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On the Universality of Small-World Graphs

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1 Introduction

The *clustering* of a graph is defined as the fraction of existing edges between the neighbors of a vertex and the maximum number of edges that could possibly exist among the neighbors, averaged over all the vertices of the graph. Unlike other important graph parameters the clustering is a local measure.

A small-world graph is a graph which has a relatively low diameter (with respect to a regular graph with the same number of vertices and edges) and a large clustering (in relation to a random graph also with the same number of vertices and edges). These graphs are of considerable interest because they model important real-life networks [7] as the WWW, transportation and communication networks, or even the neural system of the worm *Caenorhabditis elegans* and social networks like the graph associated to the Erdős number [2].

In this paper we show that it is possible to obtain a small-world graph starting from any graph with a relatively large diameter by interconnecting a certain subset of vertices. We show that this interconnection may reduce the diameter of the graph until a given arbitrarily small threshold without significatively changing the clustering of the graph.

2 Diameter reduction

The diameter of a graph $G = (V, E)$ may be reduced by choosing a subset of vertices $H \subseteq V(G)$, $h = |H|$, such that any vertex of G is at a distance at most k from some vertex in H (in this case H is called k-dominating set [4] and its vertices *hubs*). After joining the vertices of this set by a connected graph G_H of diameter D_H we obtain a new graph of diameter $D_f \le 2k + D_H$.

To minimize the effects that the addition of new edges has on the clustering, we join the hubs using a star graph, in such a way that every new edge joins one of the hubs, called the *root*, to all the other hubs, as seen in Figure 1. Therefore we connect the hubs using the minimum possible number of edges and with a structure that has a small diameter $D_H = 2$.

We recall the following result:

Proposition 2.1 ([5]) *For $k \ge 1$, if G is a connected graph of order $n \ge k + 1$, then there exists a k-dominating set H with cardinality $h \le n/(k + 1)$.*

Therefore, given a connected graph G of order n, it is possible to find a set of at most $h = n/(k + 1)$ hubs such that no vertex in G is at distance greater than k from one of the hubs of the set.

If we use a star graph to interconnect the hubs, the resulting graph will have diameter $D_f \le 2k + 2$. The same result holds if the initial graph G is not connected and the star graph joins all its connected components.

By proposisiton 2.1 we have $h \le n/(k+1)$ and, since $D_f \le 2(k+1)$, it results $f_n = h/n \le 1/(k+1) \le 1/(D_f/2)$.

Theorem 2.1 *Let $G = (V, E)$ be a connected graph of diameter D and order n. Let $f_d, f_n \in R^+$, $f_d \ge 1$ and $f_n \le 1$, and such that D/f_d is an even integer. Then if $D \ge 2f_d/f_n$, it is possible to decrease the diameter of G by a factor of at least f_d by connecting a fraction of the initial vertices not greater than f_n.*

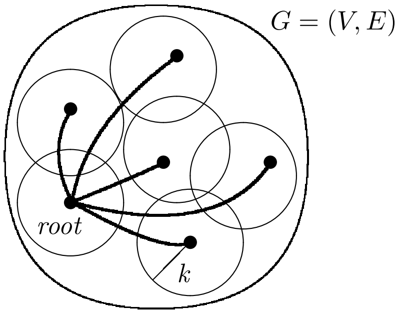

Figure 1: Connecting the hubs using a star graph.

Therefore, it is possible to reduce the diameter of any graph by only connecting a small number of vertices or hubs. The condition $D \geq 2f_d/f_n$ is easily fulfilled, as we need initial graphs with a relatively large diameter to obtain a small-world graph. On the other hand, and because we are considering a star graph to interconnect the vertices of H, the number of new added edges will be in general small.

Next we show that, after reducing its diameter following the former method, the clustering of the new graph is not far from the clustering of the initial graph. In that way, it is possible to obtain a small-world graph from any arbitrary graph.

3 Clustering bounds

We study the change in the clustering of a graph (induced by the new connections of the hubs) by considering the change that occurs at the clustering of each hub and its adjacent vertices. Clearly, the clustering of any other vertex of the graph will not be modified.

Thus, we can give an upper bound on the clustering C_f of the final graph by considering that, in the worst case, the change in the clustering of a vertex is from 0 to 1 in at most $h + \Delta$ vertices (i.e. all vertices in H, and those vertices that are simultaneously adjacent to the root hub and at least one other hub). On the other hand, we can only have a lowering of the clustering in vertices of H. So, taking into account Theorem 2.1, we obtain the following results.

Lemma 3.1 *Let $G = (V, E)$ be a graph of order n, maximum degree Δ and clustering C_0, and let H be a k-dominating set of G with cardinality h. Define $f_n = h/n$. Then, by joining the vertices of H with a star graph, the clustering C_f of the resulting graph verifies*

$$C_0 - f_n \leq C_f \leq C_0 + f_n + \frac{\Delta}{n}$$

Theorem 3.1 *Let $G = (V, E)$ be a connected graph of diameter D and order n. Let $f_d, f_n \in R^+$, $f_d \geq 1$ and $f_n \leq 1$, and such that D/f_d is an even integer. Then if $D \geq 2f_d/f_n$, it is possible to decrease the diameter of G by a factor of at least f_d in such a way that the final clustering C_f is bounded by*

$$C_0 - f_n \leq C_f \leq C_0 + f_n + \frac{\Delta}{n}$$

The bounds in Theorem 3.1 can be substantially improved by a careful computation of the change in the clustering of the affected vertices. This leads to the following theorem.

Theorem 3.2 *Let $G = (V, E)$ be a connected graph of diameter D, order n, clustering C_0, minimum clustering (of one of its vertices) c_{min}, maximum clustering c_{max}, minimum degree δ and maximum* *degree Δ. Let $f_d, f_n \in R^+$, $f_d \geq 1$ and $f_n \leq 1$ and such that D/f_d is an even integer. Then if $D \geq 2f_d/f_n$, and there exists a k-dominating set with at least one vertex which is not adjacent to the rest of the vertices (k being an integer such that $2(k+1) = D/f_d$), it is possible to decrease the diameter of G by a factor of at least f_d in such a way that the final clustering C_f is bounded by*

$$C_f \leq C_0 + (f_n - \frac{1}{n})\frac{2(1-c_{min})}{\delta+1} + \frac{1}{n} + \frac{2}{\delta}\cdot\frac{\Delta}{n} \leq C_0 + (f_n - \frac{1}{n})\frac{2}{\delta+1} + \frac{1}{n} + \frac{2}{\delta}\cdot\frac{\Delta}{n}$$

when $\delta \geq 3$ or by

$$C_f \leq C_0 + (f_n - \frac{1}{n})\frac{2(1-c_{min})}{\delta+1} + \frac{\Delta+1}{n} \leq C_0 + (f_n - \frac{1}{n})\frac{2}{\delta+1} + \frac{\Delta+1}{n}$$

when $\delta < 3$, and in any case by

$$C_f \geq C_0 - (f_n - \frac{1}{n})\frac{2c_{max}}{\delta+1} - \frac{1}{n} \geq C_0 - (f_n - \frac{1}{n})\frac{2}{\delta+1} - \frac{1}{n}$$

Corollary 3.1 *Any graph (connected or not) with relatively large diameter and clustering can be changed to a small-world graph with independence of its original topology.*

Example 3.1 *Consider a graph with the same parameters as those analyzed by Watts and Strogatz [7], i.e. a vertex-symmetric graph of order $n = 1000$, diameter $D = 100$, degree $\Delta = 10$ and clustering $C_0 = 0.67$. In this case, to reduce the diameter by a factor f_d of at least 16.67, the fraction f_n of hubs should be less than or equal to $1/3$ provided that $D \geq 2f_d/f_n = 100$. In that case, the clustering of the final graph would be $0.6254 \leq C_f \leq 0.6898$, and it is possible to obtain a final diameter of 6 with a change in the clustering of at most 6.2%.*

Therefore, if Watts and Strogatz had analyzed any other topology with the same parameters as the circulant graph in [7], it would have been also possible to obtain a small-world graph. However, while Watts and Strogatz reconnected the edges of the graph at random, we have defined a deterministic strategy. So Theorem 3.1 does not prove that any randomly reconnected graph should obey the same pattern as the graph studied by Watts and Strogatz. On the contrary, given a graph with the parameters described in [7], Theorem 3.1 ensures the existence of at least one deterministic interconnection strategy which leads to a reduction greater than 16 of the diameter and a variation of the clustering not greater than 6%, which is enough to guarantee the existence of a small-world graph.

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