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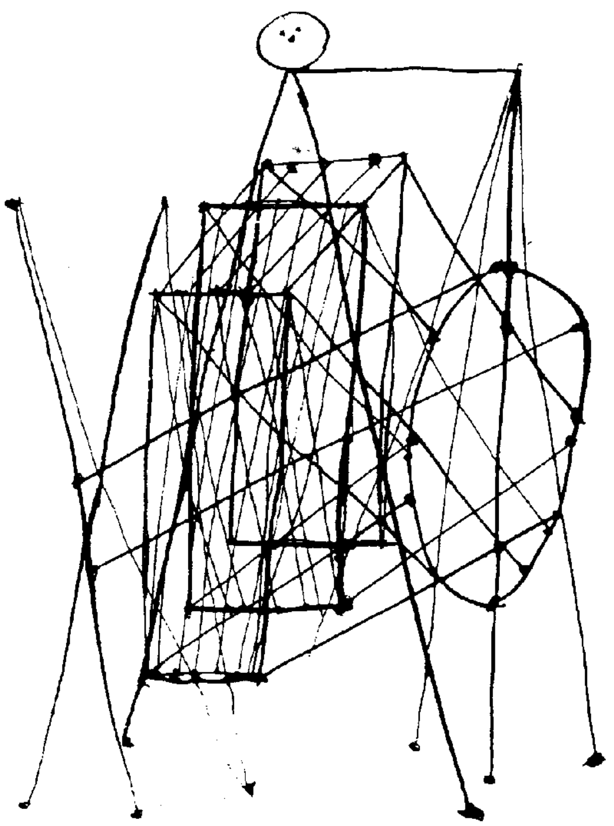

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Covering the vertices of a cycle prefix digraph

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Abstract

Cycle prefix digraphs are an interesting family of directed Cayley coset graphs that have been proposed as a model of interconnection networks for parallel architectures. They have many remarkable communication properties such as symmetry, large number of nodes for a given degree and diameter, simple shortest path routing, Hamiltonicity, optimal connectivity, etc. The digraphs can be decomposed into vertex–disjoint cycles of the same length as it was shown in [3]. However, the distribution of tasks of different size in a network should be based in a decomposition of the digraph in vertex–disjoint paths of different length. In this paper we show that the cycle prefix digraph $\Gamma_\Delta(D)$ can be decomposed into one single path with $(D+1)!$ vertices and $\binom{\Delta-D+k}{k+1}$ paths of $(D-k)D!$ vertices, for $k = 0 \ldots D$.

1 Introduction

Thorough this paper, $\Gamma_\Delta(D)$ will denote a *cycle prefix digraph* of degree Δ and diameter D. These digraphs are defined on an alphabet of $\Delta+1$ symbols as follows: Each vertex $x_1x_2\cdots x_D$ is a sequence of distinct symbols from the alphabet. The adjacencies are given by

$$x_1x_2\cdots x_D \;\rightarrow\; \begin{cases} x_2x_3x_4\cdots x_Dx_{D+1}, & x_{D+1} \neq x_1, x_2, \ldots, x_D \\ x_2x_3x_4\cdots x_Dx_1 & \\ x_1x_2\cdots x_{k-1}x_{k+1}\cdots x_Dx_k, & 2 \cdot\ k \cdot\ D-1 \end{cases}$$

The first kind of adjacency, that introduces a new symbol, will be called a *shift*. The other adjacencies will be called *rotations*: r_k is the adjacency rotating the symbol in position k to the end of the word. Some relevant results and properties concerning cycle prefix digraphs may be found in [1], [2] and [4].

To obtain a decomposition into vertex–disjoint paths of a cycle prefix digraph, we use the *shift tree* $\mathcal{T}$, which was introduced in [2]. Since $\Gamma_\Delta(D)$, $\Delta \geq D$ decomposes into $\binom{\Delta+1}{D}$ subdigraphs isomorphic to $\Gamma_{D-1}(D-1)$, the tree $\mathcal{T}$ is an ordered way to have a representative of each of these subdigraphs. In the next section we present a construction of the tree $\mathcal{T}$ and the disjoint paths in it. The last section is devoted to decompose a cycle prefix digraph into paths with different number of vertices. Using the recursivity of the digraphs we prove that the number of vertices of the paths is a multiple of $|\Gamma_{D-1}(D-1)| = D!$.

2 The shift tree

Lemma 1 *For any cycle prefix digraph $\Gamma_\Delta(D)$ with $\Delta \geq D$, and any vertex $\mathbf{x}$, there exists a tree $\mathcal{T}$ rooted at $\mathbf{x}$ with $\binom{\Delta+1}{D}$ vertices, depth D, and maximum degree $\Delta + 1 - D$, such that any two vertices in $\mathcal{T}$ differ at least in one symbol and all its adjacencies are of type shift.*

Proof. See [3] ■

Because in $\mathcal{T}$ any two vertices differ at least in one symbol, each vertex of the tree is a representative of each of the $\binom{\Delta+1}{D}$ subdigraphs that are isomorphic to $\Gamma_{D-1}(D-1)$.

Figure 1 shows the shift tree corresponding to the cycle prefix digraph of degree $\Delta = 6$ and diameter $D = 4$.

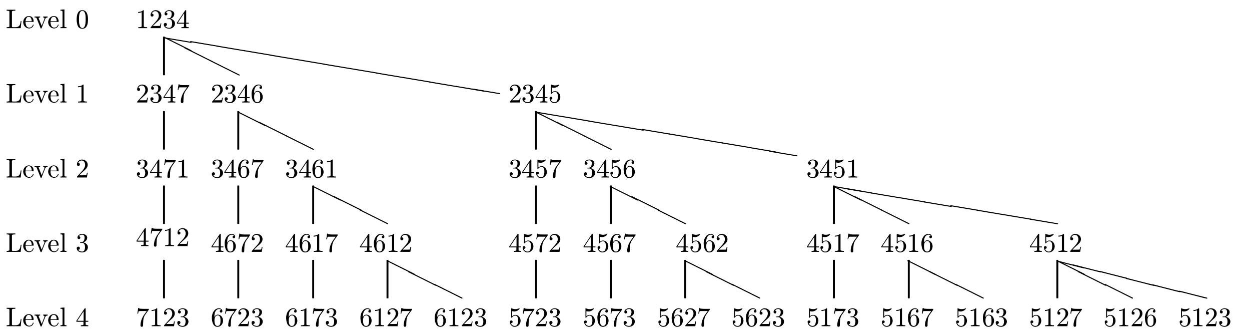

Figure 1: The shift tree associated to $\Gamma_6(4)$

Proposition 1 *Given $\Gamma_\Delta(D)$ and its shift tree $\mathcal{T}$, there are $\binom{\Delta-D+k}{k+1}$ vertex–disjoint paths with $D-k$ vertices, $k = 0 \ldots D+1$.*

Proof. We give a constructive proof. The tree $\mathcal{T}$ has the following vertex–disjoint paths:

One path with $D+1$ vertices

Starting at the root of the tree , the path is: $12\ldots D \to 2\ldots Da_1 \to \cdots \to a_1 12\ldots(D-1)$, with $a_1 = D+1$.

$\binom{\Delta-D+k}{k+1}$ paths with $D-k$ vertices

We start each path of length $D-k-1$ at each vertex of level $k+1$ that has not been used before. The paths are: $(k+2)\ldots Da_1\ldots a_{k+1} \rightarrow (k+3)\ldots Da_1\ldots a_{k+1}(k+1) \rightarrow \ldots Da_1\ldots a_{k+1}(k+1)(k+3) \rightarrow \cdots \rightarrow a_1\ldots a_{k+1}(k+1)\ldots(D-1)$

where $a_1 \in \{D+1,\ldots,\Delta+1\}$ and $a_2,\ldots a_{k+1} \in \{D+2,\ldots,\Delta+1\}$ and $a_1 < a_2, < \ldots < a_{k+1}$

The total number of paths is the number of vertices at level $k+1$ except for the vertices of this level already used:

$$\binom{\Delta-D+k+1}{k+1} - \sum_{j=0}^{k-1}\binom{\Delta-D+j}{j+1} - 1 = \binom{\Delta-D+k}{k+1}$$

■

Notice that two adjacent vertices in any of the above paths have $D-1$ common symbols. When $\Delta = D$, there is only one path which is actually a cycle with $D+1$ vertices.

3 The covering

We will use the recursive structure of the cycle prefix digraphs, see [2]. It is known that the cycle prefix digraph $\Gamma_\Delta(D)$ decomposes into $\binom{\Delta+1}{D}$ subdigraphs, each isomorphic to $\Gamma_{D-1}(D-1)$. As $|\Gamma_{D-1}(D-1)| = D!$ and because cycle prefix digraphs are Hamiltonian, see [5], each subdigraph $\Gamma_{D-1}(D-1)$ has an Hamiltonian cycle of length $D!$.

Next we will show that two Hamiltonian cycles of the same length and containing vertices that differ in one symbol can be properly connected to obtain a cycle of double length. Finally we present the decomposition of the cycle prefix digraph into vertex–disjoint paths.

Lemma 2 *Let* $\mathbf{x} = x_1x_2\ldots x_{D-1}x_D$ *and* $\mathbf{x}' = x_2x_3\ldots x_{D-1}x_Dy$ *be two vertices of* $\Gamma_\Delta(D)$ *,* $y \neq x_1$*. Then there is a cycle of length* $2D!$ *whose vertices are all the permutations of symbols* $\{x_1,x_2,\ldots,x_{D-1},x_D\}$ *and* $\{x_2,x_3,\ldots,x_{D-1},x_D,y\}$*.*

Proof. Consider the Hamiltonian cycle in the subdigraph isomorphic to $\Gamma_{D-1}(D-1)$ referred to vertex $\mathbf{x}$. The vertices of this cycle are all the permutations of symbols $\{x_1,x_2,\ldots,x_{D-1},x_D\}$. As it was shown in [5], the cycle has D adjacencies of type r_1, in particular:

$$\cdots \rightarrow x_1z_2\ldots z_{D-1}z_D \Rightarrow z_2z_3\ldots z_Dx_1 \rightarrow \cdots$$

with $z_2,\ldots,z_D \in \{x_2,\ldots,x_{D-1},x_D\}$ and $\Rightarrow$ representing an adjacency of type r_1.
The adjacency r_1 can be replaced by a Hamiltonian path in $\Gamma_{D-1}(D-1)$ with symbols $\{x_2,\ldots,x_{D-1},x_D,y\}$. This Hamiltonian path begins with vertex $z_2\ldots z_Dy$, ends with $yz_2\ldots z_D$ and contains all symbols $\{x_2,\ldots,x_{D-1},x_D,y\}$. The cycle has $2D!$ vertices.■

Theorem 1 *The cycle prefix digraph* $\Gamma_\Delta(D)$ *, can be decomposed into the vertex–disjoint union of* $\binom{\Delta-D+k+1}{k+1}$ *paths each with* $(D-k+1)D!$ *vertices,* $k = 0\dots D$.

Proof. The result follows from the existence of the paths in the tree $\mathcal{T}$ and the connection between hamiltonian cycles corresponding to vertices which differ in one symbol as it was proved in Lemma 2. ■

Example. Covering the vertices of $\Gamma_6(4)$. The paths obtained from the shift tree $\mathcal{T}$ of Figure 1 are:

One path of 5 vertices $1234 \to 2345 \to 3451 \to 4512 \to 5123$

Two paths of four vertices: $2346 \to 3461 \to 4612 \to 6123$ and $2347 \to 3471 \to 4712 \to 7123$.

Three paths of three vertices: $3456 \to 4562 \to 5623$; $3457 \to 4572 \to 5723$ and $3467 \to 4672 \to 6723$

Four paths of two vertices: $4516 \to 5163$; $4517 \to 5173$; $4567 \to 5673$ and $4617 \to 6173$

Vertices 5126, 5127, 5167, 5627 and 6127

$\Gamma_6(4)$, therefore, decomposes into the vertex–disjoint union of the following paths:

Number of paths	Vertices in a path
1	$5! = 120$
2	$4.4! = 96$
3	$3.4! = 72$
4	$2.4! = 48$
5	$1.4! = 24$

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