J. Phys. A: Math. Theor. **40** (2007) 4483–4492 doi:10.1088/1751-8113/40/17/006

Synchronizability of complex networks

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Received 10 January 2007, in final form 2 March 2007
Published 11 April 2007
Online at stacks.iop.org/JPhysA/40/4483

Abstract
Recent interest in the study of networks associated with complex systems has led to a better understanding of the factors and parameters relating the topology of the associated networks with the dynamics of the systems, and in particular their synchronization. In this paper, by using known and new results from spectral graph theory, we characterize relevant factors which affect the synchronization of complex networks.

PACS numbers: 89.20.Hh, 89.75.Da, 89.75.Fb, 89.75.Hc, 89.75.−k, 05.45.Xt

1. Introduction

Natural and technical systems that change with time can be modelled mathematically by considering dynamical systems, either stochastic or deterministic. While linear dynamical deterministic systems are well known, many aspects of stochastic and nonlinear systems are not totally understood. One of them is the relationship between topology and synchronization of the associated network of special interest after many recent studies and models for real networks associated with complex systems have characterized their relevant topological parameters and graph invariants. In this paper we find bounds that connect the synchronizability of the network, through the eigenvalues of its Laplacian, with some important network parameters like the diameter, average distance, betweenness, isoperimetric number and maximum and minimum degrees.

Different theoretical frameworks have been considered to study the synchronization of complex networks, and in many of them synchronization is measured by considering the Laplacian spectra. For example, in the classical approach of Barahona and Pecora [1], a network of coupled identical oscillators has the following equations of motion:

$$\dot{x}_i(t) = F(x_i(t)) + \sigma \sum_{j=1}^{N} L_{ij} H(x_j(t)), \qquad i = 1, \dots, N, \tag{1}$$

where $x_i(t) = (x_{i1}(t), \ldots, x_{iN}(t))^T \in R^N$ are the state variables at each node i, $\dot{x}(t) = F(x(t))$ controls the dynamics of each node, $H(x(t))$ is an output function (the same at each node), σ is the coupling strength and L is the Laplacian matrix. The Laplacian is a symmetric matrix with zero row-sum that accounts for the topology of the network, defined to be $L_{ij} = -1$ if nodes i and j are connected, $L_{ii} = \delta_i$ if node i has degree δ_i (i.e. is connected to δ_i other nodes) and $L_{ij} = 0$ otherwise.

The linear stability of the synchronization state $\{x_i(t) = x^*(t), i = 1, 2, \ldots, N\}$ is given by equations that can be diagonalized into blocks as $\dot{y}(t) = [DF(s) + \theta DH(s)]y(t)$, where y represents the different perturbation modes from the synchronized state, $\theta = \sigma\theta_i$ for the i block, where θ_i is the ith eigenvalue of the Laplacian matrix. The largest Lyapunov exponent for this equation $\Delta(\theta)$ provides the linear stability of the synchronized state for any linear coupling. In [2], it is shown that this state is stable if $\Delta(\sigma\theta_i) < 0$ for $i = 2, \ldots, N$, where θ_i is an eigenvalue of the Laplacian. On the other hand, it has been found [1] that for many chaotic oscillators there exists a range of values (α_1, α_2) where this condition is satisfied. In this case, there exists a value of the coupling strength σ such that the synchronization state is linearly stable if and only if $\theta_N/\theta_2 < \alpha_2/\alpha_1 \equiv \beta$, where β is a constant independent of L. The values for β are between 5 and 100 for many chaotic oscillators [2]. Therefore, for large values of θ_N/θ_2 it is not possible to obtain synchronization, independently of the value of the coupling strength. In [1], the authors considered this framework to study the synchronization of a small-world network, known as 'pristine world', in which links have been added to shorten the average distance in a certain lattice (cycle of N nodes where each one is also linked to its $2k$ nearest neighbours to obtain a network with N nodes and Nk links), resulting in a good synchronizability.

In another study, Wu and Chua [3] show that an array of identical coupled resistive Chua oscillators can synchronize if the coupling conductances are sufficiently large. Using algebraic graph theory they provide a bound on the conductances. More precisely, the algebraic connectivity of the graph gives an upper bound on the value of the coupling conductance G_C that enables network synchronization. This fact suggests that a larger algebraic connectivity facilitates network synchronization, i.e. the more connected the graph is, the easier it is to obtain its global synchronization.

Finally, Wang and Chen [4] study the asymptotical synchronization of a dynamical network model characterized by equation (1) with $H(x_j(t)) = -x_j(t)$. The dynamical system reaches a state of asymptotical synchronization if $x_1(t) = \cdots = x_N(t) = s(t)$ when $t \to \infty$, where $s(t)$ can be an equilibrium point, a periodic orbit, or a chaotic attractor. The system is exponentially stable if $1/\theta_2$ is bounded by a constant.

In all of these models we can see that the synchronization of the network depends on the second eigenvalue θ_2 of the Laplacian matrix. When its value tends to 0, or $1/\theta_2$ is large, the network cannot reach a synchronization state. On the other hand, in the first model, the highest eigenvalue θ_N, and more precisely the ratio θ_N/θ_2, should also be small to attain synchronizability. However, these two Laplacian eigenvalues by themselves do not provide information about the relationship between the network topology and the dynamics of synchronization. In the following sections we will give results, in the form of bounds, that will provide a connection between the synchronizability of a network and several of its main invariants and parameters.

2. Spectral bounds for network synchronization

To characterize the synchronization of a network, we will find the relationship between the inverse of the second eigenvalue of the Laplacian matrix $1/\theta_2$ and the ratio θ_N/θ_2, with other graph parameters including the minimum and maximum degrees, diameter, average distance, isoperimetric number, betweenness centrality and clustering.

2.1. Bounds for $1/\theta_2$

The second eigenvalue θ_2 of the graph Laplacian is known as algebraic connectivity [5] as its value is related to the connectivity of the graph: if it is 0 the graph is not connected, if the value is near to 0 the graph can be easily split into several components by deleting some edges or vertices. Other more intuitive graph-theoretical measures can be related to the algebraic connectivity (see [5]).

2.1.1. Minimum degree of the graph. The minimum degree of a graph, δ, is related to the minimum connectivity of the graph. As $\theta_2 \leqslant (N\delta)/(N-1)$, we obtain

$$\frac{N-1}{N\delta} \leqslant \frac{1}{\theta_2}. \tag{2}$$

Therefore, if $\delta = 1$ and N is large we have a lower bound of 1. As the minimum degree increases this lower bound approaches 0; consequently the minimum degree alone cannot provide information on network synchronization. We will use this bound in section 3 to improve a bound for θ_N/θ_2 given in [6].

2.1.2. Edge connectivity. The edge connectivity of a graph, $e(G)$, is the minimum number of edges which must be deleted in a graph G to disconnect it. From [5], we obtain

$$\frac{1}{\theta_2} \leqslant \frac{1}{2e(G)\left(1-\cos\frac{\pi}{N}\right)}. \tag{3}$$

When N is large the bound becomes large independently of the value of $e(G)$. This parameter, by itself, is not of much help when studying the synchronization of a network.

2.1.3. Diameter. The graph diameter provides an inverse measure of the vertex connectivity. Intuitively, we can say that two nodes in a network are weakly connected if their shortest connection is through many other nodes, and therefore their distance is large. If this happens for all pairs of nodes, the diameter D of the graph is large. One bound relating the algebraic connectivity with the diameter has been given by Mohar in [7] as $D \geqslant 4/(N\theta_2)$ and we obtain

$$\frac{1}{\theta_2} \leqslant \frac{ND}{4}. \tag{4}$$

For graphs with a large diameter or graphs with many vertices, equation (4) gives a large upper bound. However, if both values are small then the network will be easy to synchronize. A lower bound for the inverse algebraic connectivity can be obtained from the diameter bound $D \leqslant 2\left\lceil \frac{\Delta+\theta_2}{4\theta_2} \ln(N-1)\right\rceil$ from [7], where Δ is the maximum degree of the graph:

$$\left(\frac{4}{\ln(N-1)}\left\lfloor\frac{D}{2}\right\rfloor - 1\right)\frac{1}{\Delta} \leqslant \frac{1}{\theta_2}. \tag{5}$$

From equation (5), we see that if $D \gg \ln(N-1)$ then the bound is greater than 1, and it will be difficult to reach a synchronization state unless the maximum degree Δ is very large.

Recent studies show that many real networks associated with complex systems have a logarithmic diameter $D \sim \ln N$ (small-world phenomenon). In this case, the lower bound can be written approximately as $1/\Delta \leqslant 1/\theta_2$. Since the maximum degree Δ is also large, in scale-free networks the lower bound will approach 0 and synchronization is possible. This bound can also be obtained from equation (2).

2.1.4. Average distance. Like the diameter, the average distance $\bar{l}$ among the vertices of the graph is an inverse measure for the connectivity of the graph G. In [7], there is a bound (obtained by B D McKay, but unpublished) which allows us give this upper bound for the inverse algebraic connectivity:

$$\frac{1}{\theta_2} \leqslant \frac{(N-1)\bar{l}}{2} - \frac{N-2}{4}. \tag{6}$$

From this bound, we see that networks that have a small number of nodes and a small average distance $\bar{l}$, should have a small inverse algebraic connectivity, so that the network can synchronize. A lower bound can be obtained from $\bar{l} \leqslant \frac{N}{N-1}\left(\left\lceil \frac{\Delta+\theta_2}{4\theta_2} \ln(N-1)\right\rceil + \frac{1}{2}\right)$:

$$\left(\left\lfloor \frac{2\bar{l}(N-1)-N}{2N\ln(N-1)} \right\rfloor - \frac{1}{4}\right)\frac{4}{\Delta} \leqslant \frac{1}{\theta_2}, \tag{7}$$

and we see that as N becomes large, the bound takes a lower value if the average distance is also small. However, if N is small and $\bar{l}$ relatively large, the bound is also large and the synchronization of the network would be more difficult to reach. The maximum degree can also affect the synchronization; larger degrees make it easier.

2.1.5. Isoperimetric number of a graph. The isoperimetric number of a graph is the number of edges that must be removed from a graph to obtain two connected components that are as large as possible. More precisely, it is defined as

$$\mathrm{i}(G) = \min_{|X| \leqslant \frac{N}{2}} \frac{|\delta X|}{|X|},$$

where X is a subset of vertices and δX is the boundary of X, i.e. the set of edges in G between vertices in X and vertices not in X.

Mohar [9] provides two different bounds for the algebraic connectivity of a graph based on the isoperimetric number. The first bound is a lower bound: $\mathrm{i}(G) \geqslant \theta_2/2$ which leads to

$$\frac{1}{2\mathrm{i}(G)} \leqslant \frac{1}{\theta_2} \tag{8}$$

and tells us that an isoperimetric number approaching 0 implies a large value for the inverse algebraic connectivity, and therefore the network is difficult to synchronize.

From the Cheeger inequality, which relates the isoperimetric number of a graph with its algebraic connectivity θ_2 and its maximum degree Δ [9], it is also possible to obtain an upper bound on $1/\theta_2$. From $\mathrm{i}(G) \leqslant \sqrt{\theta_2(2\Delta-\theta_2)}$, we have the inequality $\theta_2^2 - 2\Delta\theta_2 + \mathrm{i}(G)^2 \leqslant 0$ which leads to

$$\frac{1}{\Delta + \sqrt{\Delta^2 - \mathrm{i}(G)^2}} \leqslant \frac{1}{\theta_2} \leqslant \frac{1}{\Delta - \sqrt{\Delta^2 - \mathrm{i}(G)^2}}. \tag{9}$$

If $\mathrm{i}(G) \approx \Delta$, both bounds approach the value $1/\Delta$ and as the inverse of the algebraic connectivity will be less than 1, network synchronizability will be enhanced. On the other hand, if $\mathrm{i}(G) \sim 0$ the upper bound goes to ∞ while the lower bound is $\frac{1}{2\Delta}$, and we know from the results above that the network cannot synchronize.

From all these bound, we see that a small isoperimetric number $\mathrm{i}(G)$ means that fewer edges need to be removed to disconnect the graph into two large components and network synchronization is more difficult, while a large isoperimetric number makes harder to split the graph and favours it.

2.1.6. Average and maximum betweenness centrality. Vertex betweenness was first proposed by Freeman [10] in 1977 in the context of social networks and has been considered more recently as an important parameter in the study of networks associated with complex systems [11]. Betweenness is usually defined as the fraction of shortest paths between vertex pairs that go through the vertex considered. Therefore, in many models, betweenness is a measure of the influence of a node in the dissemination of information over a network [10, 12], and its possible relationship with the synchronization capability of a network is not a surprise.

To be more precise, if $\sigma_{uv}(w)$ denotes the number of shortest paths (geodetic paths) from vertex u to vertex v that go through w, and σ_{uv} is the total number of geodetic paths from u to v, then we define $b_w(u, v) = \sigma_{uv}(w)/\sigma_{uv}$ and the betweenness of vertex w is $B_w = \sum_{u,v\neq w} b_w(u, v)$. The betweenness of a graph $G = (V, E)$ of order N is $\overline{B} = \left(\sum_{u\in V} B_u\right)\big/N$ and the maximum betweenness of G is $B_{\max} = \max\{B_w | w \in V\}$. The average betweenness $\overline{B}$ is related to the average distance $\bar{l}$ of the graph as $\overline{B} = (N - 1)(\bar{l} - 1)$ [13].

The maximum betweenness centrality, $B_{\max}$, is larger than $\overline{B}$, so the corresponding lower bounds can also be used to obtain lower bounds. We have obtained [13] the bound $B_{\max} + 2 \geqslant \frac{N}{\sqrt{\theta_2(2\Delta-\theta_2)}}$ that relates $B_{\max}$ to the isoperimetric number. Therefore, we have

$$\frac{1}{\Delta + \sqrt{\Delta^2 - (N/(B_{\max}+2))^2}} \leqslant \frac{1}{\theta_2} \leqslant \frac{1}{\Delta - \sqrt{\Delta^2 - (N/(B_{\max}+2))^2}}. \tag{10}$$

If $\Delta(B_{\max} + 2) \sim N$, then $1/\theta_2 \sim 1/\Delta$ and there is synchronization, but if $B_{\max} + 2 \sim 2N(N - 1) + 2$ (as in a star graph) then $1/2\Delta \leqslant 1/\theta_2 \leqslant \infty$ and the bounds cannot be used to describe the synchronizability of a network because the range between bounds is too large.

2.1.7. Clustering. The clustering of the graph, a measure of the number of mutual neighbours of adjacent nodes, can affect the network synchronization; a large value means that there are many vertices that are close to each other, and this enhances network synchronization if there is a small variation on the clustering parameter for individual nodes.

2.2. Bounds for θ_N/θ_2

From the bounds for $1/\theta_2$ found in the previous section and the bound $\Delta < \theta_N \leqslant 2\Delta$ provided by Fiedler [5], we can find new lower and upper bounds for the ratio θ_N/θ_2. First, we use bounds given by Mohar [7].

If there are two subsets of vertices, B and C, at distance $r + 1$:

$$4(r-1)^2 \frac{|B||C|}{(N - |B| - |C|)\cdot(|B| + |C|)} < \frac{\theta_N}{\theta_2}, \tag{11}$$

where $|B|$ are $|C|$ are the cardinalities of the subsets. If both subsets have only one vertex and they are at maximum distance D, then the bound is

$$\frac{2(D-2)^2}{(N-2)} < \frac{\theta_N}{\theta_2}, \tag{12}$$

which tells us that for large N and a small diameter, the bound will tend to 0 and synchronization is possible.

Graphs with a diameter close to N would lead to large lower bound and the networks will hardly synchronize.

We find in [7] another lower bound relating θ_N/θ_2 to the average distance

$$\bar{l} < \frac{N}{N-1}\left\lceil 1 + \sqrt{\frac{\theta_N}{\theta_2}}\sqrt{\frac{\alpha^2-1}{4\alpha}}\right\rceil\left(\frac{1}{2} + \left\lceil \log_\alpha \frac{N}{2}\right\rceil\right), \tag{13}$$

where $\alpha > 1$ is a parameter. From this equation, we can see that a large average distance hinders the network synchronization.

The following lower and upper bounds are given in [8]:

$$\left(1-\frac{1}{N}\right)\frac{\Delta}{\delta} \leqslant \frac{\theta_N}{\theta_2} \leqslant (N-1)\Delta\mathcal{B}^E_{\max}D\bar{l}, \tag{14}$$

where $\mathcal{B}^E_{\max}$ is the normalized maximum edge betweenness of the network [13, 14]. The usefulness of the bounds is discussed in [8]. The lower bound tells us that the heterogeneity of degrees affects the synchronization of the network. A large difference between the maximum and minimum degrees makes the network more difficult to synchronize. However, this does not mean, as we will see below, that homogeneous networks can always synchronize. Simulation studies on model networks confirm this result, see [16].

In some cases it is possible to improve these bounds by considering bounds already known for θ_2 and θ_N.

From equation (2) and from $\theta_N \geqslant \Delta N/(N-1)$ (see [5]), we have

$$\frac{\Delta}{\delta} \leqslant \frac{\theta_N}{\theta_2}, \tag{15}$$

which is a small improvement on the lower bound given in equation (14). A better bound is also obtained from equation (2) and from $\theta_N \geqslant (\Delta+1)$, see [15]:

$$\left(1-\frac{1}{N}\right)\frac{(\Delta+1)}{\delta} \leqslant \frac{\theta_N}{\theta_2}. \tag{16}$$

For scale-free networks, for which $\delta = 1$, Δ is large and N is also large ($1-1/N \simeq 1$) the bound becomes $(\Delta+1) \leqslant \theta_N/\theta_2$, and the network has a very low synchronization capability.

Another upper bound for θ_N/θ_2 can be obtained from equation (7) and $\theta_N \leqslant 2\Delta$:

$$\frac{\theta_N}{\theta_2} \leqslant \Delta\left(N\left(\bar{l}-\frac{1}{2}\right)N+1-\bar{l}\right). \tag{17}$$

We can see that if some of the parameters Δ, $\bar{l}$, or N are large, then the upper bound is large (and not useful). However, if all three are small simultaneously, then the bound is small and the network will synchronize.

Another diameter-based upper bound can be obtained from equation (4):

$$\frac{\theta_N}{\theta_2} \leqslant \frac{\Delta N D}{2}. \tag{18}$$

As before, if the maximum degree, number of nodes and diameter are small then the network will synchronize.

With respect to lower bounds for θ_N/θ_2, we can use equations (5) and 7 to obtain

$$\left(\frac{4}{\ln(N-1)}\left\lfloor\frac{D}{2}\right\rfloor-1\right)\frac{\Delta+1}{\Delta} \leqslant \frac{\theta_N}{\theta_2}, \tag{19}$$

$$\left(\left\lfloor\frac{2\bar{l}(N-1)-N}{2N\ln(N-1)}\right\rfloor-\frac{1}{4}\right)\frac{4(\Delta+1)}{\Delta} \leqslant \frac{\theta_N}{\theta_2}. \tag{20}$$

Finally, from the bounds of equation (9) based on the isoperimetric number, we obtain

$$\frac{2}{1+\sqrt{1-(\mathrm{i}(G)/\Delta)^2}} \leqslant \frac{\theta_N}{\theta_2} \leqslant \frac{2}{1-\sqrt{1-(\mathrm{i}(G)/\Delta)^2}}. \tag{21}$$

If $\mathrm{i}(G) \sim \Delta$ then both bounds in (9) reach 2 and the network should synchronize, while if $\mathrm{i}(G) \sim 0$ the bounds go to infinity and the network cannot synchronize.

3. Synchronization in small-world scale-free networks

There have been several recent experimental studies about how different parameters affect synchronization in dynamical systems, particularly for small-world and scale-free networks.

In [1], the authors find that a low value for the average distance $\bar{l}$ facilitates network synchronization in small-world networks. In [8], the authors study different known models for small-world scale-free networks and they observe that synchronization is enhanced compared to regular meshes. In [16], it is claimed that heterogeneity in the degree distribution can reduce the average distance $\bar{l}$ improving the network synchronizability. However, these authors show that increasing the degree heterogeneity and decreasing $\bar{l}$ makes the network synchronization difficult to reach, as most of the network charge is concentrated in the nodes with largest degree (hubs) and B_{max} is high. Factors that can lead to synchronization are also studied in [1, 8, 16]: maximum degree, average distance of shortest paths, degree heterogeneity and betweenness centrality for a small-world network (Watts–Strogatz model). From their simulation results they conclude that in this sort of network, synchronization is enhanced by degree heterogeneity, a small average distance, a small maximum betweenness centrality, and a large maximum degree. Note that the effect of degree heterogeneity is different from the results in [8] for other networks, but the role of B_{max} is the same. There is also an example in [17] that shows that with B_{max} small the network does not synchronize.

On the other hand, the influence of the clustering of the graph in the synchronization of the network has been studied in [18] for networks with a Poisson degree distribution and for scale-free networks. In both cases, a high value of the clustering parameter inhibits global synchronization and the different clusters oscillate at distinct frequencies. Moreover, for scale-free networks, synchronization is promoted in the hubs although is not global.

Finally, in [19] the authors provide a method that, given a degree distribution, a network with this distribution can be produced with an algebraic connectivity inversely proportional to the number of links of the network. The isoperimetric number and Cheeger inequalities are also used to produce a bound for this parameter and then construct a network that will synchronize. A similar result can be found in [20].

In another context, Wang and Chen [4] have studied the synchronization of small-world scale-free networks. They find synchronization in a modified Watts and Strogatz model which consists of adding new links with probability p without rewiring the original links. The algebraic connectivity increases when this probability increases. This is an expected result as with a large value for p the graph has many links and approaches a complete graph. The same authors in [21] study the synchronizability of the Barabási–Albert preferential attachment model, and they see that it remains almost unchanged during the growing process. Moreover, network synchronization is resilient to the deletion of random nodes, but fragile if the hubs are selected for deletion.

Small-world scale-free networks have a small diameter and average distance [22]; this enhances synchronizability. The study of some other properties will allow us to use the bounds found in the previous sections to understand the conditions that can lead to the synchronization of a given network.

3.1. Difference between the maximum and minimum degrees

From the bound given by equation (16),

$$\left(1 - \frac{1}{N}\right)\frac{(\Delta + 1)}{\delta} \leqslant \frac{\theta_N}{\theta_2}, \qquad (22)$$

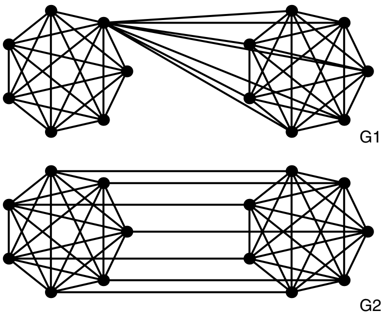

Figure 1. Two different graphs with maximum isoperimetric number.

Table 1. Invariants for the graphs of figure 1.

	δ	Δ	D	$\bar{l}$	i(G)	$B_{\max}$	θ_N	θ_2
G_1	$n-1$	$2n-1$	2	$\frac{3n-2}{2n-1}$	1	$2n(n-1)$	$2n$	1
G_2	n	n	2	$\frac{3n-2}{2n-1}$	1	$n-1$	$n+2$	2

if a network has a large difference between the minimum and maximum degrees, δ and Δ, then it is difficult to synchronize. As very often this is the case for scale-free networks and N is large, these category of networks are difficult to synchronize as it is found in [8].

Consider the union of two complete graphs K_n with n nodes connected according to the different patterns shown in figure 1. The new graphs will have $N = 2n$ nodes.

In the first case G_1, one node from the first complete graph is connected to all the nodes of the second complete graph. In the second case G_2, each node from the first complete graph is connected to the corresponding node in the second graph.

The values for the relevant network parameters are shown in table 1.

For these two graphs, we find the lower bound for the ratio θ_N/θ_2 which for G_1 is

$$\frac{N-1}{N}\frac{(\Delta+1)}{\delta} = 2 + \frac{1}{n-1} \leqslant \frac{\theta_N}{\theta_2}, \tag{23}$$

which tends to 2 when n is large. For G_2, we obtain

$$\frac{N-1}{N}\frac{(\Delta+1)}{\delta} = 1 + \frac{n-1}{2n^2} \leqslant \frac{\theta_N}{\theta_2} \tag{24}$$

and the bound goes to 1 when n is large and therefore the network can synchronize better.

We note, from the bound, that the heterogeneity in the degree distribution [8, 23] affects the synchronizability through the ratio between the highest and lowest degree nodes.

3.2. Isoperimetric number

When the difference between δ and Δ in the network is small (homogeneous networks), then we should check if the network can be easily disconnected in two parts with sizes as large as possible (i.e. if the isoperimetric number is small). From the bounds given by equations (8) and (21) the network cannot synchronize.

In figure 2, we have an example of such a graph with its relevant invariants and parameters listed in table 2. Note that although most of the parameters should favour synchronization, the network has a low synchronizability as the isoperimetric number is very small.

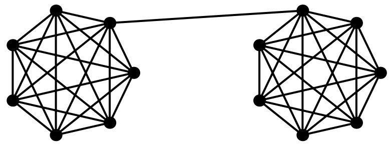

Figure 2. A graph with a low isoperimetric number.

Table 2. Invariants for the graph of figure 2.

δ	Δ	D	$\bar{l}$	i(G)	B_{max}	C
$n-1$	n	3	$\frac{4n-3}{2n-1}$	$\frac{1}{n}$	$2n(n-1)$	$1-\frac{1}{n}$

3.3. Number of nodes with degree 1

This is another factor to be considered for synchronization. If a scale-free network has a high ratio of degree 1 nodes, even when the maximum degree is small, it will be hard for the network to synchronize. Let X_0 be the set of nodes of degree 1, and let $N_0 = |X_0| \leqslant \frac{|V|}{2}$. From the definition of isoperimetric number

$$\mathrm{i}(G) = \min_{|X| \leqslant \frac{N}{2}} \frac{|\delta X|}{|X|} \leqslant \frac{|\delta X_0|}{|X_0|} = 1 \tag{25}$$

and from equation (8), as $1/2 \leqslant 1/\theta_2$, the network is difficult to synchronize.

In conclusion, and from the study of the different network parameters and invariants, a network will synchronize if the diameter is small, the average distance is also small, the maximum betweenness B_{max} is small, the clustering parameter is high, the isoperimetric number is high, the ratio Δ/δ is small and finally if the number of nodes with degree 1 is also relatively small. The heterogeneity of the degree distribution and other factors suggested by some authors can be related to those mentioned above. How easy or difficult the synchronizability of a network is can be seen from the bounds relating the former parameters with the values of $1/\theta_2$ and θ_N/θ_2, obtained from the Laplacian of the network, which characterize the synchronizability of a given network.

Acknowledgments

This research was supported by the Secretaria de Estado de Universidades e Investigación (Ministerio de Educación y Ciencia), Spain, and the European Regional Development Fund (ERDF) under project TEC2005-03575.

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