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On the Spectrum of a Weakly Distance-Regular Digraph

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Abstract

The notion of distance-regularity for undirected graphs can be extended for the directed case in two different ways. Damerell adopted the strongest definition of distance-regularity, which is equivalent to say that the corresponding set of distance matrices $\{\boldsymbol{A}_i\}_{i=0}^D$ constitutes a commutative association scheme. In particular, a (strongly) distance-regular digraph Γ is stable, which means that $\boldsymbol{A}_i^\top = \boldsymbol{A}_{g-i}$, for each $i = 1, \ldots, g-1$, where g denotes the girth of Γ. If we remove the stability property from the definition of distance-regularity, it still holds that the number of walks of a given length between any two vertices of Γ does not depend on the chosen vertices but only on their distance. We consider the class of digraphs characterized by such a weaker condition, referred to as *weakly distance-regular digraphs*, and show that their spectrum can also be obtained from a smaller 'quotient digraph'. As happens in the case of distance-regular graphs, the study is greatly facilitated by a family of orthogonal polynomials called the distance polynomials.

1 Introduction

The concept of a distance-regular digraph was introduced by Damerell [2] as a generalization of the notion of distance-transitivity given by Lam [4]. Distance-regular digraphs are defined by using a regularity type condition concerning the cardinality of some vertex subsets. More precisely, a connected digraph $\Gamma = (V, E)$ with diameter D is *distance-regular* if, for any pair of vertices $u, v \in V$ such that $\operatorname{dist}(u, v) = k$, $0 \le k \le D$, the numbers[1]

$$s_{i1}^k(u, v) := |\Gamma_i^+(u) \cap \Gamma_1^+(v)|, \tag{1}$$

for each i such that $0 \le i \le k+1$, do not depend on the chosen vertices u and v, but only on their distance k; in which case they are referred to as the *intersection numbers*. Damerell proved that every distance-regular digraph Γ with girth g is *stable*; that is, $\operatorname{dist}(u, v) + \operatorname{dist}(v, u) = g$, for any pair of vertices $u, v \in V(\Gamma)$ at distance $0 < \operatorname{dist}(u, v) < g$.

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[1] For any fixed integer $0 \le k \le D$, we will denote by $\Gamma_k^+(v)$ (respectively, $\Gamma_k^-(v)$) the set of vertices at distance k from v (respectively, the set of vertices from which v is at distance k).

If we change $\Gamma_1^+(v)$ by $\Gamma_1^-(v)$ in the definition of distance-regularity, the stability property does not necessarily hold, and a class of digraphs with less structure appears. We focus our attention on such digraphs, here referred to as weakly distance-regular, and study some of their properties, which are closely related to the properties enjoyed by the distance-regular graphs (see [1, 3]).

2 Weakly distance-regularity

Definition 2.1 *A digraph Γ of diameter D is weakly distance-regular if, for each non-negative integer $l \leq D$, the number $a_{uv}^{(l)}$ of walks of length l from vertex u to vertex v only depends on their distance $\mathrm{dist}(u,v) = k$, for any $l = 0, 1, \ldots, D$.*

Theorem 2.2 *Let Γ be a connected digraph of diameter D. Then, Γ is a weakly distance-regular digraph if and only if any of the following statements hold:*

(*a*) *The distance matrix $\boldsymbol{A}_k$ is a polynomial of degree k in the adjacency matrix $\boldsymbol{A}$; that is, $\boldsymbol{A}_k = p_k(\boldsymbol{A})$, for each $k = 0, 1, \ldots, D$, where $p_k \in \mathbb{Q}[x]$ and $p_0 = 1$, $p_1 = x$.*

(*b*) *For any two vertices $u, v \in V(\Gamma)$ at distance $\mathrm{dist}(u,v) = k$, the numbers*

$$p_{i1}^k(u,v) = |\Gamma_i^+(u) \cap \Gamma_1^-(v)| \qquad (k-1 \leq i \leq D) \tag{2}$$

do not depend on the vertices u and v, but only on their distance k; in which case they are denoted by p_{i1}^k.

The polynomials p_k such that $\boldsymbol{A}_k = p_k(\boldsymbol{A})$, $0 \leq k \leq D$, will be referred to as the *distance polynomials* of Γ. It turns out that such polynomials are orthogonal with respect to the following scalar product

$$\langle f, g \rangle_\Gamma := \frac{1}{N} \operatorname{tr}(f(\boldsymbol{A}) g(\boldsymbol{A})^*), \tag{3}$$

which is well defined in the quotient ring $\mathbb{C}[x]/\mathcal{I}$, where $\mathcal{I} = (m_\Gamma)$ is the ideal generated by the minimum polynomial of Γ and N is the order of Γ. Then, the representation of xp_i in terms of the basis $\{p_k\}_{k=0}^D$ must be of the form

$$xp_i = p_i x = \sum_{k=0}^{\min\{i+1,D\}} \gamma_i^k p_k \qquad (0 \leq i \leq D)$$

where γ_i^k is the corresponding Fourier coefficient which must be equal to the intersection number $p_{i1}^k = p_{1i}^k$ (a real number). As a consequence,

$$p_{i1}^k = \frac{1}{N_k} \sum_{\mathrm{dist}(u,v)=i} s_{k1}^i(u,v) \qquad (k-1 \leq i), \tag{4}$$

where N_k denotes the number of (ordered) vertex pairs u, v such that $\mathrm{dist}(u,v) = k$.

Corollary 2.3 *Let Γ be a weakly distance-regular digraph with adjacency matrix $\boldsymbol{A}$. Then Γ is distance-regular if and only if any of the following conditions hold:*

(*a*) $\boldsymbol{A}^\top = \boldsymbol{A}_{g-1}$, *where g is the girth of Γ.*

(*b*) $\boldsymbol{A}$ *is normal.*

3 The spectrum

In this section we study how to compute the spectrum of a weakly distance-regular digraph Γ in terms of its defining parameters. We show that the whole spectrum of Γ can be retrieved from the information given by either of two matrices, the recurrence (intersection) matrix and the multiplicity matrix, which have size much more smaller that the adjacency matrix.

Theorem 3.1 *Let Γ be a weakly distance-regular digraph with N vertices, degree Δ and diameter D. Let $\{p_k\}_{k=0}^D$ be the distance polynomials of Γ and let $\boldsymbol{B} = (p_{1j}^k)$ be its intersection matrix. Then, the following statements hold:*

(*a*) *The minimum polynomials of Γ and $\boldsymbol{B}$ are equal to*

$$\det(x\boldsymbol{I} - \boldsymbol{B}) = \frac{1}{\alpha_D^D}(x-\Delta)\sum_{k=0}^{D} p_k,$$

where α_D^D is the leading coefficient of p_D.

(*b*) *If $\lambda_0 = \Delta, \lambda_1, \ldots, \lambda_d$ are the distinct eigenvalues of Γ, then their multiplicities $m(\lambda_i)$ are given by*

$$m(\lambda_i) = N(\boldsymbol{P}_{\mathrm{ev}}^{-1})_{i0} \qquad (0 \le i \le d), \tag{5}$$

where $\boldsymbol{P}_{\mathrm{ev}}$ is the matrix whose (i,j)-element is $p_i(\lambda_j)$, $0 \le i,j \le d$.

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