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LARGE REGULAR INTERCONNECTION NETWORKS

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ABSTRACT

Distributed computer systems can be modeled using graph theory to evaluate potential network topologies. This paper presents a new construction method for interconnection networks. The method generalizes the chordal ring networks of Arden and Lee, and produces the largest known graphs for many degrees and diameters. Numerical results are presented both for individual chordal rings and for graph products of chordal rings. A new construction method for degree five graphs is also given.

1. INTRODUCTION

This paper looks at the design of interconnection networks for large systems of microcomputers. An optimal design would involve many factors, including the network topology, the physical layout of the network, and the expected requirements for interprocessor communication. In this paper, the concern is with the network topology, which must be carefully designed in order to take full advantage of the capabilities of a distributed computer system.

With a message-passing protocol, communication between computers may require the use of several intermediate computers as relays. Message queueing may occur at each intermediate computer, causing delays and a loss in efficiency for the system. An optimal network might have every computer directly connected to every other one. This is usually not possible, because the number of connections for each computer is limited. The network must be designed so that messages travel quickly between computers, with the restriction that each computer be connected to only a few others.

A graph theory model of a distributed computer system has been used by many authors to evaluate network topologies. In this model, nodes represent the computers and edges represent the communication links. The degree of a node is the number of nodes it is connected to by a single edge. Each node is assumed to have a degree not exceeding some prescribed number d, which is the degree of the graph. If all nodes have the same degree the graph is called regular. The distance between two nodes is the minimum number of edges which must be used to travel between the nodes. The maximum inter-nodal distance is the diameter of the graph, which is denoted k.

An optimal network topology should have small inter-nodal distances. There are several relevant composite measures of distances in a graph, such as the diameter or the average distance; in this paper the diameter will be used. There are several related problems involving the degree, diameter, and number of nodes in a graph. Perhaps the most natural is to find the minimum diameter k for a graph with n nodes and degree d. However, the value of this minimum k would be a function of n and d, for hundreds or thousands of relevant values of n, with different graphs for each. Instead the dual problem of finding the maximum number of nodes in a graph with degree d and diameter k will be considered. There is almost as much information with this, and the results are easier to present.

In the following section, previous work by other authors on maximizing the number of nodes in a graph with degree d and diameter k is discussed. Section 3 looks at a generalization of the chordal rings of Arden and Lee [2] which gives larger graphs for many values of d and k. Section 4 presents the quantitative results of this research. The final section summarizes the paper.

2. PREVIOUS WORK

The maximum number of nodes in a graph with degree d and diameter k is denoted $n(d,k)$. An upper bound on $n(d,k)$ is easily calculated. From any given node at most d nodes can be reached in a distance of one and, for $j \geq 1$, at most $d(d-1)^{j-1}$ nodes can be reached in a distance of j. Thus

$$n(d,k) \leq 1 + d + \dots + d(d-1)^{k-1}$$
$$= \frac{d(d-1)^k - 2}{d - 2} \qquad (1)$$

Expression 1 is called the Moore bound, and any graph which has that number of nodes is called a Moore graph. Most Moore graphs fall into two classes: 1) the odd polygons, where $d = 2$; 2) the complete graphs, where $k = 1$. In [13] it was shown that for $k = 2$ there are only a few other Moore graphs: the Petersen graph (figure 1) where $d = 3$; the Hoffman-Singleton graph, where $d = 7$; and possibly a graph with $d = 57$. In [6] and [9] it was shown that there are no other Moore graphs. In [7] it was shown that except for the square there are no graphs with a number of nodes equal to one less than the Moore bound. No better upper bound on

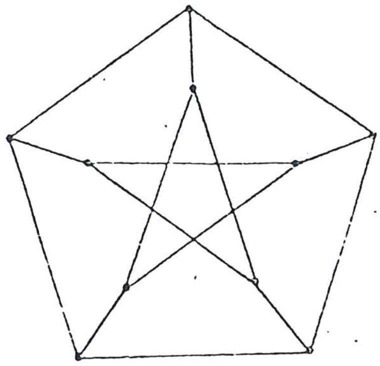


Figure 1. Petersen Graph

n(d,k) has been established.

A lower bound on $n(d,k)$ will be denoted by $b(d,k)$. Values of $b(d,k)$ can be obtained by exhibiting a graph with degree d, diameter k, and $b(d,k)$ nodes. A number of authors have written papers on finding improved values of $b(d,k)$. One of the first papers on the subject was by Elspas [11]. He defined the problem explored here and provided the first results. In particular, he showed that $n(3,3) = 20$, $n(4,2) = 15$, and $n(5,2) = 24$, which are the only known values of $n(d,k)$ other than those associated with Moore graphs.

Friedman [12], Korn [15], and Storwick [17] gave related construction methods to obtain improved values of $b(d,k)$. These methods involved connecting many copies of hierarchical graphs in specified structures. Formulas were obtained expressing $b(d,k)$ as a function of d and k. All of the formulas had $b(d,k)$ proportional to $d^{k/2}$.

Tutte [18] and Balaban [4], [5] obtained improved values of $b(3,k)$ for specific values of k while working on a related problem. Larger values of $b(3,k)$ were obtained by Arden and Lee [3] by using a construction method called multitree structured networks. A special construction was devised by Akers [1], in which $b(d,d-1)$ is equal to the binomial coefficient $\binom{2d-1}{d}$.

Imase and Itoh [14] gave a simple number-theoretic construction based on de Bruijn sequences [10] which has significantly more nodes than the other constructions for large degrees and diameters. Consider the nodes to be numbered from 0 to $n - 1$. An edge is added connecting nodes i and j if

$$j \equiv i \cdot (d/2) + a \pmod{n}, \quad a = 0,1,\ldots,d/2 - 1.$$

The resulting graph has diameter $k = \lceil \log_{d/2} n \rceil$ where $\lceil x \rceil$ denotes the smallest integer not less than x), so as many as $(d/2)^k$ nodes can be used in a graph with diameter k.

The best published results for many degrees and diameters are those of Leland et al [16]. Graphs with small degrees and diameters were constructed using heuristic methods. These were then combined by using various graph products to form larger graphs.

3. CHORDAL RINGS

In this section a new construction method is proposed which produces graphs with larger values of $b(d,k)$. This method is a generalization of the chordal ring networks of Arden and Lee [2]. Their work will be summarized before proceeding to the generalization.

In their construction, a chordal ring network is a ring on n nodes, plus $n/2$ additional links connecting pairs of nodes. Every node has one of these chordal links, so the graph is regular and has degree 3. Let the nodes be numbered consecutively 0, 1, ..., n-1 around the ring. Each odd node i is connected to the even node $i + w \pmod{n}$, where w is an odd integer. Figure 2 illustrates a chordal ring network with $n = 14$, $w = 5$.

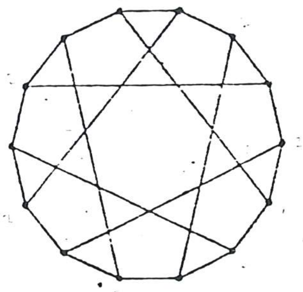


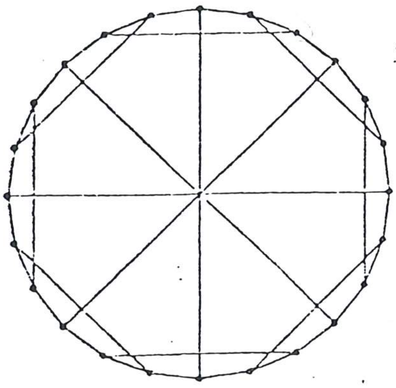
Figure 2. Example of Arden and Lee Chordal Ring

The largest chordal ring for given diameters is derived in their paper. For $k \geq 5$ and odd the maximum number of nodes is $k^2 + 3k - 6$, and the optimal w is $k + 4$. For $k \geq 8$ and even the maximum number of nodes is $k^2 + 3k - 12$, and the optimal w is $k + 5$. These are fewer nodes than can be obtained by other methods. The paper also gives a distributed routing algorithm for internode communication.

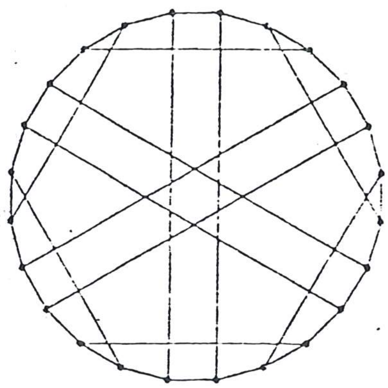
The chordal ring networks will be generalized in two ways. First, more complex chordal connection methods will be used for the degree 3 case. Second, chordal rings of higher degrees will be investigated.

The Arden and Lee ring can be thought of as having a pattern of length two. That is, the chord of length w is repeated every two nodes around the ring. The fundamental generalization is to look at other pattern lengths, or <u>orders</u>.

A chordal ring of order r will be defined as having the property that r is the smallest integer such that if i is connected to j, then $i + r \pmod{n}$ is connected to $j + r \pmod{n}$ for all i and j. An Arden and Lee chordal ring is thus a degree 3 chordal ring of order 2. Figure 3 illustrates two 24 node degree 3 chordal rings, the first with order 3 and the second with order 4.



Order 3



Order 4

Figure 3. Higher Order Chordal Rings

A chordal ring of degree 3 must have an even number of nodes, since any regular graph with an odd degree must have n even. In addition, the number of nodes must be a multiple of the order.

The objective is to find the largest chordal ring for given diameters. With the higher order chordal rings, there does not appear to be a simple formula giving the maximum number of nodes as a function of either the chord lengths or the diameter. As a result, all values of n less than the Moore bound for a given k must be considered, in order to determine if there is a chordal ring on n nodes with diameter k. Given a value of n, the ideal algorithm for this determination would proceed as follows:

1. Find the acceptable orders for this n.
2. Find the allowable chord lengths for each order.
3. Calculate the diameter of the graph for every set of chord lengths.
4. Choose a set of chord lengths which give the smallest diameter.

The determination of the acceptable orders is easy. Any r such that $1 \leq r \leq n/2$ and such that r divides n may be used. If $r = 1$ (a chordal ring in which every node is connected to the node opposite it) it is easy to see that $k = \lceil n/4 \rceil$. If $r = 2$ the results of [2] can be applied. Thus we need to only consider the case of $r \geq 3$.

Next, the chord lengths which produce allowable chordal rings must be determined. Not all chord lengths are acceptable--for example, in the order 2 case the chord length must be odd. If the nodes are numbered from 0 to n - 1, each node can be considered to belong to one of r residue classes, depending on its value modulo r. If one node in residue class p is connected (by a chord) to a node in residue class q, then by definition of a chordal ring all nodes in residue class p are connected to nodes in residue class q. As a result, the next step is to determine which residue classes can be connected, and what the corresponding chord lengths are.

Suppose residue class p is assigned to be connected to itself. The chord length w is then congruent to 0 (mod r). If node i is in class p, it is connected to node i + w. However, node i - w, also in class p, is connected to node i. Since the graph is of degree 3, i + w and i - w must be the same node. Then $w = n/2$, and node i is connected to the node opposite it.

If class p is connected to class q different from p, this problem does not arise. The corresponding chord length w must be congruent to q - p (mod r). However, if r is odd at least one class must be connected to itself, because an even number of classes must be paired.

For $r = 2$ there is only one method of pairing the classes, which is the one used in [2]. For $r = 3$ there is also only one: one class must be connected to itself, and the other two are paired. (If all three classes are connected to themselves this becomes an order 1 chordal ring.) However, for $r = 4$ there are two different pairing methods. Let {i} denote the class containing i. One method connects {0} to {1} and {2} to {3}, while the other connects {0} to {2} and {1} to {3}. A third method--{0} to {3} and {1} to {2}--is the same as the first via rotation. In general, for r even there are

$$\frac{(r-1)!}{(r/2-1)!\,2^{r/2-1}}$$

pairing methods, with some reduction for duplication due to rotation and reflection.

Given a pairing method for the residue classes, the chord lengths must be selected. For each pair of residue classes, there are n/r chord lengths with the allowed congruences. With r even there are r/2 pairs of classes, so there are $(n/r)^{r/2}$ possible chordal rings with a given residue class pairing.

For small values of n and r it is feasible to evaluate all possible sets of chord lengths to determine which set is best, but for large values of n r this is impossible. For example, with $n = 1000$ and $r = 10$ there are over 10^{11} different chordal rings. To find the optimal one the diameter of each would have to be calculated, which is computationally infeasible. Instead, some search method which examines only a relatively few possibilities must be employed. Such a method is likely to lead to sub-optimal solutions.

Once a search method is chosen, the diameter of each chordal ring must be calculated. This can be done by using any shortest-path algorithm, and can be speeded up by taking advantage of the fact that all edge lengths are one. In addition, instead of calculating the distances between all pairs of nodes only the distance from representatives of each residue class to all other nodes need to be calculated.

Global search, local search, and random search methods were used to find the chord lengths for degree 3 chordal rings between 6 and 10. The largest graphs found are listed in Table 1. The length of the chord from each residue class is included. The largest chordal rings found with diameters 4 and 5 had 30 and 56 nodes, respectively; larger graphs are given in section 4 of this paper.

Chordal rings can also be extended to larger degrees. Now each residue class must be paired with $d - 2$ other classes. A class may be paired with itself without each node being connected to its opposite node. Node i would be connected to both $i + w$ and $i - w$, where $w \equiv 0 \pmod r$. However, this would take care of two of the necessary $d - 2$ pairings for $\{i\}$.

Figure 4 shows an example of a higher degree chordal ring--a 36 node, degree 4 graph with diameter 3. This ring has order 3, and each of the three residue classes has a chord to the other two.

4. QUANTITATIVE RESULTS

Table 2 lists the largest known graphs with degrees between 3 and 10 and diameters between 2 and 10. The largest degree two graphs are polygons and have $2k + 1$ nodes, while the largest diameter one graphs are complete graphs and have $d + 1$ nodes. Of the 72 values in Table 2, 52 are new. Chordal rings account for 37 of these, for which the computer search was limited to 7000 nodes. Eight of them are graph products of chordal rings, using the methods and notation of [16]. The numbers in parentheses refer to the d and k values of the product operands. Five of the graphs with diameter 8 are Uhr products involving large degree, diameter 2 graphs. A diameter 2 graph with degree d and $d^2 - d + 1$ nodes always exists if $d - 1$ is a power of a prime number, using the construction in [8]. Two of the graphs (degree 3, diameters 4 and 5) were found by heuristic methods, which were based upon chordal rings. These last two are shown in Figures 5 and 6.

Another graph product can be used to construct the largest known graphs in certain small degree, large diameter cases. This product combines m copies of a graph G. Each node in the product graph is of the form $(v_1, v_2, \ldots, v_m)$, where v_i is a node of the component graph. This node is connected to

$$(v_2, v_3, \ldots, v_m, v_1)$$

$$(v_m, v_1, \ldots, v_{m-2}, v_{m-1})$$

and to all nodes of the form

$$(v_1', v_2, \ldots, v_m),$$

where v_1 and v_1' are connected by an edge in the component graph. If the component graphs each have degree d_G, diameter k_G, and n_G nodes, the product graph has n_G^m nodes, degree $d_G + 2$, and diameter $mk_G + m - 1$.

Using this product, the largest known infinite family of degree five graphs can be constructed. Taking the product of m copies of the Petersen graph, a degree 5 graph can be formed which has $10^m = 10^{(k+1)/3}$ nodes. By comparison, a de Bruijn graph has only 2^k nodes.

Diameter	Number of Nodes: Chordal Ring	Number of Nodes: Moore Bound	Order	Chord Lengths
6	100	190	5	50, 11, 89, 21, 79
7	180	382	6	153, 16, 116, 27, 64, 164
8	280	766	7	140, 101, 73, 17, 179, 207, 263
9	462	1534	11	231, 37, 16, 139, 247, 425, 79, 446, 383, 215, 323
10	708	3070	12	632, 685, 23, 208, 601 107, 483, 500, 76, 225, 433, 275

TABLE 1 DEGREE 3 CHORDAL RINGS

d \ k	2	3	4	5	6	7	8	9	10
3	10* [10]	20* [20]	38 h [34]	60 h [56]	100 C [84]	180 C [122]	280 C [176]	462 C [311]	708 C [525]
4	15* [15]	36 C [35]	92 C [67]	188 C [134]	378 C [261]	856 C [425]	1872 C [910]	3708 C [1360]	7000 C [2312]
5	24* [24]	60 C [48]	164 C [126]	400 C [262]	1014 C [505]	2604 C [1260]	7000 C [2450]	8556 U (4,4) [4690]	16928 L (4,4)x(4,4) [9380]
6	31 S [31]	94 C [65]	284 C [164]	820 C [600]	2604 C [1152]	7000 C [2520]	13272 U (5,2)x(24,2) [6561]	27060 U (5,4) [19683]	59049 dB [59049]
7	50* [50]	122 C [88]	420 C [252]	1550 C [992]	5304 C [2880]	8930 U (6,3) [4680]	27001 U (6,2)x(30,2) [12250]	80940 U (6,4) [43200]	187488 R (5,2)x(6,6) [86400]
8	57 S [57]	176 C [105]	609 C [384]	2550 U (7,2) [2550]	7000 C [5760]	16384 dB [16384]	122550 U (7,2)x(50,2) [65536]	262144 dB [262144]	1048576 dB [1048576]
9	74 S [74]	212 C [150]	882 C [600]	3900 C [3306]	12500 LQ (7,2)x(2,2) [12500]	31152 U (8,3) [20160]	163191 U (8,2)x(54,2) [76500]	382500 R (7,2)x(8,5) [382500]	1050000 R (7,2)x(8,6) [1048576]
10	91 S [91]	238 C [200]	1216 C [864]	6000 C [5550]	25000 LQ (7,2)x(3,2) [25000]	78125 dB [78125]	399822 U (9,2)x(74,2) [390625]	1953125 dB [1953125]	9765625 dB [9765625]

Key:

- * Proven Optimal
- C Chordal Ring
- dB de Bruijn graphs [14]
- h Heuristic methods
- L Leland product [16]
- LQ Li product [16]
- R Regular Leland product [16]
- S Storwick [17]
- U Uhr product [16]
- Bracketed values are from Leland, et al [16]

TABLE 2 LARGEST NUMBER OF NODES IN KNOWN GRAPHS WITH DEGREE d AND DIAMETER k

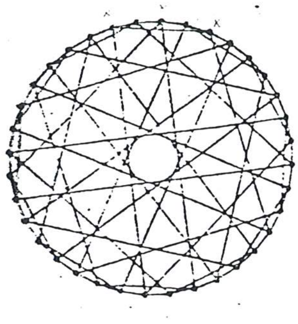


Figure 4. Degree 4 Chordal Ring

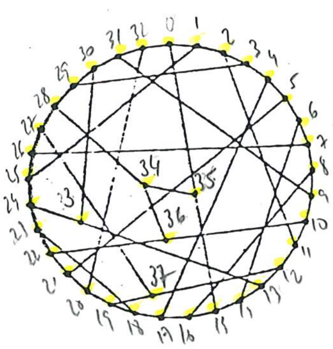


Figure 5. 38 Node, Diameter 4 Graph

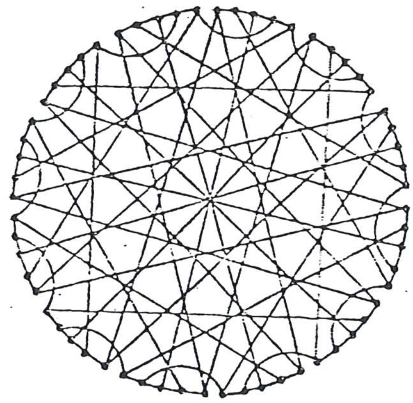


Figure 6. 60 Node, Diameter 5 Graph

5. SUMMARY

This paper has presented a new candidate design for interconnection networks. It produces graphs with more nodes for given degrees and diameters than other methods have produced. However, these graphs are still much smaller than the Moore bound indicates may be possible. Either much larger graphs remain to be discovered, or a tighter upper bound should be found.

6. ACKNOWLEDGEMENTS

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