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Complex Networks: Deterministic Models.

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Abstract. The recent discovery that many networks associated with complex systems belong to a new category known as scale-free small-world has led to a surge in the number of new models for these systems. Many studies are based on probabilistic and statistical methods which capture well some of the basic properties of the networks. More recently, a deterministic approach has proven useful to complement and enhance the probabilistic and simulation techniques. In this paper, after a short introduction to the main concepts and models, we survey recent deterministic models.

Keywords.
Complex networks, small-world networks, scale-free networks, deterministic models, graph theory.

1. Introduction

Recent research shows that many networks associated with complex systems, like the World Wide Web, the Internet, telephone networks, transportation systems (including the power distribution network), and biological and social networks, belong to a class of networks known as small-world scale-free networks [4,35,41,42,57,52]. These networks exhibit both strong local clustering (nodes have many mutual neighbors) and a small average path length and diameter (maximum distance between any two nodes). Another important common characteristic is that the number of links attached to the nodes usually obeys a power-law distribution (is scale-free). Moreover, introducing a new measuring technique, it has recently been discovered that many real networks are self-similar, see [70]. Along these observational studies, researchers have developed different models and techniques -borrowed in some cases from statistical physics, computer science and graph theory- which should help us to understand and predict the behavior and characteristics of the systems. The origin of the interest in these studies may be found in the papers by Watts and Strogatz on small-world networks [75] and Barabási and Albert on scale-free networks [9]. Since then the study of complex networks has received a considerable boost as an interdisciplinary subject. Several excellent general reviews and books are available, and therefore in this paper we refer to them for the reader who would like to obtain more information on the topic, see references [74,71,3,31,11,58,73,8,20,32,67].

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To describe these complex networks several models have been proposed and analysed through simulations and considering probabilistic methods. The first, which triggered a sharp interest in the studies of the different properties of small-world, was the simple computational method to produce small-world netkorks proposed by Watts and Strogatz in their often cited paper [75]. Shortly after, Barabási and Albert [9,12] introduced a network model which uses two main mechanisms to produce a power-law distribution for the degrees: growth and preferential attachment. Dorogovtsev, Mendes, and Samukhin [33] use a "master-equation" to obtain an exact solution for a class of growing network models, Krapivsky, Redner, and Leyvraz [48] examined the effect of a nonlinear preferential attachment on network dynamics and topology. Models that incorporate aging and cost and capacity constraints were studied by Amaral et al. [6] to explain deviations from the power-law behavior in several real-life networks. Dorogovtsev and Mendes [28] also considered the evolution of networks with aging of sites. Bianconi and Barabási [16] introduced a model addressing the competitive aspect of many real networks such as the WWW. Additionally, in real systems microscopic events affect the network evolution, including the addition or rewiring of new edges or the removal of vertices or edges. Albert and Barabási [2] discussed a model that incorporates new edges between existing vertices and the rewiring of old edges. Dorogovtsev and Mendes [29] considered a class of undirected models in which new edges are added between old vertices and existing edges can be removed. It is now well established that preferential attachment can explain the power-law characteristic of networks, but some other alternative mechanisms affecting the evolution of growing networks can also lead to the observed scale-free topologies. Kleinberg et al. [46] and Kumar et al. [49,50] suggested certain copying mechanisms in an attempt to explain the power-law degree distribution of the World Wide Web. Chung et al. [26] also introduced a duplication model for biological networks. Krapivsky and Render [47] use an edge redirection mechanism which is equivalent to the model of Kumar et al [49,50]. All of these models have been studied intensively. Barthélémy and Amaral studied the origins of the small-world behavior in Ref. [15]. Barrat and Weigt addressed analytically as well as numerically the structure properties of the Watts-Strogatz model [13]. Amaral *et al.* investigated the statistical characteristics of many real-life networks [6]. Latora and Marchiori introduced the concept of efficiency of a network and found that small-world networks are both globally and locally efficient [51]. References [65,56,66,61] deal with the percolation properties of the networks and in particular the spread of information and disease along the shortest path in the graph or the spanning trees. More recently, researchers have also focused their attention on other aspects characterising properties of small-world scale-free networks [62,53,39,38,19,36,17,44].

While most of the models referenced above are stochastically produced and analysed, small-world scale-free networks can be created also by deterministic methods. Deterministic models have the strong advantage that it is often possible to compute analytically their properties, which may be compared with experimental data from real and simulated networks. Deterministic networks can be created by various techniques: modification of some regular graphs [23], addition and product of graphs [25], and other mathematical methods, like those which appear in [84]. Another important technique producing families of deterministic small-world scale-free networks is based on recursive techniques. Recursive and general scale-free constructions are given in

[12,25,43,27,68,63]. Recursive methods based on the existence of cliques in a given network have appeared in [22,84,7,34,80,81,79].

After a short introduction to the basic definitions and models on probabilistic constructions, we present a review of recent deterministic models, mainly graph constructions which share the property that the graphs contain many complete subgraphs. Although with different names (hierarchical, pseudo-fractal, Apollonian, geometrical, recursive cliques) they all consider the same principle: the successive addition of vertices, each one connected to all the vertices of a subgraph isomorphic to a clique or complete graph. The rule used to add vertices produce different final networks sharing many basic properties: they are small-world, scale-free, with high clustering, and small average distance.

2. Basic properties of complex networks

Many measures and parameters have been considered and studied to analyze complex networks. The use of certain subsets is in most cases sufficient to capture the rough structure of a given network. Recent research focuses on three main concepts, namely the mean distance of the network (average path length) -and in some cases the diameter-, the clustering coefficient and the degree distribution.

Although the interest in the analysis of networks has always been present in the scientific community through the work in social sciences, the paper from Watts and Strogatz [75] extended this interest to other scientists. In their work they produce networks with a small average path length, similar to that of a random graph, and a relatively large clustering coefficient, as occurs in many structured networks, and show that real networks like the WWW, a power grid, and the neuronal network of the worm *C. Elegans* have a similar relation. A few months later, Barabási and Albert [9] discovered that many of these networks have a degree distribution that follows a power law (are scale-free), and introduced a model to produce that distribution. In this section we provide a short introduction to these concepts.

First we give the following definitions, see also [76].

A network is represented by a graph $G = (V, E)$ with vertex (node) set $V = V(G)$, and edge (link) set $E = E(G)$. The *order* of the graph, $n = |V|$, is the number of vertices or nodes of it. The *degree* of a vertex i, which we denote k_i, is the number of edges incident to i and the degree of a graph G is $\Delta = \max_{i \in V} k_i$. A graph is Δ-*regular* if the degree of all its vertices is Δ.

A *complete graph* K_d (also referred in the literature as a *d-clique*) is a graph with d vertices, where there are no loops or multiple edges and every vertex is joined to every other by an edge. Generally speaking, two graphs are said to be *isomorphic* if the vertices and edges of one graph match up with vertices and edges of the other, and the edge matching is consistent with the vertex matching.

The basic family of graphs considered by Watts and Strogats (and other studies) are known as circulant graphs. They considered the particular case $C_{n,\Delta}$, Δ even, which has n nodes labelled with the integers modulo n, and Δ links per node such that each node i is adjacent to the nodes $i \pm 1, i \pm 2, \cdots, i \pm \frac{\Delta}{2} \pmod{n}$.

Diameter and mean distance

In a graph the *distance* between two vertices i and j, $d(i,j)$, is defined as the number of edges along the shortest path between i and j. The maximum distance between any pair of vertices, $D = \max_{i,j \in V} d(i,j)$, is the *diameter* of the graph. The mean distance of the graph is defined to be $\dfrac{1}{n(n-1)} \displaystyle\sum_{i,j \in V} d(i,j)$. In some probabilistic models the average path length (APL) of the network is introduced as the average value of $d(i,j)$, with i and j chosen uniformly at random. In a social network, for example, the APL is associated with the average number of acquitances existing in the shortest chain which connects any two persons of the network. Note that all these definitions only make sense if we require that the graph is connected.

As Watts and Strogatz noticed in [75], the average path length of most real complex networks is relatively small, even when the networks are sparse (they have many fewer edges than a complete graph with the same number of nodes). Some authors call this the small-world effect, and hence the name small-world networks. However, random networks also have a small diameter (and mean distance) [18] and they are different from real networks. For this reason it is perhaps better to refer to small-world networks as those networks that also have a relatively large clustering with respect to a similar random network.

Clustering coefficient

Clustering measures the "connectedness" of a graph and is another of the parameters used to characterize small-world networks. For example, in a friendship network, it's very likely that two of your friends are also friends with each other, reflecting the clustering nature of this social network.

The *clustering coefficient* was introduced to quantify this concept. First, for each node i of a graph G, $\mathcal{C}_i$ is defined as the fraction of the $\frac{k_i(k_i-1)}{2}$ possible edges among the neighbours of i that are present in G. More precisely, ϵ_i is the number of edges connecting the k_i vertices adjacent to the vertex i, the clustering coefficient of the vertex is $C_i = \frac{2\epsilon_i}{k_i(k_i-1)}$. Then the *clustering coefficient* of G, denoted $\mathcal{C}_G$, is the average over all nodes $i \in V(G)$ of C_i.

Obviously, the clustering coefficient varies between 0 and 1. A value near 0 means that most of the vertices connected to any given vertex i are not connected to each other. Conversely, a value near 1 means that those neighbours tend to be connected to one another.

Degree Distribution

One simple and important characteristic of a given vertex is its degree. The degree k_i of a vertex gives the total number of its connections. The average of k_i over all i is called the average degree of the network, and is denoted by $< k >$. The spread of vertex degrees over a graph can be characterized by a distribution function $P(k)$, which gives the probability that a randomly selected vertex has degree k.

A structured graph, for example a circulant graph $C_{n,\Delta}$, which is regular will have a degree distribution containing a single sharp spike (delta distribution). In a random

network (in the Erdős-Rényi model) the degree sequence will obey the well known Poisson distribution with a peak value in $< k >$ and exponential declines. (The probability of finding vertices with k edges is negligible for $k \gg < k >$.) In the past few years, many observational results showed that for most real large-scale networks the degree distribution deviates significantly from the Poisson distribution and in many instances the degree distribution can be better described by a power law, $P(k) \propto k^{-\gamma}$. Because these power-laws are free of any characteristic scale, such a network with a power-law degree distribution is called a scale-free network.

3. Complex Network Models

3.1. Watts-Strogatz small-world graphs

Watts and Strogatz suggest a simple method for constructing graphs with the small world property [75]. The method is as follows. Start with a circulant graph $C_{n,\Delta}$, then choose vertex 0 and the edge that connects it to 1. With probability p, reconnect this edge to a vertex chosen uniformly at random over the entire set of vertices (without duplicating any edge); otherwise do not change it. The process is repeated for all the remaining vertices in succession $(1, \ldots, n)$ considering each vertex in turn until one lap is completed. Next, do the same process with the edges connecting i to $i+2$, $i = 0, 1, \ldots, n$, as before, randomly rewire each of these edges with probability p, and continue this process, circulating around the ring and proceeding outward to more distant neighbours, $i + 2, i + 3, \cdots, i + \Delta/2$, after each lap, until each edge in the original graph $C_{n,\Delta}$ has been considered once. Therefore the rewiring process stops after $\Delta/2$ laps. With this process, for $p = 0$, the original graph is unchanged whereas for $p = 1$ when all edges are rewired randomly (in that case we obtain a random graph which has a Gaussian degree distribution). Intermediate values for p lead to different states of disorder. With p around 0.01 small-world graphs are obtained with a large clustering coefficient, similar to the starting graph, and a small average path length and diameter, as in a random graph [18]. (See Fig. 2.)

Watts and Strogatz realized that their model captures some aspects of many real networks, namely, they have a low average path length and diameter, in relation to a random network with a similar order and size, while they have a relatively high clustering (the clustering coefficient of a random network is almost zero), see Table 1.

Table 1. Values of parameters for some real small-world scale-free networks, see [75].

Network	**order**	**APL**	$< k >$	**Clustering**	γ
WWW	153,127	3.1	35.21	0.11	1.94 [1]
Internet (domain) [35]	3,015	3.52	4.75	0.18	2.1
Power grid	4,014	18.7	2.67	0.08	4
Silwood Pk food web [55]	154	3.40	4.75	0.15	4.75
C. Elegans	282	2.65	14	0.28	-
Movie actors	225,226	3.65	61	0.79	2.3

3.2. Barabási-Albert scale-free graphs.

To explain the origin of the power-law degree distribution of real networks, Barabási and Albert proposed and analyzed a simple graph model (BA) based on two main concepts growth and preferential attachment. In this model a graph is dynamically formed by a continuous addition of new vertices. Each new vertex is joined to several existing vertices selected proportionally to their degree.

The generation algorithm of a BA scale-free graph is as follows:

Growth: Start with a small number, m_0, of vertices. At each step, a new vertex is introduced and is connected to $m < m_0$ already existing vertives.

Preferential attachment: The probability that the new vertex will be connected to an existing vertex i depends on its degree k_i according to $P(k_i) = k_i / \sum_{j \in V} k_j$.

Considering these two rules, they proved analytically that the graph evolves into a scale-invariant state: The shape of the degree distribution does not change over the time and it is described by a power law $P(k) \propto k^{-\gamma}$, with $\gamma = 3$. This means that scale-free graphs have a few nodes with a high degree (called *hubs*). The analytical results can be contrasted easily with numerical simulations and compared with a random network produced according to the Erdős-Rényi method (start with a given number of vertices and add edges connected at random), see Fig. 1. The BA model does not allow, however, the analytical computations of the average path length and the clustering coefficient. It is, therefore, a minimal model capturing the mechanisms responsible for the power law degree distribution, but with some evident limitations when compared to some real-world networks (as it does not explain their relatively high clustering).

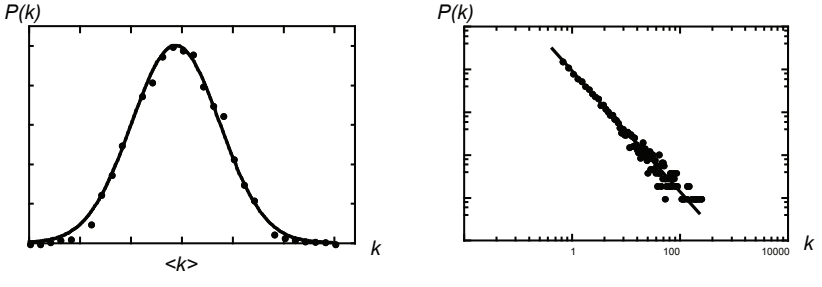

Figure 1. A Gaussian distribution of degrees for an Erdős-Rényi random network and a power law distribution obtained with preferential attachment [9].

The BA model has allowed a thorough study of some important properties of real world networks. One of them, robustness to the random failure of nodes [4], is a result of the networks tend to stay connected, maintaining an small APL, when a node is deleted at random (this is because the probability of deleting a hub is small). However, they are particularly vulnerable to targeted attacks addressed to remove hubs.

Other models have been proposed to overcome some of the limitations of the BA model and to produce scale-free networks with a small average path length and relatively high clustering, see the references given in the introduction.

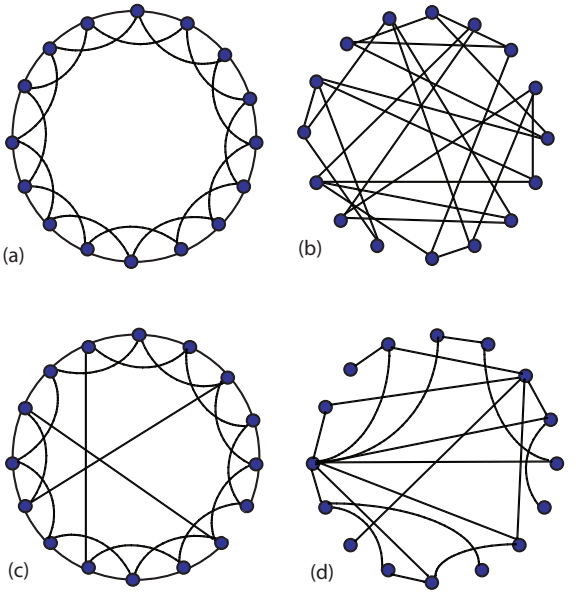

Figure 2. (a) $C(16, 4)$, a circulant graph. (b) Random graph. (c) Small-world graph. (d) Scale-free graph.

4. Deterministic small-world scale-free graphs

In contrast to the random models of Watts and Strogatz and Barabási and Albert, and many other modifications and variations, it is possible to produce small-world scale-free graphs deterministically. The deterministic model very often allows a complete analytical study of the relevant parameters of the graphs and can be contrasted with the random model.

As most real life networks are clustered, this characteristic can be reproduced in the deterministic model by considering complete graphs (or cliques). This is one of the reasons why many deterministic models are based on complete graphs.

In this section we will introduce several deterministic models and compare, in some cases, with their randomized versions.

4.1. Deterministic WS small-world graphs

In [23], small-world networks were constructed by choosing h nodes of $C_{n,\Delta}$ to be *hubs* and then using a graph with a very small diameter (star graph, complete graph, optimal double loop, etc.) of order h to interconnect the hubs. In this way, the clustering parameter of the final graph is high and very near to that of the original graph while the diameter is reduced considerably. This deterministic construction allows an analytical computation of all its main characteristics which can be compared with the numerical simulation, see Fig. 3.

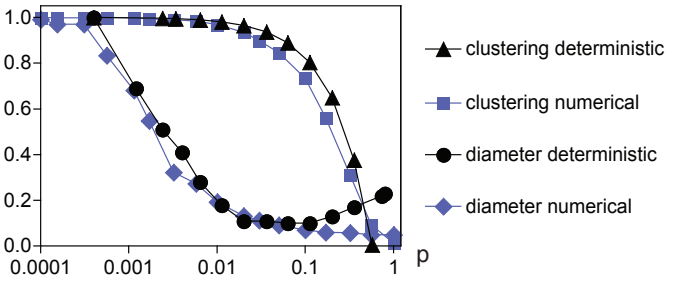

Figure 3. Comparison of the values of the diameter and clustering obtained according to the simulation model of the Watts and Strogatz [75] with those from the deterministic model introduced in [23].

4.2. Small-world scale-free networks from graph products and sums

Two simple deterministic construction techniques for small-world networks were introduced in [25]. The first method uses a replacement of vertices of a graph by clustered graphs (graph product). If the original graph has a low diameter and we use cliques to replace the vertices, a graph with low diameter and high clustering is obtained.

In a second construction a small-world network is obtained by connecting the nodes of a network of diameter d to a complete network of any size, which may be different from node to node. In this case the resulting network has diameter $d+2$, high clustering and nodes may have a different number of neighbors. This technique is very flexible as it allows different final degree distributions, including scale-free networks.

4.3. Hierachical networks

In [12], Barabási et al. introduced a simple hierarchical family of networks and showed it had a small-world scale-free nature. In [40] the authors compute some other properties of this family of graphs (like the spectrum.) The model is generalized in [68] and further studied in [63]. Hierachical graphs have been used to model metabolic networks in [69]. Several authors claim that a signature for a hierarchical network is that other than the small-world scale-free characteristics the clustering of the vertices of the graph follows $C_i \propto 1/k_i$. Hierarchical networks can be constructed starting from a complete graph K_n and connecting to a selected root node $n-1$ replicas of K_n. Next, $n-1$ replicas of the new whole structure are added, to the root. At this step the graph will have n^3 vertices. The process continues until we reach the desired graph size. There are many variations for these hierarchical networks, depending on the initial graph , the introduction of extra edges among the different copies of the complete subgraphs, etc. However, given the starting graph, they have no parameters to adjust and the main characteristics become fixed. In [14] a general model is considered and a labeling system introduced to allow the study of routing and other communication properties of hierachical networks.

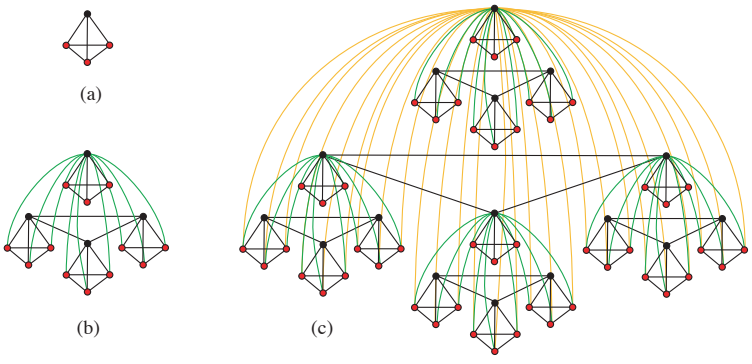

Figure 4. Recursive construction of an hierarchical network based on K_4, from [14].

4.4. Deterministic recursive clique-trees

While in the hierarchical models the deletion of some edges leads to the decomposition of the graph into different complete graphs, another construction, also based on complete graphs, intermixes them producing a more complex structure.

A generic recursive $d-$clique-tree is a graph theoretical construction which starts at $t = 0$ with a complete graph $K(d, 0) = K_d$. For any step $t \geq 1$, the clique-tree $K(d, t)$ is constructed from $K(d, t-1)$ by selecting one or more existing d-cliques in $K(d, t-1)$ and adding, for each clique, a new vertex connected to all the vertices of the clique. See Fig. 5. Note that a recursive clique-tree is a graph which contains numerous cycles and hence it is not a tree in the strict sense.

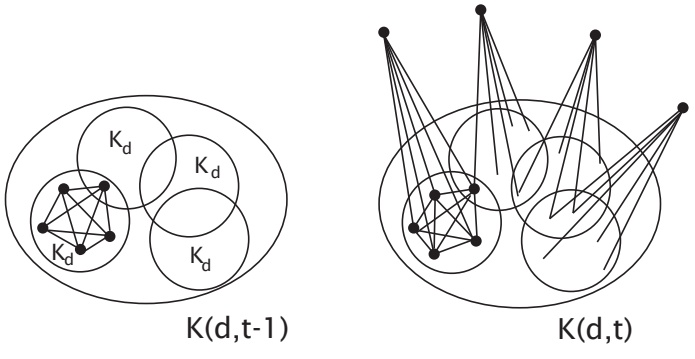

Figure 5. Iterative construction of a deterministic clique-tree graph.

Several modifications of this general construction have been considered and are discussed next. Different networks are assocaited to the choice of the value d. Authors

have considered $d = 2$, $d = 3$ and the general case d ($d \in N, d \geq 2$). It is also important the way to select the existing d-cliques which will be joined to a new vertex: If we select *all* the cliques (even those that have been used before) we have *complete recursive clique-trees* [22] which include, as a particular case, *pseudo-fractal networks* [27] (when $d = 2$). If we just select cliques which have *never been used before* we obtain (for any $d \geq 3$) *high dimensional Apollonian networks* [34,81], which include for $d = 3$ *Apollonian networks* [7,34]. Table 2 summarizes these deterministic constructions.

Table 2. Deterministic recursive clique-tree constructions.

	Adding at the same time a vertex to each d-clique **with** repetition	Adding at the same time a vertex to each d-clique **without** repetition
Case $d = 2$	*Pseudofractal scale-free* Dorogotsev, Goltsev, Mendes Phys.Rev.E 65 (2002) 066122	*Deterministic SW network* Zhang, Rong, Guo Physica A cond-mat/0503637
Case $d = 3$		*Apollonian network* Andrade, Herrmann, Andrade, Silva Phys.Rev.Lett. 94 (2005) 018702 Doye, Massen Phys. Rev. E 71 (2005) 016128.
General case $d = 2 \ldots \infty$ (includes cases d=2,3)	*Recursive clique-trees* Comellas, Fertin, Raspaud Phys.Rev.E 69 (2004) 037104.	*High dimensional Apollonian network* Zhang, Comellas, Fertin, Rong J. Phys. A. 39 (2006) 1811 (introduced by Doye and Massen, Phys. Rev. E 71 (2005) 016128.)

4.5. Random recursive clique-trees

The same clique-based principles used for the deterministic constructions can be used to produce graphs generated randomly. Interestingly, the final graphs differ in several values of the parameters. In some cases the discrepancy can be explained by a biased choice of the substructures selected in the process of growing the random graph, see [24].

If we select randomly a clique (allowing even those that have been used before), we have, for generic d, *random recursive clique-trees* studied in the Appendix of this survey. If we select randomly a clique (avoiding repetitions), we have, for generic d, *high dimensional random Apollonian networks* [80]. which include, as the particular case $d = 3$ *random Apollonian networks* [82].

We note that it is possible, for random constructions, to introduce a parameter to control part of the structural properties of the growing network. By tuning this parameter, one can allow the introduction at each step of one vertex attached to one clique or different vertices attached to different cliques up to the deterministic case. For $d = 2$, avoiding clique repetitions, this has been studied in [80]. A similar study could easily be done for the general case.

Finally, the next table compares the values of the (asymptotical) degree distribution (scale-free in most cases, in which case we give the γ exponent) of Apollonian graphs [7], random Apollonian graphs [82], their high dimensional versions [81,80] (they include

Table 3. Random recursive clique-tree constructions.

	Adding a single vertex to a random clique **with** repetition	Adding a single vertex to a random clique **without** repetition
Case $d = 2$		*Random SW network* Ozik, Hunt, Ott Phys.Rev.E 69 (2004) 02618
Case $d = 3$		*Random Apollonian network* Zhou, Yan, Wang Phys.Rev.E 71 (2005) 046141
General case $d = 2 \ldots \infty$ (includes cases d=2,3)	*Random recursive clique-tree* see Appendix	*HD random Apollonian network* Zhang, Comellas, Rong Physica A. cond-mat/0502591

as particular cases the former two families of graphs). We also present the random versions of the pseudo fractal scale-free graphs (introduced by Dorogovtsev, Goltsev and Mendes [27], their generalization, complete recursive clique-trees [22] and the so-called deterministic and random small-world graphs from [77,78]. We observe that these last two cases do not produce scale-free networks.

Table 4. Comparison of deterministic and random Apollonian graphs and recursive clique-trees.

Graph family	$P(k)$ or γ-exponent	Clustering
Deterministic SW [78]	$2^{-\frac{k}{2}}$	$0.69 = \ln 2$
Random SW [77]	$\frac{3}{4}(\frac{2}{3})^{-k}$	$0.65(= \frac{3}{2}\ln 3 - 1)$
Apollonian [7,34]	$2.58(= 1 + \frac{\ln 3}{\ln 2})$	0.83
Random Apollonian [82]	$\frac{3N-5}{N} \approx 3$	$0.74(= \frac{46}{3} - 36\ln\frac{3}{2})$
High-Dim. Apollonian [81]	$1 + \frac{\ln(d+1)}{\ln d}$ (2 to 2.58)	0.83 to 1
High-Dim. Random Apollonian [80]	$\frac{2d-1}{d-1}$ 2 to 3	0.74 to 1
Pseudo fractal scale-free [27]	$1 + \frac{\ln 3}{\ln 2} = 2.58$	$0.80(= \frac{4}{5})$
Random pseudo fractal scale-free	$\frac{5}{2} = 2.5$	
Determ. recursive clique-trees [22]	$1 + \frac{\ln(d+1)}{\ln d}$ (2 to 2.58)	0.80 to 1
Random rec. clique-trees [see Appendix]	$\frac{2d-1}{d-1}$ (2 to 3)	0.74 to 1

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APPENDIX: Deterministic vs random recursive clique-trees.

In this appendix we compute analytically the order, size and degree distribution of deterministic recursive clique-trees and their random variations.

5.1. Deterministic recursive clique-trees

The results for the values of the order, size, clustering, degree distribution and diameter of recursive clique-trees appeared in [22]. In this section we provide an alternative simpler method for the calculation of some of these parameters.

We denote a deterministic recursive d-clique-tree network after t iterations by $K(d,t)$, $d \geq 2, t \geq 0$. Then this deterministic recursive clique-tree network at step t is constructed as follows: For $t = 0$, $K(d,0)$ is the complete graph K_d (or d-clique), and $K(d,0)$ has d vertices and $d(d-1)/2$ edges. For $t \geq 1$, $K(d,t)$ is obtained from $K(d,t-1)$ by adding, for each of its existing subgraphs isomorphic to a d-clique, a new vertex and joining it to all the vertices of this subgraph (see Fig. 5). Then, at $t = 1$, we add one new vertex and d new edges to the graph, creating d new d-cliques. Therefore, at $t = 2$ we add $d+1$ new vertices, each of them connected to all the vertices of one of the $d+1$ cliques K_d and we introduce $d(d+1)$ new edges, and so on.

Note that the addition of each new vertex leads to d new d-cliques and d new edges. So the number of new vertices and edges at step t_i is $L_v(t_i) = (d+1)^{t_i-1}$ and $L_e(t_i) = d(d+1)^{t_i-1}$, respectively. Therefore, a deterministic recursive clique-tree $K(d,t)$ is a growing network, whose number of vertices increases exponentially with time.

Thus we can easily see that at step t, the network $K(d,t)$ has

$$N_t = d + \sum_{t_i=1}^{t} L_v(t_i) = \frac{(d+1)^t - 1}{d} + d \tag{1}$$

vertices and total number of edges is:

$$|E|_t = \frac{d(d-1)}{2} + \sum_{t_i=1}^{t} L_e(t_i) = \frac{d(d-1)}{2} + (d+1)^t - 1 \tag{2}$$

Table 5. Number of new edges added to $K(d,t)$ at each step and the total number of K_d's at this step.

Step	New edges	Number of K_d
0	$\frac{d(d-1)}{2}$	1
1	d	$d+1$
2	$d(d+1)$	$d(d+1)+(d+1) = (d+1)^2$
3	$d(d+1)^2$	$d(d+1)^2+(d+1)^2 = (d+1)^3$
...	...	...
i	$d(d+1)^{i-1}$	$(d+1)^i$
i+1	$d(d+1)^i$	$d(d+1)^i+(d+1)^i = (d+1)^{i+1}$
...	...	...

The average degree $\overline{k}_t$ is then

$$\overline{k}_t = \frac{2|E|_t}{N_t} = \frac{2d(d+1)^t + d^3 - d^2 - 2d}{(d+1)^t + d^2 - 1}. \tag{3}$$

For large t it is approximately $2d$. We can see when t is large enough the resulting networks are sparse graphs as for many real-world networks whose vertices have a lot less connections than the maximum possible.

We see that the dimension d is a tunable parameter controlling all the relevant characteristics of the network.

When a new vertex i is added to the graph at step t_i ($t_i \geq 1$), it has degree d and forms d new d-cliques. From the iterative algorithm, we can see that each new neighbor of i generated $d-1$ new d-cliques with i as one vertex of them. In the next iteration, these d-cliques add to the already existing cliques and introduce new vertices that are connected to the vertex i. Let $k_i(t)$ be the degree of i at step t ($t > t_i + 1$). Then,

$$\Delta k_i(t) = k_i(t) - k_i(t-1) = d\Delta k_i(t-1) \tag{4}$$

combining the initial condition $k_i(t_i) = d$ and $\Delta k_i(t_i + 1) = d$, we obtain

$$\Delta k_i(t) = d^{t-t_i} \tag{5}$$

and the degree of vertex i becomes

$$k_i(t) = \sum_{t_m=t_i}^{t} \Delta k_i(t_m) = d\left(\frac{d^{t-t_i} - 1}{d-1} + 1\right). \tag{6}$$

The distribution of all vertices and their degrees at step t is given in Table 6.

Table 6. Distribution of vertices and their degrees for $K(d,t)$ at step t.

Num. vert.	Degree
$d+1$	$d + \sum_{j=1}^{t-1} d^j$
$d+1$	$d + \sum_{j=1}^{t-2} d^j$
$\cdots$	$\cdots$
$(d+1)^{t-2}$	$2d$
$(d+1)^{t-1}$	d

Therefore, the degree spectrum of the graph is discrete and some values of the degree are absent. To relate the exponent of this discrete degree distribution to the standard γ exponent as defined for continuous degree distribution, we use a cumulative distribution $P_{cum}(k) \equiv \sum_{k' \geq k} N(k',t)/N_t \sim k^{1-\gamma}$. Here k and k' are points of the discrete degree spectrum. The analytic computation details are given as follows.

For a degree k

$$k = d\left(\frac{d^{t-l} - 1}{d-1} + 1\right),$$

there are d^{l-1} vertices with this exact degree, all of which were introduced at step l.

All vertices introduced at step l or earlier have this or a higher degree. So we have

$$\sum_{k' \geq k} N(k', t) = d + \sum_{s=1}^{l} L_v(s) = \frac{(d+1)^l - 1}{d} + d.$$

As the total number of vertices at step t is given in Eq. (1) we have

$$\left[d\left(\frac{d^{t-l} - 1}{d-1} + 1\right)\right]^{1-\gamma} = \frac{\frac{(d+1)^l - 1}{d} + d}{\frac{(d+1)^t - 1}{d} + d}$$

$$= \frac{(d+1)^l + d^2 - 1}{(d+1)^t + d^2 - 1} \tag{7}$$

Therefore, for large t we obtain

$$(d^{t-l})^{1-\gamma} = (d+1)^{l-t}$$

and

$$\gamma \approx 1 + \frac{\ln(d+1)}{\ln d} \tag{8}$$

so that $2 < \gamma < 2.58496$.

Notice that when t gets large, the maximal degree of a vertex is roughly equal to $d^{t-1} \sim N_t^{\ln d / \ln(d+1)} = N_t^{1/(\gamma-1)}$.

5.2. Random recursive clique-trees

In this section we study the random version of the construction of the last section.

A random recursive d-clique-tree network after t iterations is denoted by $R(d, t)$, $d \geq 2, t \geq 0$ and it is constructed as follows: For $t = 0$, $R(d, 0)$ is the complete graph K_d (or d-clique), and it has d vertices and $d(d-1)/2$ edges. For $t \geq 1$, $R(d, t)$ is obtained from $R(d, t-1)$ by adding to one randomly selected subgraph isomorphic to a d-clique a new vertex and joining it to all the vertices of this subgraph. This construction differs from the deterministic complete clique-tree of the last subsection in what at each iteration step only one vertex is added (and joined to a randomly selected clique). The selection of cliques which have been used before is allowed.

Since the network size is incremented by one with each step, we use the step value t to represent a vertex created at this step. We can see easily that at step t, the network has of $N = d + t$ vertices.

We can compute the degree distribution as follows: First note that, after a new vertex is added, its degree is d and the number of d-cliques that can be chosen in the following step increases by d. If the degree of a vertex increases by 1, then this vertex belongs to $d-1$ more cliques which could be chosen at any following step. When a vertex i attains degree k_i the number of K_d to which it belongs is:

$$d + (k_i - d)(d-1) = k_i(d-1) - d^2 + 2d$$

Table 7. Distribution of vertices and number of K_d's for the random recursive clique tree $R(d,t)$ at each step t.

Step (i)	Num. vertices	Number of K_d
0	d	1
1	$d+1$	$(d+1)$
2	$d+2$	$(d+1)+d$
3	$d+3$	$(d+1)+d+d$
...	...	...
t	$d+t$	$(d+1)+d+\cdots+d=$ $=td+1$

The first term is the degree of the vertex when it is introduced to the network (equal to the number of K_d to which it belongs). The second is the increase of degree up to when it reaches degree k_i times $d-1$ (the number of cliques introduced at each increase of degree by one unit). Note that after t steps the number of $(d+1)$-cliques available for selection is $td+1$.

If we consider k to be continuous, we can write for a vertex i

$$\frac{\partial k_i}{\partial t} = \frac{k_i(d-1) - d^2 + 2d}{td+1}. \tag{9}$$

The solution of this equation, with the initial condition that vertex i was added to the network at t_i with degree $k_i(t_i) = d$, is

$$k_i(t) = \frac{d(d-2)}{(d-1)} + \frac{d}{(d-1)}\left(\frac{dt+1}{dt_i+1}\right)^{\frac{(d-1)}{d}} \tag{10}$$

The probability that a vertex has a degree $k_i(t)$ smaller than k, $P(k_i(t) < k)$, is

$$P(k_i(t) < k) = P\left(t_i > \frac{(dt+1)\left(\frac{d}{d-1}\right)^{\frac{d}{d-1}}}{d\left(k - \frac{d(d-2)}{(d-1)}\right)^{\frac{d}{d-1}}} - \frac{1}{d}\right). \tag{11}$$

Assuming that we add the vertices to the network at equal intervals, the probability density of t_i is

$$P_i(t_i) = \frac{1}{d+t}. \tag{12}$$

Substituting this into Eq. (3) we obtain that

$$P\left(t_i > \frac{(dt+1)\left(\frac{d}{d-1}\right)^{\frac{d}{d-1}}}{d\left(k - \frac{d(d-2)}{(d-1)}\right)^{\frac{d}{d-1}}} - \frac{1}{d}\right) = \tag{13}$$

$$= 1 - P\left(t_i \leq \frac{(dt+1)\left(\frac{d}{d-1}\right)^{\frac{d}{d-1}}}{d\left(k - \frac{d(d-2)}{(d-1)}\right)^{\frac{d}{d-1}}} - \frac{1}{d} \right) =$$

$$= 1 - \frac{(dt+1)\left(\frac{d}{d-1}\right)^{\frac{d}{d-1}}}{(d+t)d\left(k - \frac{d(d-2)}{(d-1)}\right)^{\frac{d}{d-1}}} + \frac{1}{(d+t)d}.$$

Thus the degree distribution is

$$P(k) = \frac{\partial P(k_i(t) < k)}{\partial k} = \frac{(dt+1)d^{\frac{d}{d-1}}}{(d+t)} \left((d-1)k - d(d-2)\right)^{\frac{1-2d}{d-1}}. \tag{14}$$

For large t

$$P(k) = d^{\frac{2d-1}{d-1}} \left((d-1)k - (d(d-2)\right)^{\frac{1-2d}{d-1}} \tag{15}$$

and if $k \gg d$ then $P(k) \sim k^{-\gamma}$ with a degree exponent $\gamma(d) = \frac{2d-1}{d-1}$.

When $d = 2$ one has $\gamma(2) = 3$, while as d goes to infinity $\gamma(\infty) = 2$.

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