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Broadcasting in Cycle Prefix Digraphs ⋆

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Cycle prefix digraphs are directed Cayley coset graphs that have been proposed as a model of interconnection networks for parallel architectures. In this paper we present new details concerning their structure that are used to design a communication scheme leading to upper bounds on their broadcast time. When the diameter is two, the digraphs are Kautz digraphs and in this case our algorithm improves the known upper bounds for their broadcast time and is indeed optimal for small values of the degree.

1 Introduction.

A wide variety of new models for parallel architectures and distributed computing have been introduced in recent years. One important issue in their design and possible implementation is the topology of the associated network. The consideration of highly symmetric and dense interconnection networks, particularly for massively parallel computers, leads to a question of special interest in graph theory: the construction of vertex symmetric digraphs with order as large as possible for a given maximum degree and diameter. One important reason to consider vertex symmetric digraphs is that in the associated network each node is able to execute the same communication software. Other aspects of interest are their modularity and simple definition.

Most large vertex symmetric digraphs correspond to Cayley (coset) digraphs and have been found either by random computer search, special graph prod-

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ucts, or direct constructions. Very often, vertex symmetric digraphs may be described as digraphs on alphabets. A digraph on an alphabet is constructed as follows: the vertices are labeled with words on the given alphabet and the arcs are defined according to a rule that relates two different words. This representation of the digraph facilitates the direct calculation of the diameter and other distance-related parameters. An interesting family of vertex symmetric digraphs was defined by Faber and Moore as Cayley coset digraphs [5] and may be viewed as well as a family of digraphs on alphabets [4,6]. This latter representation gives them the name that we will use through this paper: *cycle prefix digraphs*.

On the other hand, much work has been done related to the dissemination of information in interconnection networks. The importance of this research area lies in the fact that the ability of a network to effectively disseminate information is an important measure of the suitability of the network for parallel computing. Broadcasting is one of the most studied problems in communication networks and refers to the sending of a message from one node of the network to all the other nodes as quickly as possible, subject to the constraints that each call involves only two nodes, a node which already knows the message can only inform one of the nodes to which it is connected, and each call requires one unit of time.

In this paper we are interested in the study of communication schemes for cycle prefix digraphs. We present an efficient broadcasting scheme based on the recursive structure that we describe below.

The paper is organized as follows. The next section will introduce the notation and give some general definitions. Section 3 presents the cycle prefix digraphs and some of their properties. Then we focus on the recursive structure of the digraphs which will suggest the broadcast strategy presented in Section 4 together with the upper bounds on the broadcast time. Finally, in Section 5, we compare the broadcast time obtained with the known upper bounds for de Bruijn digraphs of similar order for the same diameter and degree. When the diameter is two, we also make a comparison with the upper bounds known for the corresponding Kautz family. We see that our scheme improves considerably the bounds given by Bermond and Peyrat [1] and by Heydemann, Opatrny and Sotteau [9], and is indeed optimal for small values of the degree.

2 Notation

Let $\Gamma = (V, A)$ be a digraph with vertex set V and arc set A. The *out-degree* of a vertex u, $\delta^+(u)$, is the number of vertices adjacent from u and its *in-degree* $\delta^-(u)$, is the number of vertices adjacent to u. A digraph is regular of

degree Δ or Δ-*regular* if the in-degrees and out-degrees of all vertices equal Δ. We define a *path* in Γ from $v_0 \in V$ to $v_k \in V$ as a sequence of abutting arcs $(v_0, v_1), (v_1, v_2), \cdots, (v_{k-1}, v_k)$ such that for each $i \in 0, 1, \cdots, k-1$, $(v_i, v_{i+1}) \in E$. A digraph is (*strongly*) *connected* if there is a path from every vertex to every other vertex.

A digraph is *vertex symmetric* if its automorphism group acts transitively on its set of vertices. A *Cayley coset digraph*, $\mathcal{C}ay(G, H, S)$, is a digraph defined by a finite group G, a subgroup H and a set of generators $S \subseteq G \setminus H$. The vertex set is given by the set of left cosets $\{gH | g \in G\}$ and there is an arc (g_1H, g_2H) whenever $g_1sH = g_2H$ for some $s \in S$. Cayley coset digraphs are $|S|$-regular, connected and vertex symmetric. Moreover, every vertex symmetric digraph is a Cayley coset digraph, as is shown in [13]. In particular, a Cayley coset digraph Γ is a *Cayley digraph* iff $H = \{e\}$, where e is the identity of G.

$\Gamma = \mathcal{C}ay(G, H, S)$ is a *hierarchical* or *quasi-minimal* Cayley coset digraph iff there is an ordering of the elements of S, say $\{s_1, s_2, \cdots, s_k\}$ such that for every $i = 1, 2, \cdots, k-1$, the group $\langle H, s_1, s_2, \cdots, s_i \rangle$ is a proper subgroup of $\langle H, s_1, s_2, \cdots, s_i, s_{i+1} \rangle$. Γ is *minimal* iff for no $S' \subset S$, $\langle H, S' \rangle = G$. A hierarchical Cayley (coset) digraph can be decomposed into a collection of isomorphic subdigraphs along with edges connecting them. Each subdigraph is a smaller Cayley (coset) digraph.

Broadcasting in a graph is the process of spreading a message known initially by one vertex, subject to the following rules. The transfer of the message from one vertex to another (termed a *call*) takes one unit of time. A vertex can only call an adjacent vertex. A vertex can participate in at most one call per unit of time. A *broadcast scheme* is a formal description of this process.

Given a connected digraph Γ and a vertex u, the *broadcast time of* u, denoted $b(u)$, is the minimum number of time units required to broadcast a message originating at u. The *broadcast time of the graph* Γ is defined $b(\Gamma) = \max\{b(u) | u \in \Gamma\}$. For any vertex u in a connected graph with $|V|$ vertices, $b(u) \geq \lceil \log_2 |V| \rceil$, since during each time unit the number of vertices informed can at most double. For a vertex symmetric graph, the broadcast time is equal to the broadcast time of any of its vertices.

The *de Bruijn* digraph $B(\Delta, D)$, $\Delta \geq 2$, has vertices labeled with words $x_1x_2 \cdots x_D$ where x_i belongs to an alphabet of size Δ. There is an arc from any vertex $x_1x_2 \cdots x_D$ to the Δ vertices $x_2 \cdots x_Dx_{D+1}$, where x_{D+1} is any letter of the alphabet. $B(\Delta, D)$ is Δ-regular, has Δ^D vertices and diameter D.

The *Kautz digraph* $K(\Delta, D)$, $\Delta \geq 2$, has vertices labeled with words $x_1x_2 \cdots x_D$ where x_i belongs to an alphabet of $\Delta+1$ letters and $x_i \neq x_{i+1}$ for $1 \leq i \leq D-1$. A vertex $x_1x_2 \cdots x_D$ is adjacent to the Δ vertices $x_2 \cdots x_Dx_{D+1}$, where x_{D+1} can be any letter different from x_D. Hence, the digraph $K(\Delta, D)$ is Δ-regular,

has $\Delta^D + \Delta^{D-1}$ vertices and diameter D. For $D = 2$ the Kautz digraphs are vertex symmetric. Figure 1 shows $K(2,2)$ (in the figures of this paper a line represents two opposite arcs).

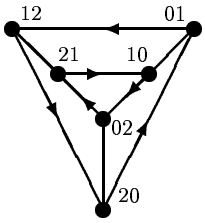

Fig. 1. *K(2,2)*, a 2-regular vertex symmetric digraph with diameter 2.

3 Cycle prefix digraphs. Recursive structure.

The *cycle prefix digraphs* $\Gamma_\Delta(D)$, $\Delta \geq D$, were introduced as Cayley coset digraphs by Faber and Moore in 1988 [5]. These digraphs may also be defined on an alphabet of $\Delta + 1$ symbols as follows [4,6]: Each vertex $x_1x_2\cdots x_D$ is a sequence of distinct symbols from the alphabet. The adjacencies are given by

$$x_1x_2\cdots x_D \;\rightarrow\; \begin{cases} x_2x_3x_4\cdots x_Dx_{D+1}, & x_{D+1} \neq x_1, x_2, \ldots, x_D \\ x_1x_2\cdots x_{k-1}x_{k+1}\cdots x_Dx_k, & 1 \leq k \leq D-1 \end{cases}$$

The first kind of adjacency, that introduces a new symbol, will be called a *shift*. The other adjacencies will be called *rotations*. $\Gamma_\Delta(D)$ has order $(\Delta+1)_D$, diameter D and is Δ-regular ($\Delta \geq D$).

Cycle prefix digraphs have several relevant properties which are of interest when modelling an interconnection network. The digraphs are Hamiltonian, as was proved by M. Jiang and F. Ruskey in [11]. In [4], Comellas and Fiol proved that, for $D \geq 3$, the digraphs are $D-$reachable (every pair of vertices, not necessarily different, may be joined by a path of exactly D arcs). Together with new families constructed from them, cycle prefix digraphs have provided important improvements to the table of largest known vertex-symmetric (Δ, D) digraphs, see [4]. It has been shown by Chen, Faber and Knill [3] that the wide diameter of $\Gamma_\Delta(D)$ is $D+2$ (the wide diameter of a graph is considered an important measure of communication efficiency and reliability, see [10]). By extending a former result for hierarchical Cayley graphs from Hamidoune, Lladó and Serra [8] to this particular family of Cayley coset digraphs, Knill, in [12], showed that the cycle prefix digraphs have optimal connectivity.

Some hierarchical Cayley graphs can be recursively decomposed. This is the case for the pancake and star graphs (see GowriSankaran [7] and Berthomé,

Ferreira and Perennes [2]). By using the description as digraphs on an alphabet it is also possible to give a recursive decomposition for the cycle prefix digraphs. The following lemma summarizes this result.

Lemma 1 *The cycle prefix digraph* $\Gamma_\Delta(D)$ *decomposes into* $\binom{\Delta+1}{D}$ *subdigraphs, each isomorphic to* $\Gamma_{D-1}(D-1)$.

PROOF. Let us consider all vertices of $\Gamma_\Delta(D)$ which have the same set of symbols, together with their corresponding edges. Clearly, they form a subdigraph that is isomorphic to $\Gamma_{D-1}(D-1)$. Since there are $\binom{\Delta+1}{D}$ possible different ways of choosing this set of symbols from the given alphabet, we can express $\Gamma_\Delta(D)$ as a vertex disjoint union of $\binom{\Delta+1}{D}$ subdigraphs. □

Figure 2 shows one of the subdigraphs that can be obtained from $\Gamma_\Delta(4)$. Note that, for clarity, we have omitted some edges that join the four terminal subgraphs, $\Gamma_2(2)$ ($\approx K(2,2)$), in the recursive structure.

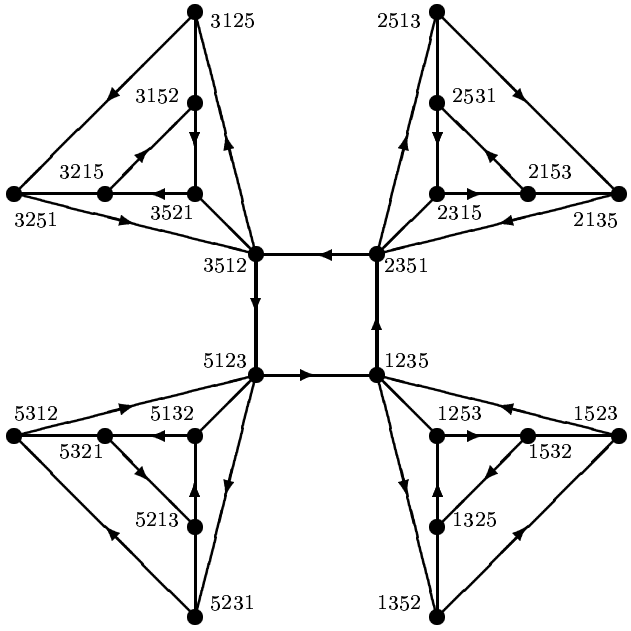

Fig. 2. One subdigraph of $\Gamma_\Delta(4)$ isomorphic to $\Gamma_3(3)$.

4 Broadcasting in cycle prefix digraphs

The broadcast algorithm we have designed uses the decomposition into subdigraphs of $\Gamma_\Delta(D)$ given in Section 3. The first phase of the algorithm distributes the message from the initial vertex to $\binom{\Delta+1}{D}-1$ other vertices, each in a different subdigraph of the decomposition. In phase two, the message is sent within

each subdigraph. The two phases are recursively executed, in parallel, in each subdigraph.

The main point is to build a structure in $\Gamma_\Delta(D)$ containing a set of vertices such that any two vertices of the set differ in at least one symbol. This set will have $\binom{\Delta+1}{D}$ elements. With this set it is possible to construct a tree to be used in the first phase of the broadcast scheme.

Lemma 2 *For any cycle prefix digraph $\Gamma_\Delta(D)$ with $\Delta \geq D$, and any vertex $\mathbf{x}$, there exists a tree $\mathcal{T}$ rooted at $\mathbf{x}$ with $\binom{\Delta+1}{D}$ vertices, depth D, and maximum degree $\Delta+1-D$, such that any two vertices in $\mathcal{T}$ differ in at least one symbol.*

PROOF. We give a constructive proof. Without loss of generality, we may choose $\mathbf{x} = 12\cdots D$ as the root of $\mathcal{T}$. The successive adjacencies are always of type *shift* as follows:

Level 1: $\mathbf{x}$ is adjacent to the $\Delta+1-D$ vertices $\mathbf{x}^1_{s_1} = 23\cdots Ds_1, \quad D+1 \leq s_1 \leq \Delta+1$. The upper index of $\mathbf{x}^1_{s_1}$ corresponds to the level and the lower index is the last symbol of the vertex.

Level 2: Vertices at this level are denoted $\mathbf{x}^2_{s_1s_2} = 3\cdots Ds_1s_2$. They are adjacent from $\mathbf{x}^1_{s_1}$ and s_2 is 1 or is a symbol from $\{D+1,\ldots,\Delta,\Delta+1\}\setminus\{s_1\}$ such that $s_2 > s_1$. At this level there are:
- $\Delta+1-D$ vertices which end with 1.
- $\binom{\Delta+1-D}{2}$ other vertices.

The total number of vertices at level two is $\Delta+1-D+\binom{\Delta+1-D}{2} = \binom{\Delta+2-D}{2}$.

Level 3: Vertices $\mathbf{x}^3_{s_1s_2s_3} = 4\ldots Ds_1s_2s_3$ are adjacent from $\mathbf{x}^2_{s_1s_2}$ by shifts that add s_3, which is 2 or a symbol from $\{D+1,\ldots,\Delta,\Delta+1\}\setminus\{s_1,s_2\}$ such that $s_1 < s_2 < s_3$ or $s_1 < s_3$ (if $s_2 = 1$). There are:
- $\binom{\Delta+2-D}{2}$ vertices with $s_3 = 2$.
- $\binom{\Delta+1-D}{2}$ vertices with $s_2 = 1$ and $s_1 < s_3$.
- $\binom{\Delta+1-D}{3}$ vertices with $s_1 < s_2 < s_3$:

The total number of vertices at this level is $\binom{\Delta+2-D}{2}+\binom{\Delta+1-D}{2}+\binom{\Delta+1-D}{3} = \binom{\Delta+3-D}{3}$.

...

Level k: Vertices are $\mathbf{x}^k_{s_1s_2\ldots s_k} = (k+1)\cdots Ds_1s_2\cdots s_k$ and they are adjacent from vertices $\mathbf{x}^{k-1}_{s_1s_2\ldots s_{k-1}}$ at level $k-1$. s_k is either $k-1$ or a symbol from $\{D+1,\ldots,\Delta,\Delta+1\}\setminus\{s_1,s_2,\ldots,s_{k-1}\}$. All symbols $s_1,s_2,\ldots,s_k$ satisfy the inequality $s_l < s_m$ if $l < m$ and $s_l \neq l-1$ and $s_m \neq m-1$. There are:
- $\binom{\Delta+(k-1)-D}{k-1}$ vertices with $s_k = k-1$.
- $\binom{\Delta+1-D}{k}$ vertices with $D+1 \leq s_1 < \ldots < s_k \leq \Delta+1$.
- $\binom{\Delta+1-D}{k-1}$ vertices with $D+1 \leq s_1 < \ldots < s_{i-1} < s_{i+1} < \ldots < s_k \leq \Delta+1$ and $s_i = i-1$.

– $\binom{\Delta+1-D}{k-2}$ vertices with $D+1 \leq s_1 < \ldots s_{i-1} < s_{i+1} \ldots s_{j-1} < s_{j+1} < \ldots < s_k \leq \Delta+1$ and $s_i = i-1$ and $s_j = j-1$ $(i \neq j)$.

  ...

– $\binom{\Delta+1-D}{1}$ vertices with $s_i = i-1$ for $i = 2 \ldots k$.

The total at level k is $\binom{\Delta+(k-1)-D}{k-1} + \sum_{j=1}^{k-1} \binom{\Delta+1-D}{j} = \binom{\Delta+k-D}{k}$

...

Level D: Vertices are $\mathbf{x}^D_{s_1 s_2 \ldots s_D} = s_1 s_2 \cdots s_D$ and they are adjacent from vertices at level $D-1$, $\mathbf{x}^{D-1}_{s_1 s_2 \ldots s_{D-1}}$ and s_D is such that $s_1 < s_2 < \ldots < s_{D-1} < s_D$ except when $s_j = j-1$, $j = 2 \ldots D-1$. The process finishes at this level and a similar count than in level k gives $\binom{\Delta}{D}$ vertices.

Therefore the maximum degree of $\mathcal{T}$ is $\Delta+1-D$, its depth is D and $\mathcal{T}$ has $\sum_{k=0}^{D} \binom{\Delta+k-D}{k} = \binom{\Delta+1}{D}$ vertices. This is precisely the number of choices of D different elements from an alphabet of $\Delta+1$ symbols.

It is not difficult to see that any two vertices in $\mathcal{T}$ differ in at least one symbol. First, notice that any vertex at level k of the tree has the form $\mathbf{x} = (k+1) \cdots D s_1 s_2 \cdots s_k$ ($\mathbf{x} = s_1 s_2 \cdots s_D$, if $k = D$) with $s_i < s_j$ if $i < j$ and $s_i \neq i-1$ and $s_j \neq j-1$. The symbol k is not contained in any of the vertices of level k. Let $\mathbf{x} = (p+1) \cdots D s_1 s_2 \cdots s_p$ and $\mathbf{x}' = (q+1) \cdots D s'_1 s'_2 \cdots s'_q$ be two different vertices of $\mathcal{T}$ not at level D. If $p \neq q$, say $p < q$, then $\mathbf{x} = (p+1) \cdots q(q+1) \cdots D s_1 s_2 \cdots s_p$, but as q is not a symbol of $\mathbf{x}'$, the vertices differ at least in this symbol. If one of the vertices is at level D it will not contain the symbol D, but as all other vertices contain it, the two vertices will differ at least in that symbol. On the other hand, if the vertices are both at the same level, in order to differ at least in one symbol $\{s_1, \cdots, s_p\}$ should be a different set of symbols than $\{s'_1, \cdots, s'_p\}$, $(p \leq D)$. Let us suppose that $s_1 \neq s'_1$ and $s_1 < s'_1$, then, because of the construction $s_1 \notin \{s'_1, \ldots, s'_p\}$ and both vertices will not share the same set of symbols. Otherwise, let i, $1 < i \leq p$ be the first position for which $s_i \neq s'_i$, if $s_i = i-1$ (or $s'_i = i-1$) then symbol s_i is not in $\mathbf{x}'$ (or s'_i is not in $\mathbf{x}$). Therefore $s_i < s'_i$ and in this case s_i will not be in $\mathbf{x}'$ because after position i, the construction only allows to add symbols s' such that $s' > s'_i$. □

Remark 3 *If $\Delta = D$, instead of a tree we have a path of length D.*

Example 4 *The tree associated with $\Gamma_5(4)$ has depth $D = 4$, maximum degree $\Delta+1-D = 2$, and contains $\binom{\Delta+1}{D} = \binom{6}{4} = 15$ vertices. The tree is shown in Figure 3.*

From the proof of the preceeding lemma, the vertices of level one are obtained by shifts that add one of the symbols $\{5, 6\}$ to the end of the root 1234. These vertices are 2345 and 2346

At level two there is one vertex ending with 1 adjacent from each of the vertices

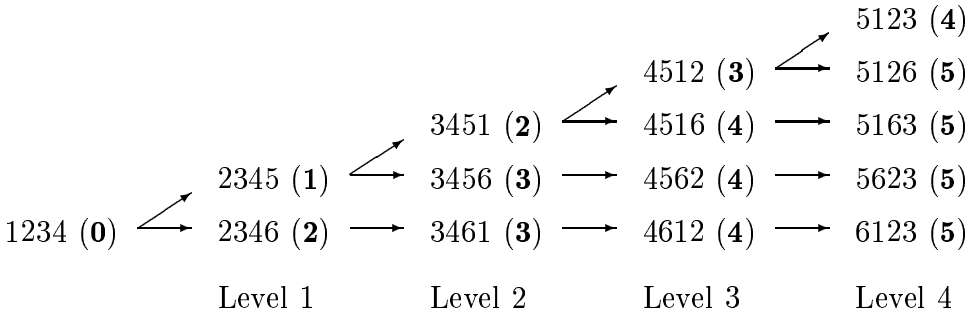

Fig. 3. The basic tree for $\Gamma_5(4)$. Boldface numbers indicate the broadcast order using the scheme of Theorem 5.

*of level one (*3451, 3461*). The other vertex of this level must be* $34s_1s_2$ *with* $s_1 < s_2$ *and* $s_1, s_2 \in \{5, 6\}$*. The only possible choice for* s_1 *and* s_2 *gives* 3456 *which is adjacent from* 2345*.*

*We obtain three vertices of level three (*4512, 4562, 4612*) from vertices of level two by shifts that add* 2*. The other vertices of this level must end with one of the symbols in* $\{D+1, \ldots, \Delta, \Delta+1\} \setminus \{s_1, s_2\} = \{5, 6\} \setminus \{s_1, s_2\}$*. Therefore, vertex* $4s_1s_2s_3$ *must be such that* $s_1 \in \{5, 6\}$ *and when* $s_2 = 1$ *the condition* $s_1 < s_3$ *leads to the choice* $s_1 = 5, s_3 = 6$*, giving vertex* 4516*. When* $s_2 \neq 1$ *it is not possible to give symbols satisfying* $s_1 < s_2 < s_3$*.*

The vertices of the last level start with a symbol from $\{D+1, \ldots, \Delta+1\} = \{5, 6\}$*. Four vertices will end with* 3 *and are adjacent from each vertex of level* 3*, (*5123, 5163, 5623, 6123*). The other vertex, following arguments similar to those for level three, is* 5126*.*

Theorem 5 *The broadcast time for* $\Gamma_\Delta(D)$*,* $\Delta \geq D$*, is bounded as follows.*

$$b(\Gamma_\Delta(D)) \leq \Delta + \frac{D(D-1)}{2}$$

PROOF. We start the broadcast process using the tree constructed according to Lemma 2. The originator sends the message to its adjacent vertices in lexicographic order. Any vertex in the tree proceeds in the same way, as shown in Figure 3. Therefore, to broadcast a message from the origin to all vertices of the tree takes Δ time units.

From each vertex of the tree the message is then broadcast to a subdigraph (isomorphic to $\Gamma_{D-1}(D-1)$). To broadcast in this digraph, we will follow the same scheme but now instead of a tree we have a path (Remark 3). The process is recursively repeated for each new vertex reached. We denote by $\beta(\Gamma_k(k))$ the time that this broadcasting scheme takes to broadcast in $\Gamma_k(k)$. As the broadcasting process first uses a path P_k of length k we have the following

expression:

$$\beta(\Gamma_k(k)) = b(P_k) + \beta(\Gamma_{k-1}(k-1)) \qquad k > 3$$

Using this recurrence relation, and the facts that $b(P_k) = k$ and $\beta(\Gamma_2(2)) = b(K(2,2)) = 3$, we obtain:

$$\begin{aligned}\beta(\Gamma_k(k)) &= b(P_k) + b(P_{k-1}) + \cdots + b(P_3) + \beta(\Gamma_2(2)) \\ &= k + (k-1) + \cdots + 3 + 3 = \frac{k(k+1)}{2}\end{aligned}$$

Combining the results, the broadcasting time of a cycle prefix digraph is:

$$b(\Gamma_\Delta(D)) \leq b(\mathcal{T}) + \beta(\Gamma_{D-1}(D-1)) = \Delta + \frac{D(D-1)}{2} \qquad \square$$

5 Conclusions

In this paper we have presented a recursive decomposition of the cycle prefix digraph that yields an efficient broadcasting scheme.

It is interesting to compare the broadcast times that result from this scheme with those known for other digraphs with similar order for the same degree and diameter. In [1] Bermond and Peyrat give $b(B(\Delta, D) \leq \frac{(\Delta+1)(D+1)}{2}$, $2 \leq D \leq 14$, as the smallest upper bound on the broadcast time of de Bruijn digraphs. Our scheme leads to much better broasdcast times for comparable cycle prefix digraphs (see Table 1).

	$\Gamma_\Delta(3)$		$B(\Delta,3)$	
Δ	$\lvert V\rvert$	$\beta(\Gamma_\Delta(3))$	$\lvert V\rvert$	$\beta'(B(\Delta,(3))$
3	24	6	27	8
4	60	7	64	10
5	120	8	125	12
6	210	9	216	14
7	336	10	343	16
8	504	11	512	18
9	720	12	729	20
10	990	13	1000	22

Table 1
Comparative values of the broadcast time of cycle prefix digraphs, $\beta(\Gamma_\Delta(3))$, and de Bruijn digraphs of diameter three, $\beta'(B(\Delta,3))$, for small values of the degree.

On the other hand, $\Gamma_\Delta(2)$ is the Kautz digraphs $K(\Delta, 2)$. Heydemann, Opatrny and Sotteau [9] give the best known upper bounds on the broadcast time of Kautz digraphs as:

$$
\begin{aligned}
b(K(2,D)) &\leq 2D, \\
b(K(3,D)) &\leq 3D, \\
b(K(\Delta,D)) &\leq \begin{cases} \frac{(\Delta+3)(D+1)}{2}, & \text{if } 4 \leq \Delta \leq 12,\ \Delta \neq 9 \\ \min\{2D\lceil \log_2 \Delta \rceil, 3D\lceil \log_3 \Delta \rceil\}, & \text{if } \Delta = 9 \text{ or } \Delta \geq 13 \end{cases}
\end{aligned}
$$

For $D = 2$ and small degrees, Theorem 5 improves these results giving a new upper bound of $\beta(K(\Delta, 2)) = \Delta + 1$. This broadcasting time is optimal for small values of the degree ($\Delta < 6$).

Finally, we mention that it should be possible to further exploit the hierarchical nature of the cycle prefix digraphs to deal with other communication problems.

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