route assignment. We say that the coloring of a given set of dipaths is *conflict-free* if any two dipaths that share an edge are assigned different colors. Since the cost of an optical switch is proportional to the number of wavelengths it can handle, and the total number of wavelengths that can be handled by a switch is limited, it is important to determine paths and wavelengths so that the total number of required wavelengths is minimized.

Given an instance I in a graph G, and a routing R for it, there are two parameters that are of interest. The *wavelength index* of the routing R, denoted $\boldsymbol{w}(G, I, R)$, is the *minimum* number of colors needed for a conflict-free assignment of colors to dipaths in the routing R of the instance I in G. The *edge-congestion* or *load* of the routing R for I, denoted by $\boldsymbol{\pi}(G, I, R)$, is the maximum number of dipaths that share the same edge. The parameters $\boldsymbol{w}(G, I)$, the *optimal wavelength index*, and $\boldsymbol{\pi}(G, I)$, the *optimal load* for the instance I in G are the minimum values over all possible routings for the given instance I in G. It is easy to see that $\boldsymbol{w}(G, I, R) \geq \boldsymbol{\pi}(G, I, R)$ for every routing R, thus $\boldsymbol{w}(G, I) \geq \boldsymbol{\pi}(G, I)$. It is known that the inequality can be strict [7]. A general upper bound for $\boldsymbol{w}(G, I)$ as a function of $\boldsymbol{\pi}(G, I)$ was given in [1]. Determining $\boldsymbol{\pi}(G, I)$ for arbitrary networks and instances is NP-hard [3], though for some specific networks such as trees and rings, and for specific instances, such as the one-to-all instance, the problem can be solved efficiently. Finding $\boldsymbol{w}(G, I)$ is also NP-hard for arbitrary G and I. In fact, it is known to be NP-hard for specific graphs such as trees and cycles [6]. Approximation algorithms for $\boldsymbol{w}(G, I)$ have been given for a variety of specific cases; see the survey paper [3].

In view of the NP-hardness of the general case, it is important to characterize the instances for which the wavelength index can be determined efficiently. In this paper, we investigate the wavelength index of *uniform communication instances* on tori. We say that an instance I is uniform if there exists a set of integers $S = \{d_1, d_2, \ldots, d_k\}$ such that I consists of all pairs of nodes in G whose distance is equal to d_i for some d_i in the set S. We denote such an instance as I_S. It is easy to see that the all-to-all instance I_A, consisting of all pairs of nodes of the network, is a special case of a uniform communication instance I_S where $S = \{1, 2, \ldots, D_G\}$, and D_G is the diameter of G. Uniform communication instances also occur in certain systolic computations.

A *torus* of size $m \times n$, denoted $T_{m \times n}$, is a network model having the vertex set $\{(i, j) : 0 \le i \le m-1, 0 \le j \le n-1\}$, where a vertex (i, j) is connected to vertices $((i+1) \bmod m, j)$, $(i, (j+1) \bmod n)$, $((i-1) \bmod m, j)$,$(i, (j-1) \bmod n)$. In the rest of the paper, all arithmetic operations involving vertices are assumed to be done modulo n or m as appropriate. By mapping vertices into the plane, we can visualize a torus as a graph in which vertices are organized into n rows and m columns, vertex (i, j) belongs to ith column and jth row. Each vertex is connected to the two vertices in the same row and adjacent columns and to the two vertices in the same column and adjacent rows. In the torus, the rows 0 and $n-1$ are adjacent, as are the columns 0 and $m-1$. For a vertex $v = (i, j)$ we call the edges from v to the vertices $(i+1, j)$, $(i-1, j)$, $(i, j+1)$, $(i, j-1)$ its right, left, up, and down edge respectively. The torus is *square* if $m = n$ and is *rectangular* otherwise. We assume without loss of generality that $m \ge n$. The diameter of the torus $T_{m \times n}$ is equal to $\lfloor \frac{m}{2} \rfloor + \lfloor \frac{n}{2} \rfloor$. The *diagonal* D_k of the torus $T_{n \times n}$ consists of the vertices $\{(i, j) \in T_{n \times n} : i + j = k\}$. Similarly, let $D_k^T = \{(i, j) \in T_{n \times n} : i - j = k\}$. The torus network is of interest because it has been used in some highly parallel machines and it is also the underlying virtual network in finite element representation of objects.

The all-to-all communication instance has been considered for several different networks, including a torus [2, 11]. For a square torus, $\boldsymbol{w}(T_{n \times n}, I_A) = n^3/8$, which is too large for the present technologies, even for small values of n. Thus, by restricting communications among nodes to pairs of nodes whose distances are in a set S, we can obtain instances that require a substantially smaller number of wavelengths. Some specific uniform instances were previously considered in [9, 10] for chordal rings and rings respectively, and general uniform instances were studied in [8] for rings. The problem of wavelength assignment for uniform instances seems to be more difficult than the all-to-all instance for tori, since a uniform instance is an arbitrary subset of the all-to-all instance. At the same time, it is also more complex than the same problem in [8] for rings, as there are many more vertices at distance d from any vertex v in the torus, as well as many more types of dipaths to each destination vertex.

As stated earlier, the task of optical routing involves both path assignment and color assignment to dipaths. In this paper, we always use shortest path routing and unless stated otherwise, the dipath from u to v is the reverse of the dipath from v to u. The colors assigned to a dipath and its reverse are always the same, and hence we speak only about the color assigned to the path between u and v.

To find the wavelength index or an upper bound on it for a uniform instance, we first consider some special cases. The next section presents results for uniform communication instances when S is a singleton set. In particular, we give necessary and sufficient conditions for the wavelength index to be equal to the load, and an upper bound on the wavelength index is derived in any case. Section 3 considers I_S for $S = \{d_1, d_2\}$. Some sufficient conditions for the wavelength index to be equal to the load are given. The main result in that section is that for any $S = \{d_1, d_2\}$, an optimal wavelength assignment is always possible, provided the torus is large enough. In Section 4 we show that the results from Section 3 can be generalized to get an optimal solution of the general case $S = \{d_1, d_2, \ldots d_k\}$ for a sufficiently large torus and give an approximation in any case. We also give some results for rectangular tori. The last section gives conclusions and some open problems.

Due to space limitations, we mostly give outlines of proofs. Detailed proofs will appear in the full version.

2 Square tori: Single path length

First, we need to derive the value of $\boldsymbol{\pi}(T_{n\times n}, I_S)$, since it gives a lower bound on the wavelength index. The following theorem is stated without proof:

Theorem 1. *Let $T_{n\times n}$ be a square torus and $S = \{d\}$.*

1. *If $d < \lfloor \frac{n}{2} \rfloor$, then $\boldsymbol{\pi}(T_{n\times n}, I_S) = d^2$.*
2. *If $d = \frac{n}{2}$, then $\boldsymbol{\pi}(T_{n\times n}, I_S) = (\frac{n}{2})^2 - \lfloor \frac{n}{4} \rfloor$.*
3. *If $d = D$, the diameter of the torus, then $\boldsymbol{\pi}(T_{n\times n}, I_S) = \frac{D}{4} = \lceil \frac{n}{4} \rceil$ if n is even and $\boldsymbol{\pi}(T_{n\times n}, I_S) = D = n-1$ if n is odd.*
4. *If $D > d > \lfloor \frac{n}{2} \rfloor$, then $\boldsymbol{\pi}(T_{n\times n}, I_S) = d(2\lfloor \frac{n}{2} \rfloor - d + 1)$.*

We determine when there is a routing and a wavelength assignment that uses exactly the number of colors given by the load of the instance, a lower bound on the wavelength index.

The paths of a routing of shortest paths in a torus can be specified using their *path-types* defined as follows. A dipath has path-type (i, j) where $0 \le |i|, |j| \le \lfloor \frac{n}{2} \rfloor$, if it uses $|i|$ horizontal edges and $|j|$ vertical edges. The horizontal edges are *right* edges if i is positive and *left* edges otherwise. Similarly, the vertical edges are *up* edges if j is positive and *down* edges if j is negative. The following theorem provides a necessary condition for the wavelength index to equal the load of a uniform instance.

Theorem 2. *Let $T_{n\times n}$ be a square torus and $S = \{d\}$ where $d < \lfloor \frac{n}{2} \rfloor$. Then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = d^2$ only if d is a factor of $4n^2$.*

Proof: Suppose $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = d^2$. Then every color is used on every edge of the network. Fix a color, say c. Let a_i be the number of dipaths of type $(i, d-i)$, $(-i, d-i)$, $(i, -(d-i))$, and $(-i, -(d-i)$, where $0 \le i \le d$, that have been assigned the color c. Clearly this accounts for ia_i right or left edges. Since the color c is used on every horizontal edge, we must have $\Sigma_{i=0}^{d} ia_i = 2n^2$. Similarly, since c must be used on every vertical edge, $\Sigma_{i=0}^{d}(d-i)a_i = 2n^2$. Adding these two equations, we obtain $d\Sigma_{i=0}^{d} a_i = 4n^2$, yielding the result. □

Next, we derive sufficient conditions for the wavelength index to equal the load. The key idea to obtain a valid wavelength assignment is as follows: We define a *band* to be a set of dipaths that are edge-disjoint and can therefore be colored with the same color. A *pattern* is defined as a set of edge-disjoint bands. We always try to find patterns that cover the edge set of the network as much as possible. The wavelength assignment problem can be solved by finding a set of patterns such that their union covers the entire set of dipaths of a given instance. Furthermore, if the set of patterns is such that every pattern contains all edges in the network, and every dipath is contained in exactly one pattern in the set, then the wavelength index equals the load. This idea was also used in [11] to solve the all-to-all instance for tori of even side.

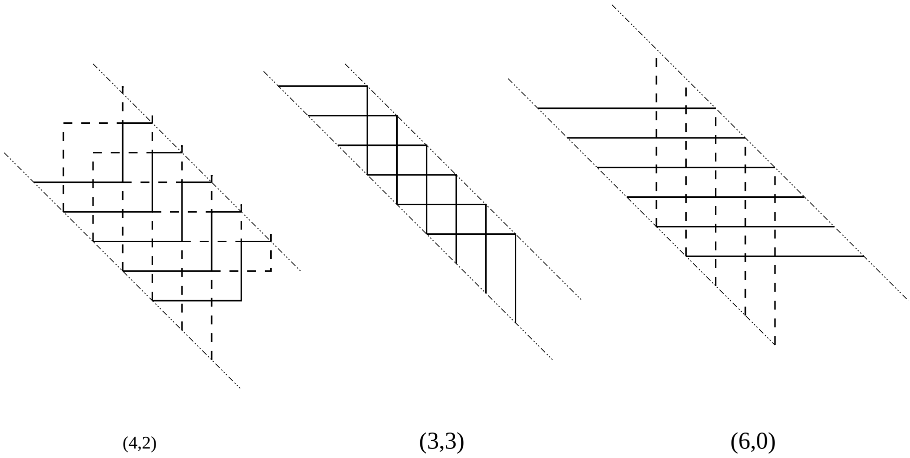

Fig. 1. Bands $A_k(i)$ for $n = 12$ and $d = 6$, and three values of i. Notice that when $i = d/2 = 3$, the band has width $d/2$ and otherwise has width d.

We define the *band* $A_k(i)$ (where $i \neq d/2$) to be the set of dipaths from each vertex $(x, y) \in D_k$ to the pair of vertices $(x + i, y + d - i)$ and $(x + d - i, y + i)$ respectively, as well as their reverses. Both of these latter vertices are in D_{k+d}. Furthermore, all edges in between the diagonals D_k and D_{k+d} are covered by the band $A_k(i)$. Notice that the dipaths in the band correspond to 4 different path-types: $(i, d - i), (d - i, i), (-i, -(d - i))$, and $(-(d - i), -i)$, the first two originating in D_k and the last two in D_{k+d}. We call these a set of *companion* path-types. Next, $A_k(\frac{d}{2})$ is defined as the set of dipaths from $(x, y) \in D_k$ to $(x - \frac{d}{2}, y + \frac{d}{2})$. The furthest intermediate vertices form the diagonal $D_{k+\frac{d}{2}}$, and all edges between the diagonals D_k and $D_{k+\frac{d}{2}}$ are covered by the band $A_k(\frac{d}{2})$. The set of companion path-types corresponding to $i = d/2$ contains only two elements, both originating in D_k. See Figure 1 for an example. Similarly, let $B_k(i)$ (where $i \neq \frac{d}{2}$) be the set of dipaths from all vertices $(x, y) \in D_k^T$ to the pair of vertices $(x - i, y + d - i)$ and $(x - d + i, y + i)$ respectively. Note that both of these vertices are in D_{k-d}^T. $B_k(\frac{d}{2})$ is defined analogously to $A_k(\frac{d}{2})$. It is straightforward to see that the width of any band $A_*(i)$ and $B_*(i)$ is d when $i \neq \frac{d}{2}$, and the width of the bands $A_*(\frac{d}{2})$ and $B_*(\frac{d}{2})$ are $d/2$. (We use $A_*(i)$ to denote any band $A_k(i)$ where $0 \leq k \leq n-1$.) It is not difficult to check that any band defined above is a set of edge-disjoint dipaths.

We define the *pattern* $P_0(i)$, $i = 0 \ldots \lceil \frac{d}{2} \rceil - 1$, as the set of bands $A_0(i)$, $A_d(i)$, $A_{2d}(i)$, $\ldots$, $A_{(\lfloor \frac{n}{d} \rfloor - 1)d}(i)$. Similarly, let $P_m(i)$ be the set of bands $\{A_{pd+m}(i) : 0 \leq m \leq d-1 \text{ and } 0 \leq p < \lfloor n/d \rfloor\}$. Finally, let $P_m(\frac{d}{2})$ be the set of bands $\{A_{p\frac{d}{2}+m}(i) : 0 \leq m \leq \frac{d}{2} - 1 \text{ and } 0 \leq p < \lfloor 2n/d \rfloor\}$. The patterns $Q_k(i)$ are defined analogously based on the bands $B_\star(i)$.

The next three theorems show that the wavelength index equals the load when the uniform instance given by a single path length d is such that d is a factor of n, the side of the torus.

Theorem 3. *Let $T_{n \times n}$ a the square torus and $S = \{d\}$ where $d < \lfloor \frac{n}{2} \rfloor$. If d is a factor of n then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = d^2$.*

Proof: Since d is a factor of n, the pattern $P_0(i)$ contains all edges in the network, where $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$. The same is true for the patterns $P_1(i), \ldots, P_{d-1}(i)$, and the corresponding patterns $Q_*(i)$. Furthermore, it is easy to check that every dipath is included in exactly one of these patterns.

Since each pattern is assigned a color, it suffices to count the number of patterns to determine the wavelengths used. If d is odd, for each value of i, $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we need d patterns of type $P_*(i)$. The same is true for patterns of type $Q_*(i)$ except we don't need patterns of this type for $i = 0$ as all dipaths of path-type $(d, 0)$ and $(0, d)$ have already been colored using the P_* patterns. Thus the total number of patterns is $2d\lfloor \frac{d}{2} \rfloor + d = d^2$. If d is even, there are $d\frac{d}{2}$ patterns of type $P_*(i)$ corresponding to $0 \leq i \leq d/2-1$, $d(\frac{d}{2}-1)$ patterns of type $Q_*(i)$ corresponding to $1 \leq i \leq d/2 - 1$, and $d/2$ patterns each corresponding to each of $P_*(\frac{d}{2})$ and $Q_*(\frac{d}{2})$. This adds to a total of $d + 2d(\frac{d}{2} - 1) + d = d^2$ as claimed. □

Theorem 4. *Let $T_{n\times n}$ be a square torus where n is even, and $S = \{\frac{n}{2}\}$. Then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = (\frac{n}{2})^2 - \lfloor \frac{n}{4} \rfloor$.*

Proof: We use the same arguments as in the previous theorems with the exception that the dipaths of path-type $(n/2, 0)$ and $(0, n/2)$ can be colored using $\lceil n/4 \rceil = \lceil d/2 \rceil$ colors, and not d colors as in the previous theorem. □

Theorem 5. *Let $T_{n\times n}$ be the square torus with n^2 vertices and $S = \{d\}$ where $d < \lfloor \frac{n}{2} \rfloor$. If d is even, $d/2$ divides n, and $n \geq d^2/2$, then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = d^2$.*

Proof: If d divides n, then the theorems above show that the wavelength index equals the load. Otherwise, let $n = kd + d/2$. We use two types of patterns to achieve the wavelength assignment. For each $0 \leq i < \frac{d}{2}$, we build a pattern with k bands of type $A_*(i)$ and one of type $A_*(\frac{d}{2})$. We shift this pattern d times, which accounts for bands of type $A_*(i)$ from kd origin diagonals and of type $A_*(d/2)$ from d origin diagonals. Repeating this for all $d/2$ possible values of i, we have bands of type $A_*(i)$ from kd origin diagonals for each i, where $0 \leq i < \frac{d}{2}$ and of type $A_*(\frac{d}{2})$ from $d^2/2$ origin diagonals.

The second type of pattern consists of one band of each type $A_*(i)$ where $0 \leq i < \frac{d}{2}$, and $\frac{n-d^2/2}{d/2}$ bands of type $A_*(\frac{d}{2})$. By shifting this pattern $d/2$ times, we claim that we can get bands of type $A_*(i)$ where $0 \leq i < \frac{d}{2}$ from $d/2$ origin diagonals, and bands of type $A_*(\frac{d}{2})$ from $n - d^2/2$ origin diagonals.

By using both types of patterns as described, and using a different color for every pattern, we assign colors to all the required dipaths from all origin vertices. □

Theorem 6. *Let $T_{n\times n}$ be a square torus and $S = \{D\}$ where D is the diameter of the torus. Then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = \frac{D}{4} = \frac{n}{4}$ if n is even and $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = D = n-1$ if n is odd.*

Proof: Omitted. □

Next, we give an approximation result for the case of arbitrary $d < \lfloor \frac{n}{2} \rfloor$.

Theorem 7. *Let $T_{n\times n}$ be a square torus and $S = \{d\}$ where $d < \lfloor \frac{n}{2} \rfloor$. Then $\boldsymbol{w}(T_{n\times n}, I_S) \leq 2\lceil \frac{d}{2} \rceil (d + \frac{n \bmod d}{\lfloor \frac{n}{d} \rfloor})$.*

Proof: Let d be odd. For every i, where $0 \leq i \leq \lfloor \frac{d}{2} \rfloor$, we use a pattern with $\lfloor \frac{n}{d} \rfloor$ bands of type $A_*(i)$. This leaves a "gap" of width $n \bmod d$. We shift this pattern d times, starting the pattern each time with the diagonal where the previous gap started, assigning a new color each time. The remaining origin diagonals, from which dipaths have not yet been assigned colors, are now covered with further patterns using $\frac{n \bmod d}{\lfloor \frac{n}{d} \rfloor}$ more colors. There are $\lceil \frac{d}{2} \rceil$ possible values of i, and symmetry considerations multiply it by 2, giving the result. A similar argument holds when d is even. □

It is not hard to see that the worst case for the above theorem occurs when the gap is of size $d-1$, i.e., $n \bmod d = d-1$, yielding the following result:

Theorem 8. *Let $T_{n\times n}$ be a square torus and $S = \{d\}$ where $d < \lfloor \frac{n}{2} \rfloor$. Then $\boldsymbol{w}(T_{n\times n}, I_S) < 1.5\boldsymbol{\pi}(T_{n\times n}, I_S)$.*

3 Square tori: Two path lengths

When the instance involves more than one path length, we can use Theorem 1 to get a general result about the load of the instance in a square torus.

Theorem 9. *Let $T_{n\times n}$ be a square torus and $S = \{d_1, d_2, \cdots, d_k\}$ where $1 \leq d_k < \cdots < d_2 < d_1 < \lceil \frac{n}{2} \rceil$. Then the load of $T_{n\times n}$ is $\boldsymbol{\pi}(T_{n\times n}, I_S) = \sum_{i=1}^{k} d^2$.*

Similarly, Theorem 3 can be generalized for S containing more than one path length.

Lemma 1. *Let $T_{n\times n}$ be a square torus and $S = \{d_1, d_2, \cdots, d_k\}$ where $1 \leq d_k < \cdots < d_2 < d_1 < \lceil \frac{n}{2} \rceil$. If $d_i | n$ for $1 \leq i \leq k$ then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = \sum_{i=1}^{k} d^2$.*

Proof: If $d_i|n$ we can apply Theorem 3 to each uniform sub-instance consisting of path length d_i and obtain a coloring with d_i^2 colors. Adding up all the contributions, the result follows. $\square$

We will consider here the double-path case $S = \{d_1, d_2\}$ where $1 \leq d_2 < d_1 < \lceil \frac{n}{2} \rceil$. We use some of the results for rings from [8]. First, we show that the wavelength index equals the load when d_1 as well as $d_1 + d_2$ divide n, even when d_2 is not a factor of n.

Lemma 2. *Let $T_{n\times n}$ be a square torus and $S = \{d_1, d_2\}$ where $1 \leq d_2 < d_1 < \lceil \frac{n}{2} \rceil$. If $(d_1 + d_2)|n$ and $d_1|n$ then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S) = d_1^2 + d_2^2$.*

Proof: We consider the case when d_1 and d_2 are both odd (the other cases are similar). Fix a value of i such that $0 \leq i \leq \lfloor \frac{d_2}{2} \rfloor$. We use a pattern that alternates bands of type $A_*(i)$ of width d_1 and d_2. This pattern can be shifted $d_1 + d_2$ times, thereby using $d_1 + d_2$ colors to color all dipaths of type $(i, d_1 - i)$ and $(i, d_2 - i)$ as well their companions. We repeat the same procedure for each value of i in the specified range. At this point, all dipaths of length d_2 have been colored. However dipaths of path-type $(i, d_1 - i)$, where $\lfloor \frac{d_2}{2} \rfloor + 1 \leq i \leq \lfloor \frac{d_1}{2} \rfloor$, and their companions have not been assigned colors. Since d_1 divides n, we can now solve these separately, by using patterns that use only bands of width d_1. Each such pattern can be shifted d_1 times, thus requiring $d_1(\lfloor \frac{d_1}{2} \rfloor - \lfloor \frac{d_2}{2} \rfloor)$ more colors. Thus $\boldsymbol{w}(T_{n\times n}, I_S) = (d_1 + d_2) + 2\lfloor \frac{d_2}{2} \rfloor (d_1 + d_2) + 2(\lfloor \frac{d_1}{2} \rfloor - \lfloor \frac{d_2}{2} \rfloor)(d_1) = d_1^2 + d_2^2$. The factor 2 comes from considering the symmetric bands of type $B_*(i)$. $\square$

Lemma 3. *Let $S = \{d_1, d_2\}$, $1 < d_1 < d_2 < \lfloor \frac{n}{2} \rfloor$, and let $n = a(pd_1) + bd_2$ where $a > b \geq 0$, $a \geq d_2$ and $a - b < pd_1 + d_2$, $1 \leq p \leq \lceil \frac{d_1}{2} \rceil$, and pd_1, d_2 and n are mutually co-prime. Then there is an optimal wavelength assignment in $T_{n\times n}$ for all dipaths corresponding to p sets of companion path-types of length d_1 and 1 set of companion path-types of length d_2.*

Proof: We use similar arguments as in Lemma 2 of [8]. We give only the idea here. We first solve the wavelength assignment problem for one set of companion path-types of length pd_1 and one such set of length d_2. We use a pattern alternating b bands of width d_2 and pd_1 followed by $a - b$ bands of width d_1. It is easy to see that this pattern covers the entire edge set. We shift this pattern $i = pd_1 + d_2 - (a - b)$ times, thereby assigning wavelengths to dipaths of length d_1 from ai origin diagonals, and dipaths of length d_2 from bi origin diagonals. As in [8], given the conditions on a and b, we can find a' and b' such that $n = a'\, d_1 + b'\, d_2$. We use a second pattern alternating a' bands of width pd_1 and d_2 followed by $b' - a'$ bands of width d_2, and shift this $j = a - b$ times. Thus we can assign wavelengths to dipaths of length pd_1 from $a'j$ origin diagonals and paths of length d_2 from $b'j$ origin diagonals. It is easy to check that $ai + a'j = bi + b'j = n$, and therefore, all dipaths of length pd_1 and d_2 have been assigned. Finally, each band of width pd_1 is sub-divided into p bands of width d_1, one for each companion set of path-types, giving the result. $\square$

This brings us to the main theorem of this section:

Theorem 10. *Let $S = \{d_1, d_2\}$ where $1 \le d_2 < d_1 < \lceil n/2 \rceil$. Then $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S)$ whenever one of the following holds:*

1. *$d_1|n$ and $d_2|n$.*
2. *$(d_1 + d_2)|n$ and $d_1|n$.*
3. *$n > (d_1\lceil \frac{d_1}{d_2-1}\rceil - 2)d_1\lceil \frac{d_1}{d_2-1}\rceil + (d_1\lceil \frac{d_1}{d_2-1}\rceil - 1)d_2$ where $d_2 > 1$.*

Proof: The first and second statements follow from Lemmas 1 and 2. The key idea for the third statement is that the set of path-types for dipaths of length d_1 and d_2 can always be divided into pairs of subsets such that there are at most $\lceil \frac{d_1}{d_2-1}\rceil$ sets of companion path-types of length d_1 and 1 of length d_2. We can then apply Lemma 3 to obtain an optimal wavelength assignment. The existence of suitable a and b required by Lemma 3 follows from the arguments in [8]. □

4 The general case

In this section, we consider the case $S = \{d_1, d_2, \ldots, d_k\}$ for $k > 2$. Although the previous two sections dealt mostly with the special cases $S = \{d_1\}$ and $S = \{d_1, d_2\}$, these results can be used to obtain the exact value of $\boldsymbol{w}(T_{n\times n}, I_S)$ in many instances of the general case.

For example, if $\sum_{i=1}^{k} d_i$ is a factor of n then $\boldsymbol{w}(T_{n\times n}, I_S) = \sum_{i=1}^{k} d_i^2$. We can also obtain a value for $\boldsymbol{w}(T_{n\times n}, I_S)$ that equals the load, by partitioning S into subsets and applying the results of Section 3. In fact, as in the two path-length case, equality between the wavelength index and the load holds for any arbitrary instance S provided the torus is large enough. If the number of path-lengths k is even, we simply pair the path-lengths, and use Theorem 10 to derive a value of n such that each pair of path-lengths can be solved optimally. If instead k is odd, then we add up two of the path-lengths and reduce to the even case.

This gives the following theorem:

Theorem 11. *Let $S = \{d_1, d_2, \ldots, d_k\}$, $1 \le d_k < \ldots < d_2 < d_1 < \lceil n/2 \rceil$. Then there exists an n_0 such that for any $n > n_0$, $\boldsymbol{w}(R_n, I_S) = \boldsymbol{\pi}(R_n, I_S) = \sum_{i=1}^{k} d_i^2$.*

An approximation on the wavelength index can be obtained in any case. The proof of the following theorem is based on Theorem 8.

Theorem 12. *Let $S = \{d_1, d_2, \ldots, d_k\}$ where $1 \le d_k < \ldots < d_2 < d_1 < \lceil n/2 \rceil$. Then $\boldsymbol{w}(T_{n\times n}, I_S) \le 3\boldsymbol{\pi}(T_{n\times n}, I_S)/2$.*

Clearly, the process of decomposition of S into sub-instances of size 1 or 2 and obtaining a solution for I_S by putting together solutions for the sub-instances can be done also in cases when neither one of the two theorems of this section applies to the entire set S.

Next, we consider a rectangular torus $T_{m\times n}$. As mentioned in the introduction, we assume that $m > n$. Many of the results from the previous sections can be generalized to rectangular tori. Using similar arguments as in the case of a square torus, we obtain the following theorem:

Theorem 13. *Let $T_{m\times n}$ be a rectangular torus with n rows and m columns, $m > n$, and $S = \{d\}$. If $d|m$, $d|n$, and $d \leq n/2$, then $\boldsymbol{w}(T_{m\times n}, I_S) = \boldsymbol{\pi}(T_{m\times n}, I_S)$. Otherwise, if d divides neither n nor m, then $\boldsymbol{w}(T_{m\times n}, I_S) < 2\boldsymbol{\pi}(T_{m\times n}, I_S)$.*

5 Conclusions and Open Problems

In the previous sections we gave exact solutions to find the wavelength index for some uniform instances and derived an approximation of the wavelength index for any uniform instance. The techniques used and results obtained for uniform instances on tori could also be used for deriving results in non-uniform cases.

There remain some open problems: For a single path length $S = \{d\}$, the sufficient and necessary conditions we give for $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S)$ do not match. Can the necessary condition $d|4n^2$ be improved? For the two path length case $S = \{d_1, d_2\}$, is it possible to substantially lower the bound in Theorem 10, part 3, on the dimension of the torus for which $\boldsymbol{w}(T_{n\times n}, I_S) = \boldsymbol{\pi}(T_{n\times n}, I_S)$? Finally, the case of a rectangular tori should be investigated in more detail.

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