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Spectral bounds for the betweenness of a graph

F. Comellas, S. Gago *

In this paper we find spectral bounds (Laplacian matrix) for the vertex and the edge betweenness of a graph. We also relate the edge betweenness with the isoperimetric number and the edge forwarding and edge expansion indices of the graph allowing a new upper bound on its diameter. The results are of interest as they can be used in the study of communication properties of real networks, in particular for dynamical processes taking place on them (broadcasting, network synchronization, virus spreading, etc).

1 Introduction

Betweenness is a good measure of the centrality of a vertex (or edge) in a graph modeling a social or communication network. It is usually defined as the fraction of shortest paths between vertex pairs that go through the vertex (or edge) considered. Therefore, in many models, betweenness is a measure of the influence of a node (or link) in the dissemination of information over a network [4, 6] and can also be used to detect communities or clusters in networks [5].

Vertex betweenness was first proposed by Freeman [4] in 1977 in the context of social networks and has been considered more recently as an important parameter in the study of networks associated to complex systems [9]. Girvan and Newman in [5] generalize this definition to edges and introduce the *edge betweenness* of an edge as the fraction of shortest paths between pairs of vertices that run along it.

To be more precise, if $\sigma_{uv}(w)$ denotes the number of shortest paths (geodetic paths) from vertex u to vertex v that go through w, and σ_{uv} is the total number of geodetic paths from u to v, then we define $b_w(u,v) = \sigma_{uv}(w)/\sigma_{uv}$ and the betweenness of vertex w is $B_w = \sum_{u,v\neq w} b_w(u,v)$. The (vertex) betweenness of a graph $G = (V,E)$ of order n is

*Departament de Matemàtica Aplicada IV, Universitat Politècnica de Catalunya,Avda. Canal Olímpic s/n, 08680 Castelldefels, Catalonia, Spain. {comellas,sgago}@ma4.upc.edu

$\overline{B} = (\sum_{u \in V} B_u)/n$ and the maximum (vertex) betweenness of G is $B_{max} = \max\{B_w \mid w \in V\}$. The (vertex) betweenness $\overline{B}$ is related with the average distance $\bar{l}$ of the graph as $\overline{B} = (n-1)(\bar{l}-1)$ [3].

The edge-betweenness is defined as follows [5]: let $\sigma_{uv}(e)$, $e \in E(G)$, be the number of shortest paths from u to v that go through edge e, and let σ_{uv} be the total number of shortest paths from u to v, then we introduce $b^e(u,v) = \sigma_{uv}(e)/\sigma_{uv}$. The betwenness for an edge of the graph is defined as $B^e = \sum_{u \neq v} b^e(u,v)$, the maximum edge-betweenness is $B^E_{\max} = \max_{e \in E} B^e$ and the average edge-betweenness or edge-betweenness of a graph G of size m is $\overline{B}^E = (\sum_{e \in E} B^e)/m$.

In the next section we obtain new spectral bounds based on the Laplacian matrix, for the vertex betweenness of a graph. In Section 3 we relate the edge-betweenness with the isoperimetric number and the edge expansion and edge forwarding indices of the graph, we give an spectral bound for the maximum edge-betweennes and we improve known upper bounds on the diameter.

2 Spectral bounds for the vertex betweenness

2.1 Notation and preliminary results

First, we introduce some notation and preliminaries which will we useful to obtain the main results.

Given two vertices $u, v \in V$ at a distance d, let us consider the sets:

$$P^i_{u,v} = \{w \in V \mid \quad d(u,w) = i, \quad d(w,v) = d-i\}, \quad 0 \leq i \leq d.$$

which we call i-layers. It is obvious that $P^0_{u,v} = \{u\}$ and $P^d_{u,v} = \{v\}$. Let n_i be the cardinal of the i-layer $P^i_{u,v}$, and let e_i be the number of edges that connect the vertices of $P^i_{u,v}$ with those of $P^{i+1}_{u,v}$.

It is not difficult to see:

Lemma 1. *For all* $1 \leq i \leq d-1$*:*

1. $1 \leq n_i \leq \alpha = \max\limits_{0 \leq i \leq d} n_i$.
2. $e_i \geq n_i$*, as there is at least one edge from every vertex in the layer* i *to vertices of the next layer.*

3. $e_i \leq n_i n_{i+1} \leq \alpha^2$, *with equality when all the vertices of the layer* $P^i_{u,v}$ *are connected to those of the next layer* $P^{i+1}_{u,v}$ *and* $\alpha = n_i = n_{i+1}$.

4. $e_{i-1} + e_i \leq n_i \Delta$.

5. $n_0 + \cdots + n_d \leq n$.

Another property of these sets which is easy to prove is:

Lemma 2. *Let G be a graph of order n, let $u, v \in V(G)$ two vertices of G at a distance $d \geq 2$ then* $\sum_{w \in P^i_{u,v}} b_w(u, v) = 1$, *for all* $1 \leq i \leq d - 1$.

2.2 Bounds from the Laplacian spectra

Let us consider the Laplacian of the graph G, $L = D - A$ (D is the diagonal matrix of vertex degrees of G). The eigenvalues of this matrix can be used to obtain bounds for the betweenness of a vertex w:

Proposition 1. *Let G be a connected graph of order n. Let $u, v, w \in V(G)$ be vertices such that $d(u, v) = 2$ and $d(u, w) = 1 = d(w, v)$. Let Δ be the maximum degree of G and let θ_n be the largest eigenvalue of the Laplacian of G, then*

$$\frac{n}{(n-1)\theta_n} \leq b_w(u, v).$$

Proof. We have that $1 \leq \sigma_{uv} \leq \min\{\delta_u, \delta_v\} \leq \Delta$ as at most one of the vertices u or v has all its neighbours in a shortest path connecting both vertices. Besides $\sigma_{uv}(w) = 1$ for all w between u and v and $1/\Delta \leq \sigma_{uv}(w)/\sigma_{uv} = 1/\sigma_{uv}$. Therefore using the inequality given by Fiedler in [2], $(n\Delta)/(n-1) \leq \theta_n$, we can write:

$$\frac{n}{(n-1)\theta_n} \leq \frac{1}{\Delta} \leq b_w(u, v).$$

□

Proposition 2. *Let G be a graph of order n, let $u, v, w \in V(G)$ be vertices of G such that $d(u, v) = d > 2$ and w is contained in some layer P^i_{uv} for some $1 \leq i \leq d - 1$. Let m be the total of vertices of the layer, Δ the maximum degree of the graph and let θ_2 be the second largest eigenvalue of the Laplacian of G, then:*

$$\left(\frac{\theta_2}{n\Delta}\right)^{d-1} \leq b_w(u, v) \leq 1 - (m-1)\left(\frac{\theta_2}{n\Delta}\right)^{d-1}.$$

Proof. It is known that we can determine the second eigenvalue θ_2 of the Laplacian matrix from a variation of the Courant-Fisher theorem, [2]:

$$\theta_2 = \min_{x\in\mathbb{R}^n} \frac{2n\sum_{uv\in E}(x_u - x_v)^2}{\sum_{u\in V}\sum_{v\in V}(x_u - x_v)^2}.$$

Now let us consider the vector of $\mathbb{R}^n$ defined as $x_u = \begin{cases} 1 & \text{if} \quad u \in P^i_{u,v} \\ 0 & \text{if} \quad u \notin P^i_{u,v} \end{cases}$ where the i-layer is such that $n_i \leq n_h = \alpha$ (with equality if all the layers have the same cardinality), then we get that

$$\theta_2 \leq \frac{2n(e_{i-1}+e_i)}{2n_i(n-n_i)} \leq \frac{nn_i\Delta}{n_i(n-n_i)} = \frac{n\Delta}{n-n_i} \leq \frac{n\Delta}{\alpha}.$$

as we know that $n \geq n_1 + \cdots + n_{d-1}$ and $n - n_i \geq \sum_{1\leq j\leq d-1, j\neq i} n_j \geq n_h = \alpha$, then

$$\frac{\theta_2}{n\Delta} \leq \frac{1}{\alpha} \Rightarrow \left(\frac{\theta_2}{n\Delta}\right)^{d-1} \leq \left(\frac{1}{\alpha}\right)^{d-1}.$$

We now consider that σ_{uv} must be greater than α, as there will be at least one shortest path for all the vertices of the corresponding layer. And we can also obtain an upper bound from the product of the total of vertices of the i-layers: $\alpha \leq \sigma_{uv} \leq \alpha^{d-1}$ and

$$\frac{1}{\alpha^{d-1}} \leq \frac{1}{\sigma_{uv}} \leq \frac{1}{\alpha}.$$

If w is at a distance h_1 of u, i.e. belongs to the layer $P^{h_1}_{u,v}$, then there will be at least one shortest path from u to v going through w. As $m = n_{h_1}$, the maximum of this number will be reached if for the remaining vertices of the layer $P^{h_1}_{u,v}$ there is only one shortest path, and all the others $\sigma_{uv} - (m-1)$ paths, go through w, so $1 \leq \sigma_{uv}(w) \leq \sigma_{uv} - (m-1)$. Using both inequalities:

$$\frac{1}{\alpha^{d-1}} \leq \frac{\sigma_{uv}(w)}{\sigma_{uv}} \leq \frac{\sigma_{uv} - m + 1}{\sigma_{uv}} = 1 - \frac{m-1}{\sigma_{uv}} \leq 1 - \frac{m-1}{\alpha^{d-1}}.$$

And finally, using the lower bound obtained for the second eigenvalue of the Laplacian

$$\left(\frac{\theta_2}{n\Delta}\right)^{d-1} \leq b_w(u,v) \leq 1 - (m-1)\left(\frac{\theta_2}{n\Delta}\right)^{d-1}.$$

□

Example 1. *Fig. 1 shows a graph where the upper bound is almost attained. The betweenness of vertex* w_1 *is* $b_{w_1}(u,v) = 30/31 = 0.9677$ *and the betweenness of vertex* w_2 *is* $b_{w_2}(u,v) = 1/31 = 0.0322$*. Other parameters are:* $\Delta = 7$*,* $n = 15$*,* $d = 4$*,* $m = 2$ *and*

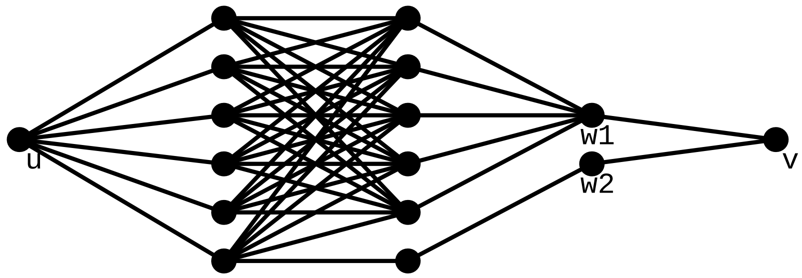

Figure 1: A graph with a vertex, w_1, which has a betweenness approaching the upper bound

$\theta_2 = 0.6301$. *Then the lower bound is* $\left(\frac{0.6301}{15 \cdot 7}\right)^3 = 1.7813 \cdot 10^{-7} < 0.0322 = b_{w_2}(u, v)$ *and the upper bound is* $1 - 1 \cdot \left(\frac{0.6301}{15 \cdot 7}\right)^3 = 0.9999 > 0.9677 = b_{w_1}(u, v)$. *Notice that if in Fig. 1 the bipartite subgraph* $K_{6,5}$ *is replaced by a larger bipartite subgraph the approximation to the upper bound would be better.*

3 Spectral bounds for the edge-betweenness

The main results of this section are a spectral bound for the maximum edge betweenness of a graph and an improved bound for its diameter. First we give several results that relate the edge betweenness with other parameters and invariants of a graph like the edge expansion index and the isoperimetric number.

The following lemma provides some basic properties for the betweenness related parameters. Let us recall that for a graph G, B^e is the betweenness of edge e, B^E is the edge betweenness of the graph, B^E_{max} is the maximum edge betweenness of G and $\overline{B}^E$ is the (average) edge betweenness of G.

Lemma 3. *Let* G *be a connected graph and let* $e \in E$ *be an edge with endvertices* $u, v \in V$, *then*

1. $b^e(u, v) = 1 = b^e(v, u)$.
2. $2 \leq B^e \leq n^2/2$ *if* n *is even and* $2 \leq B^e \leq (n^2 - 1)/2$ *if* n *is odd.*
3. $B^e \leq \max\{B_u + 2, B_v + 2\}$.

4. $B^e = 2(n-1)$ *if one of the endvertices of* e *has degree 1.*

5. $B^E_{max} \leq B_{max} + 2$.

6. $\overline{B}^E \leq \frac{m}{n}(\overline{B} + 2)$.

Proof. The second property is obvious as the minimum clearly is 2 and the maximum would be reached for an edge-bridge leaving half of the vertices of the graph in each of the two components.

The third is also easy to prove as all the shortest paths that go through vertices u or v, should not go through the edge e joining them and $B^e \leq B_u$, but if the equality is reached, we will have to count the path that goes from u to v and the reverse, so in that case $B^e \leq B_u + 2$ or $B^e \leq B_v + 2$.

The last two properties are a consequence of the third.

□

Lemma 4. *Let* G *be a graph of order* n*, then*

- *If* e *is an edge-bridge of the graph* G *connecting* G_1 *with* $G \backslash G_1$ *where* $\mid V(G_1) \mid = n_1$*, then* $B^e = 2n_1(n - n_1)$*.*
- *If* C *is a cut-set of edges of the graph* G*, connecting two sets of vertexs* X *and* $V(G) \backslash X$ *and* $\mid X \mid = n_x$*, then* $\sum_{e \in C} B^e = 2n_x(n - n_x)$*.*

Proof. The first part is clear since an edge-bridge connects at least two connected components of the graph, so all the shortest paths going from a vertex of one component to another vertex of the second component, must pass necessarily by e, and the equality holds. The second part can also be shown by a similar argument. □

3.1 Edge betweenness and other related parameters.

Let us see the connection of the edge betweenness with parameters related with the expansion in graphs like, for instance, the *edge expansion index* introduced in [11] as:

$$\beta = min\{\frac{\mid \partial X \mid}{(|X||\overline{X}|)} : X \subset V, 1 \leq \mid X \mid \leq n-1\}.$$

where X is a proper set of vertices of V, $\overline{X}$ is its complement in V, $\mid \partial X \mid$ is the number of edges connecting X with $\overline{X}$.

The *isoperimetric number* was introduced by Mohar in [8] and is defined as:

$$i(G) = min\{\frac{| \partial X |}{|X|} : X \subset V, 1 \leq | X | \leq \frac{n}{2}\}.$$

Another interesting parameter related with those above is the *edge-forwarding index*, introduced by Heydemann, Meyer and Sotteau in [7] as an extension to edges of the vertex-forwarding index defined by Chung in [1]. Let a route $\mathcal{R}$ be the set of $n(n-1)$ paths connecting the vertices of a graph, and let $R(e)$ be the number of paths of the route $\mathcal{R}$ that go through the edge e, then the edge-forwarding index is defined as

$$\Pi = \min_{\mathcal{R}} \max_{e \in E} R(e).$$

In [11] we can find the relation among these parameters:

$$i(G) \geq \beta \frac{n}{2},$$

$$\Pi \beta \geq 2.$$

Proposition 3. *Let G be a graph of order n, edge expansion index β and isoperimetric number $i(G)$, then*

$$B_{max} + 2 \geq B^E_{max} \geq \frac{2}{\beta},$$

$$B_{max} + 2 \geq B^E_{max} \geq \frac{n}{i(G)}.$$

Proof. Let X_1 denote the set of vertices of G which reach the edge expansion index bound, i.e. $\beta = | \partial X_1 | / (|X_1|(n - |X_1|))$ then $|X_1|(n - |X_1|) = | \partial X_1 | / \beta$.

As X_1 is a cut-set, from Lemma 4, we have $|X_1|(n - |X_1|) = (\sum_{e \in \partial X_1} B^e)/2$, and therefore $| \partial X_1 | / \beta = (\sum_{e \in \partial X_1} B^e)/2 \leq (| \partial X_1 | \, B^E_{max})/2$ and finally $2/\beta \leq B^E_{max}$.

To prove the second inequality we use that β and $i(G)$ are related by $i(G) \geq (n\beta)/2$. Then $1/\beta \geq n/(2i(G))$ and the result follows. □

Therefore for graphs with poor expansion properties, i.e. with small β, B_{max} will be large. The same happens for graphs with a small isoperimetric number (which can be easily disconnected).

Using relations between these parameters obtained in [8] let us see the relation with the second eigenvalue or the Laplacian, θ_2.

Corollary 1. *Let G be a graph of order n, maximum degree Δ and θ_2 the second eigenvalue of the Laplacian matrix, then*

$$\frac{n}{\sqrt{\theta_2(2\Delta - \theta_2)}} \leq B^E_{max} \leq B_{max} + 2.$$

Following the proof of Theorem 2.3 from Solé [11] we can improve the bound given there as Corollary 2.4. This bound improves also a former result from Mohar [8].

Corollary 2. *Let be G a graph of order n, maximum degree Δ and diameter D, then*

$$D \leq 2\left\lceil \frac{\ln(n/2)}{\ln \frac{B^E_{max}\Delta+n}{B^E_{max}\Delta-n}} \right\rceil \leq 2\left\lceil \frac{\ln(n/2)}{\ln \frac{\Pi\Delta+n}{\Pi\Delta-n}} \right\rceil \leq 2\left\lceil \frac{\ln(n/2)}{\ln \frac{\Delta+i(G)}{\Delta-i(G)}} \right\rceil.$$

Solé [11] *Mohar [8]*

We should note that, for a given graph, this new bound for the diameter is easier to obtain as the maximum edge betwenness is less difficult to compute than the edge forwarding index or the isoperimetric number.

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