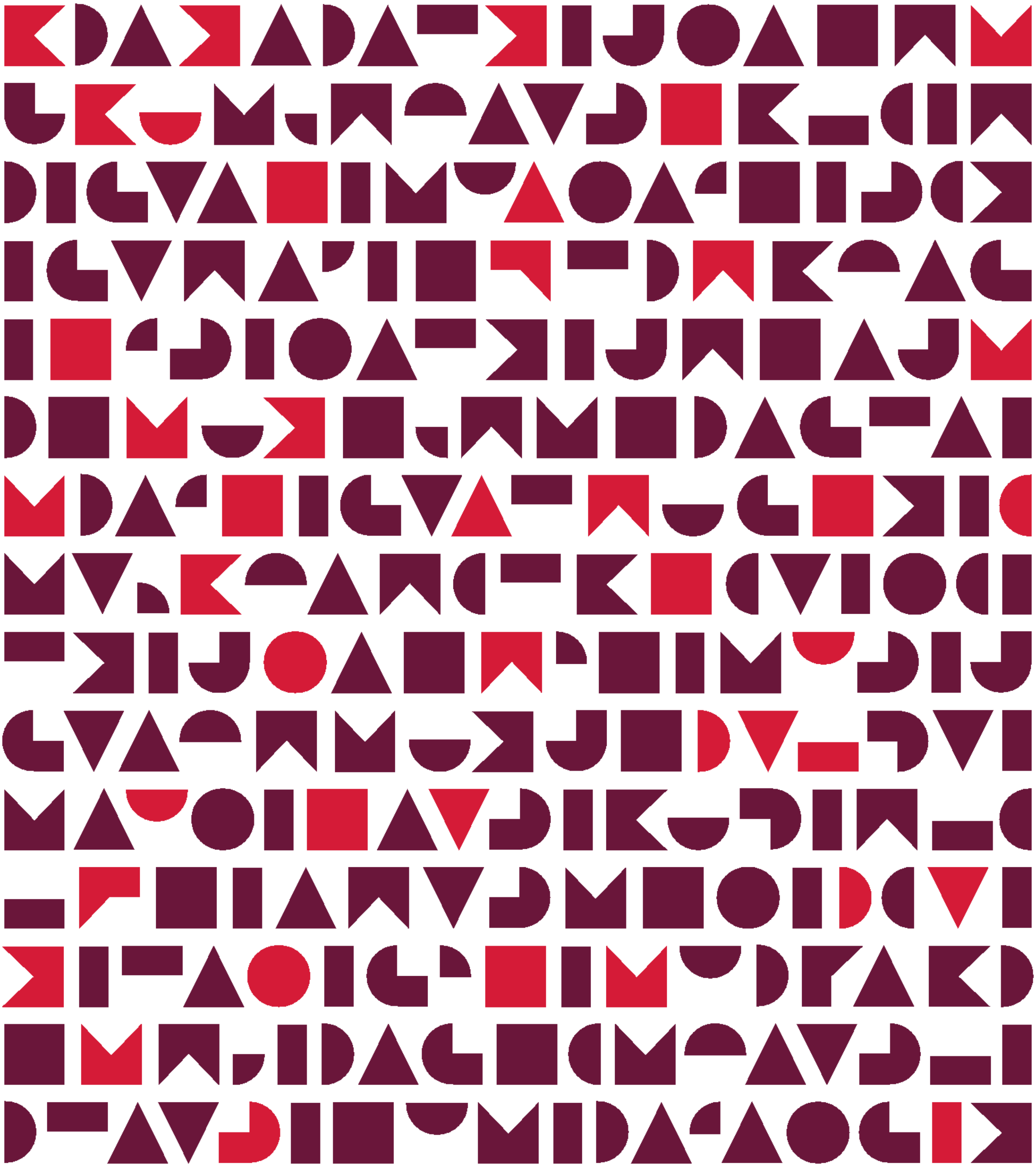

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A new operation on digraphs: the Manhattan product *

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Abstract. We give a formal definition of a new product of bipartite digraphs, the Manhattan product, and we study some of its main properties. It is shown that if all the factors of the above product are (directed) cycles, then the digraph obtained is a Manhattan street network. To this respect, it is proved that many properties of these networks, such as high symmetries and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs. Moreover, we prove that the Manhattan product of two Manhattan streets networks is also a Manhattan street network. Also, some necessary conditions for the Manhattan product of two Cayley digraphs to be again a Cayley digraph are given.

Key words: Digraph, Product, Cayley Digraph, Hamiltonian Cycle, Manhattan Street Network.

1 Introduction

The 2-dimensional Manhattan street network M_2 was introduced simultaneously, in different contexts, by Morillo *et al.* [6] and Maxemchuk [5] as an unidirectional regular mesh structure resembling locally the topology of the avenues and streets of Manhattan (or *l'Eixample* in downtown Barcelona). In fact, M_2 has a natural embedding in the torus and it has been extensively studied in the literature as a model of interconnection networks.

Before outlining the contents of the paper, recall that a digraph $G = (V, A)$ consists of a set of *vertices* V, together with a set of *arcs* A, which are ordered pairs of vertices, $A \subset V \times V = \{(u, v) : u, v \in V\}$. An arc (u, v) is usually depicted as an arrow with *tail* u (initial vertex) and *head* v (end vertex), that is, $u \to v$. The *in-degree* $\delta^-(u)$ (respectively, *out-degree* $\delta^+(u)$) of a vertex u is the number of arcs with tail (respectively, head) u. Then, G is *δ-regular*

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when $\delta^-(u) = \delta^+(u) = \delta$ for every vertex $u \in V$. Given a digraph $G = (V, A)$, its *converse* digraph $\overline{G} = (V, \overline{A})$ is obtained from G by reversing all the orientations of the arcs in A, that is, $(u, v) \in \overline{A}$ if and only if $(v, u) \in A$. The standard definitions and basic results about graphs and digraphs not defined here can be found in [1,2,7].

In this paper, we first recall the definition and some of the properties of the Manhattan street network (where the Manhattan product takes its name from). Afterwards, we introduce the Manhattan product of (bipartite) digraphs. It is shown that when all the factors are (directed) cycles, then the digraph obtained is just the Manhattan street network. Moreover, we prove that the Manhattan product of two Manhattan streets networks is also a Manhattan street network. In fact, many properties of these networks, such as high symmetries and the presence of Hamiltonian cycles, are shared by the Manhattan product of some digraphs. We also investigate when the Manhattan product of two Cayley digraph is again a Cayley digraph and characterize the corresponding group.

2 Manhattan street networks

In this section, we recall the definition and some basic properties of a class of toroidal directed networks, commonly known as Manhattan street networks [3,4].

Given n *even* positive integers $N_1, N_2, \ldots, N_n$, the *n-dimensional Manhattan street network* $M_n = M(N_1, N_2, \ldots, N_n)$ is a digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$. Thus, each of its vertices is represented by an n-vector $\boldsymbol{u} = (u_1, u_2, \ldots, u_n)$, with $0 \leq u_i \leq N_i - 1$, $i = 1, 2, \ldots, n$. The arc set $A(M_n)$ is defined by the following adjacencies (here called *i-arcs*):

$$(u_1, \ldots, u_i, \ldots, u_n) \;\rightarrow\; (u_1, \ldots, u_i + (-1)^{\sum_{j\neq i} u_j}, \ldots, u_n) \quad (1 \leq i \leq n). \quad (1)$$

Therefore, M_n is an n-regular digraph on $N = \prod_{i=1}^{n} N_i$ vertices.

The properties of M_n are the following:

- Homomorphism: There exist an homomorphism from M_n to the symmetric digraph of the hypercube Q_n^*, so that M_n is both a bipartite and 2^n-partite digraph.
- Vertex-symmetry: The n-dimensional Manhattan street network M_n is a vertex-symmetric digraph.
- Line digraph: For any N_1, N_2, the 2-dimensional Manhattan street network $M_2(N_1, N_2)$ is a line digraph.
- Diameter: For $N_i > 4$, the diameter of the n-dimensional Manhattan street network $M_n = M(N_1, N_2, \ldots, N_n)$, $i = 1, 2, \ldots, n$, is
  $(a)\, D(M_n) = \frac{1}{2}\sum_{i=1}^{n} N_i + 1$, if $N_i \equiv 0 \pmod 4$ for any $1 \leq i \leq n$;
  $(b)\, D(M_n) = \frac{1}{2}\sum_{i=1}^{n} N_i$, otherwise.

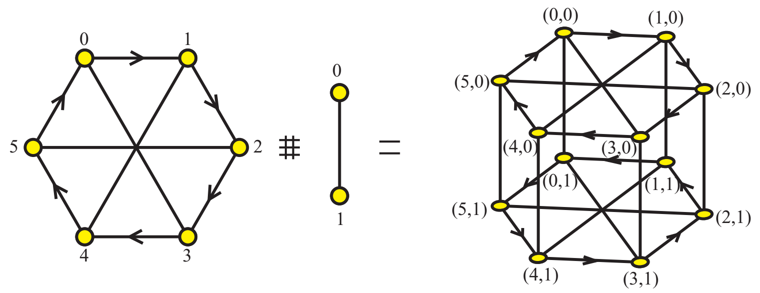

Fig. 1. The Manhattan product $\text{Cay}(\mathbb{Z}_6, \{1,3\}) \# K_2^*$ (undirected lines stand for pairs of arcs in opposite directions).

- Hamiltonicity: The n-dimensional Manhattan street network M_n is Hamiltonian.

3 The Manhattan product and its basic properties

In this section, we present an operation on (bipartite) digraphs which, as a particular case, gives rise to a Manhattan street network. With this aim, let $G_i = (V_i, A_i)$ be n bipartite digraphs with independent sets $V_i = V_{i0} \cup V_{i1}$, $N_i = |V_i|$, $i = 1, 2, \ldots, n$. Let π be the characteristic function of $V_{i1} \subset V_i$ for any i, that is,

$$\pi(u) = \begin{cases} 0 \text{ if } u \in V_{i0}, \\ 1 \text{ if } u \in V_{i1}. \end{cases}$$

Then, the *Manhattan product* $M_n = G_1 \# G_2 \# \cdots \# G_n$ is the digraph with vertex set $V(M_n) = V_1 \times V_2 \times \cdots \times V_n$, and each vertex $(u_1, u_2, \ldots, u_i, \ldots, u_n)$ is adjacent to vertices $(u_1, u_2, \ldots, v_i, \ldots, u_n)$, $1 \le i \le n$, when

- $v_i \in \Gamma^+(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is even,
- $v_i \in \Gamma^-(u_i)$ if $\sum_{j \neq i} \pi(u_j)$ is odd,

where $\Gamma^+(u_i)$ and $\Gamma^-(u_i)$ denote the sets of vertices adjacent from u_i and to u_i, respectively.

Fig. 1 shows an example of the Manhattan product of the circulant digraph on six vertices and steps 1 and 3 (in other words, the Cayley digraph on $\mathbb{Z}_6$ with generating set {1,3}) by the symmetric complete digraph on two vertices, $K_2^* = \text{Cay}(\mathbb{Z}_2, \{1\})$.

Thus, if every G_i is δ_i-regular, then M_n is a δ-regular digraph, with $\delta = \sum_{i=1}^n \delta_i$, on $N = \prod_{i=1}^n N_i$ vertices.

Some of the basic properties of the Manhattan product, which are a generalization of the properties of the Manhattan street networks given in [3], are presented in the next proposition.

Proposition 1. *The Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ satisfies the following properties:*

(a) *The Manhattan product holds the associative and commutative properties.*
(b) *There exists an homomorphism from H to the symmetric digraph of the hypercube Q_n^*. Therefore, H is a bipartite and 2^n-partite digraph.*
(c) *For any $n-k$ fixed vertices $x_i \in V_i$, $i = k+1, k+2, \ldots, n$, the subdigraph of H induced by the vertices $(u_1, u_2, \ldots, u_k, x_{k+1}, x_{k+2}, \ldots, x_n)$ is either the Manhattan product $H_k = G_1 \# G_2 \# \cdots \# G_k$ or its converse $\overline{H}_k$, depending on if $\alpha := \sum_{i=k+1}^{n} \pi(x_i)$ is even or odd, respectively.*
(d) *If each G_i, $i = 1, 2, \ldots, n$, is isomorphic to its converse, then H also is.*

As an example of a Manhattan product satisfying the Proposition 1(d), see again Fig. 1.

4 The Manhattan product and the Manhattan street networks

In this section, we show the relationship between the digraphs obtained by the Manhattan product and the Manhattan street networks.

Proposition 2. *The Manhattan product of directed cycles with even orders is a Manhattan street network. More precisely, if $G_i = C_{N_i}$, N_i even, then*

$$C_{N_1} \# C_{N_2} \# \cdots \# C_{N_n} = M(N_1, N_2, \ldots, N_n).$$

Proof. Each cycle C_{N_i} has set of vertices $V_i = \mathbb{Z}_{N_i}$, and adjacencies $\Gamma^+(u_i) = \{u_i + 1 \pmod{N_i}\}$ and $\Gamma^-(u_i) = \{u_i - 1 \pmod{N_i}\}$, such that V_{i0} and V_{i1} are the sets of even and odd vertices, respectively. Thus, the set of vertices in the Manhattan product of directed cycles is $\mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$ and each vertex $(u_1, u_2, \ldots, u_i, \ldots, u_n)$ is adjacent to the vertices $(u_1, u_2, \ldots, v_i, \ldots, u_n)$, for $1 \le i \le n$, when

- $v_i = u_i + 1$ if and only if $\sum_{j \ne i} \pi(u_j)$ is even and, then, $\sum_{j \ne i} u_j$ is also even,
- $v_i = u_i - 1$ if and only if $\sum_{j \ne i} \pi(u_j)$ is odd and, then, $\sum_{j \ne i} u_j$ is also odd,

which corresponds to the definition of the Manhattan street network.

Another expected result of the Manhattan product is the following:

Proposition 3. *The Manhattan product of two Manhattan street networks is a Manhattan street network. More precisely, if $M^1 = M(N_1^1, N_2^1, \ldots, N_{n_1}^1)$ and $M^2 = M(N_1^2, N_2^2, \ldots, N_{n_2}^2)$, then*

$$M^1 \# M^2 = M,$$

where $M = M(N_1^1, N_2^1, \ldots, N_{n_1}^1, N_1^2, N_2^2, \ldots, N_{n_2}^2)$.

The result of the above proposition can be seen as a corollary of the Proposition 2 and the associative property. Indeed,

$$\begin{aligned} M^1 \# M^2 &= M(N_1^1, N_2^1, \ldots, N_{n_1}^1) \# M(N_1^2, N_2^2, \ldots, N_{n_2}^2) \\ &= (C_{N_1}^1 \# C_{N_2}^1 \# \cdots \# C_{N_{n_1}}^1) \# (C_{N_1}^2 \# C_{N_2}^2 \# \cdots \# C_{N_{n_2}}^2) \\ &= C_{N_1}^1 \# C_{N_2}^1 \# \cdots \# C_{N_{n_1}}^1 \# C_{N_1}^2 \# C_{N_2}^2 \# \cdots \# C_{N_{n_2}}^2 \\ &= M(N_1^1, N_2^1, \ldots, N_{n_1}^1, N_1^2, N_2^2, \ldots, N_{n_2}^2) = M. \end{aligned}$$

5 Symmetries

Here we study the symmetries of the digraphs obtained by the Manhattan product.

Proposition 4. *Let G_i be vertex-symmetric digraphs such that they are isomorphic to their converses, $i = 1, 2, \ldots, n$. Then, the Manhattan product $H = G_1 \# G_2 \# \cdots \# G_n$ is vertex-symmetric.*

6 Cayley digraphs and the Manhattan product

In this section, we investigate when the Manhattan product of Cayley digraphs is again a Cayley digraph. This generalizes the case studied of Manhattan street networks [3,4], where the factors of the product are directed cycles (see Proposition 2), that is, Cayley digraphs of the cyclic groups. Because of the associative property of the Manhattan product (see Proposition 1(a)), we only need to study the case of two factors.

Theorem 1. *Let $G_1 = \mathrm{Cay}(\Gamma_1, \Delta_1)$ be a bipartite Cayley digraph of the group Γ_1 with generating set $\Delta_1 = \{a_1, a_2, \ldots, a_p\}$ and set of generating relations R_1, such that there exists a group automorphism ψ_1 satisfying $\psi_1(a_i) = a_i^{-1}$, for $i = 1, 2, \ldots, p$. Let $G_2 = \mathrm{Cay}(\Gamma_2, \Delta_2)$ be the bipartite Cayley digraph of the group Γ_2 with generating set $\Delta_2 = \{b_1, b_2, \ldots, b_q\}$ and set of generating relations R_2, such that there exists a group automorphism ψ_2 satisfying $\psi_2(b_j) = b_j^{-1}$, for $j = 1, 2, \ldots, q$. Then, the Manhattan product $H = G_1 \# G_2$ is the Cayley digraph of the group*

$$\Gamma = \langle \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q \mid R_1', R_2', (\alpha_i\beta_j)^2 = (\alpha_i\beta_j^{-1})^2 = 1, i \neq j \rangle, \quad (2)$$

where R_1' is the same set of generating relations as R_1 changing a_i by α_i (and similarly for R_2', changing b_j by β_j).

This result can be compared with the well-known following one [7]: If G_1 and G_2 are, respectively, Cayley digraphs of the groups $\Gamma_1 = \langle a_1, a_2, \ldots, a_p \mid R_1 \rangle$

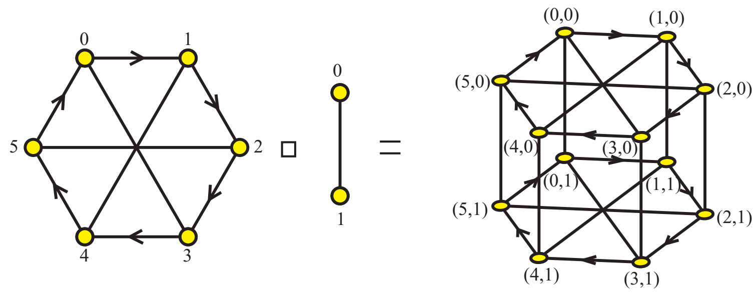

Fig. 2. The direct product $\mathrm{Cay}(\mathbb{Z}_6, \{1,3\}) \square K_2^*$ (undirected lines stand for pairs of arcs in opposite directions).

and $\Gamma_2 = \langle b_1, b_2, \ldots, b_q \mid R_2 \rangle$, then its direct product $G_1 \square G_2$ is the Cayley digraph of the group

$$\Gamma = \Gamma_1 \times \Gamma_2 = \langle \alpha_1, \alpha_2, \ldots, \alpha_p, \beta_1, \beta_2, \ldots, \beta_q | R_1', R_2', \alpha_i\beta_j = \beta_j\alpha_i, 1 \le i \le p, 1 \le j \le q \rangle,$$

with the same notation as above. Fig. 2 illustrates an example of direct product of Cayley digraphs, which can be compared with the Manhattan product of the same digraphs shown in Fig. 1.

7 An alternative definition

The results of the preceding section, specifically the structure of the color-preserving automorphisms, suggest to study some alternative definitions of the Manhattan product of digraphs when they satisfy some conditions. More precisely, if each of the factors G_i of the Manhattan product has an involutive automorphism from G_i to $\overline{G}_i$, we have the following result:

Proposition 5. *Let ψ_i be an involutive automorphism from G_i to $\overline{G}_i$, for $i = 1, 2, \ldots, n$. Then, the Manhattan product $H = G_1 \# G_2 \# \ldots \# G_n$ is the digraph with vertex set $V(M_n) = \mathbb{Z}_{N_1} \times \mathbb{Z}_{N_2} \times \cdots \times \mathbb{Z}_{N_n}$ and the following adjacencies ($i = 1, 2, \ldots, n$):*

$$(u_1, u_2, \ldots, u_i, \ldots, u_n) \quad \rightsquigarrow \quad \big(\psi_1(u_1), \psi_2(u_2), \ldots, v_i, \ldots, \psi_n(u_n)\big),$$

where $v_i \in \Gamma^+(u_i)$.

Proof. For the sake of simplicity, we write the adjacencies of the first definition and the alternative one, respectively, as ($i = 1, 2, \ldots, n$):

$$(u_1, u_2, \ldots, u_i, \ldots, u_n) \quad \to \quad \big(u_1, u_2, \ldots, \Gamma^{(-1)^{\sum_{j \ne i} \pi(u_j)}}(u_i), \ldots, u_n\big), \quad (3)$$
$$(u_1, u_2, \ldots, u_i, \ldots, u_n) \quad \rightsquigarrow \quad \big(\psi_1(u_1), \psi_2(u_2), \ldots, \Gamma^+(u_i), \ldots, \psi_n(u_n)\big), \quad (4)$$

where $\Gamma^{+1} \equiv \Gamma^+$ and $\Gamma^{-1} \equiv \Gamma^-$.

The isomorphism from the first definition to the alternative one is

$$\Phi(u_1,\dots,u_i,\dots,u_n) = \left(\psi_1^{\sum_{j\neq 1}\pi(u_j)}(u_1),\dots,\psi_i^{\sum_{j\neq i}\pi(u_j)}(u_i),\dots,\psi_n^{\sum_{j\neq n}\pi(u_j)}(u_n)\right).$$

Indeed, let us see that this mapping preserves the adjacencies. First, by (3), we have

$$\Phi\big(\Gamma^+(u_1,\dots,u_i,\dots,u_n)\big) = \Big(\psi_1^{\sum_{j\neq 1}\pi(u_j)+1}(u_1),\dots, \psi_i^{\sum_{j\neq 1}\pi(u_j)}\big(\Gamma^{(-1)^{\sum_{j\neq i}\pi(u_j)}}(u_i)\big),\dots,\psi_n^{\sum_{j\neq 1}\pi(u_j)+1}(u_1)\Big).\quad(5)$$

Whereas, by (4), we have

$$\Gamma^+\big(\Phi(u_1,\dots,u_i,\dots,u_n)\big) = \Big(\psi_1^{\sum_{j\neq 1}\pi(u_j)+1}(u_1),\dots,\Gamma^+\big(\psi_i^{\sum_{j\neq i}\pi(u_j)}(u_i)\big),\dots,\psi_n^{\sum_{j\neq n}\pi(u_j)+1}(u_n)\Big).\quad(6)$$

To check that the i-th entry in (5) and (6) represents the same set, we distinguish two cases:

- If $\sum_{j\neq i}\pi(u_j) = \alpha$ is an even number, then $\psi_i^\alpha = Id$ (as ψ_i is involutive) and $Id\big(\Gamma^+(u_i)\big) = \Gamma^+\big(Id(u_i)\big)$.
- If $\sum_{j\neq i}\pi(u_j) = \beta$ is an odd number, then $\psi_i^\beta = \psi_i$ and $\psi_i\big(\Gamma^-(u_i)\big) = \Gamma^+\big(\psi_i(u_i)\big)$ (as ψ_i is an automorphism from G_i to $\overline{G}_i$) .

In the case of the Manhattan street network M_n, $G_i = C_i$ (Proposition 2). Then, a simple way of choosing the involutive automorphisms is the following: $\psi_i(u_i) = -u_i (\mathrm{mod}\ N_i)$. In fact, it is readily checked that any isomorphism from C_i to $\overline{C}_i$ is involutive. That gives the following definition of M_n [3,4]: The Manhattan street network $M_n = M_n(M_1,\dots,M_n)$ is the digraph with vertex set $\mathbb{Z}_{N_1}\times\mathbb{Z}_{N_2}\times\cdots\times\mathbb{Z}_{N_n}$ and the adjacencies

$$(u_1,u_2,\dots,u_i,\dots,u_n) \quad\rightsquigarrow\quad (-u_1,-u_2,\dots,u_i{+}1,\dots,-u_n) \qquad (1\le i\le n).$$

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