CLASSIFICATION AND PROPERTIES OF A FAMILY OF AXISYMMETRIC ONE-SOLITON SOLUTIONS.

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ABSTRACT: We study a two-parameter family of stationary axisymmetric solutions of the Einstein equations in vacuum, which were generated from non-physical metrics by the inverse scattering technique. By using the null tetrad formalism, the family is found to contain Bel-Petrov types I, II, D and Minkowski metrics. The Ernst potential of these solutions and the use of prolate spheroidal coordinates suggest new related families of solutions which are asimptotically flat. One of them contains the Zipoy-Voorhees metric with deformation parameter $\delta = 1/2$ as a particular case.

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1. INTRODUCTION

Several generation techniques for finding new solutions of the Einstein equations from known ones when certain symmetries are assumed, have been developed in the past few years. For a review of these techniques and their relations see Cosgrove [1]. One such technique, developed by Belinsky and Zakharov [2] is based on the inverse scattering method (soliton technique) which has been applied to nonlinear equations in other fields of physics.

The soliton technique can be applied to the Einstein equations in vacuum if one assumes the existence of two commuting Killing vectors. It allows one to generate the so called n-soliton solutions with an arbitrarily large number of multipole parameters, depending on how large n is, from given "seed" solutions. Thus, for instance, the Kerr metric is the 2-soliton solution obtained from the Minkowski seed.

The stationary axisymmetric 2n-soliton solutions from the Minkowski metric have been studied by Belinsky and Zakharov [3] and Alekseev and Belinsky [4].

The stationary axisymmetric (2n+1)-soliton solutions can only be obtained from a seed with nonphysical signature. The reason is that the introduction of one soliton produces a signature change in the metric.

In this paper we study the simplest of such solutions, namely, the stationary axisymmetric one-soliton solutions deduced from a family of unphysical metrics. These solutions can be considered as physical seeds for the general (2n+1)-soliton solutions.

Our family of solutions depends on two parameters. One of them, q, is related to the seed metric and is responsible for the strength of the field, thus $q = 0$ gives the Minkowski metric. Here it is interesting to note that the Minkowski metric has been generated by the inverse scattering method from a nonphysical seed. The other parameter, D, is related to the rotation of the field, thus when $D \to \infty$ the metrics become static.

In section 2 we use the null tetrad formalism [5] to classify the family of solutions. Interesting cases are: the member $q = -1/2$ corresponding to the Euclidean seed, which is a solution of Bel-Petrov type I with an "extreme rotation" limit belonging to the Van Stockum class of type II [6]; the member $q = 0$ which is Minkowski space and the member $q = 1$ which is a type D metric in its static limit and can be compared to the Schwarzchild solution in some regions of the space-time [6].

In section 3 we use the Ernst potentials to relate our solutions to other known solutions. Using the symmetry of these potentials in terms of prolate spheroidal coordinates we find some related asymptotically flat solutions. The most interesting of those includes the Zipoy-

Voorhees metric with deformation parameter $\delta = 1/2$ which can be interpreted as the external field of a rod. The procedure used also suggests some possible physical interpretation for the one-soliton metric.

2. CLASSIFICATION OF THE SOLUTIONS

The generation of a stationary axisymmetric solution of Einstein equations in vacuum with the soliton technique requires the use of a seed metric. If we take as seed the nonphysical metric:

$$ds^2 = \rho^{2q^2-\frac{1}{2}} (d\rho^2+dz^2)+\rho^{1-2q} d\phi^2+\rho^{1+2q} dt^2 \qquad (1)$$

which can be reduced to the cosmological Kasner solution by a complex coordinate transformation, the 1-soliton solution is the two-parameter family

$$ds^2 = \frac{C\rho^{2q^2} \mathrm{Ch}(q\psi+D)}{\sqrt{\rho^2+z^2}} (d\rho^2+dz^2)$$

$$+\frac{1}{\mathrm{Ch}(q\psi+D)} [-\rho^{1+2q} \mathrm{Sh}(q\psi+\frac{1}{2}\psi+D)dt^2-\rho^{1-2q} \mathrm{Sh}(-q\psi+\frac{1}{2}\psi-D)d\phi^2$$

$$-2\rho \mathrm{Ch}\,\frac{\psi}{2} d\phi dt] \qquad (2)$$

$$e^{-\psi} = \left(\frac{\rho}{n}\right)^2, \qquad \mu = -z+\sqrt{\rho^2+z^2}$$

where q and D are two arbitrary parameters.

In this section we shall use the null tetrad formalism [5] to study solutions (2) in the Bel-Petrov classification.

First, we shall note that the $q=0$ solution is Minkowski. In fact, by a rotation on the $t\phi$-plane of angle

$$\theta = -\frac{1}{2}\,\mathrm{arcotg}(\mathrm{Sh}\ D)$$

the metric takes the form:

$$ds^2 = \frac{1}{\sqrt{\rho^2+z^2}}\,(d\rho^2+dz^2)+(z+\sqrt{\rho^2+z^2})d\phi^2-(z+\sqrt{\rho^2+z^2})dt^2 \qquad (3)$$

which by the coordinate change,

$$R = [(z^2+\rho^2)^{1/2}+z]^{1/2}$$

$$Z = \mathrm{Ch}t\ [(z^2+\rho^2)^{1/2}-z]^{1/2} \qquad (4)$$

$$T = \mathrm{Sh}t\ [(z^2+\rho^2)^{1/2}-z]^{1/2}$$

can be reduced to

$$ds^2 = dR^2 + dZ^2 + R^2 d\phi^2 - dT^2$$

Therefore the inverse scattering technique generates the Minkowski metric from a nonphysical seed that can be deduced from the "isotropic" $(q=0)$ Kasner metric.

Now, for the general case, we shall use the null tetrad formalism. It is convenient to write the general form of a stationary axisymmetric metric as:

$$ds^2 = 2\, e^{-2F} dx d\bar{x} + \frac{2e^H}{S+T}\left[ST d\phi^2 - (S - T) d\phi dt - dt^2\right] \quad (5)$$

with:

$$x = \rho + iz \quad ; \quad (x^0, x^1, x^2, x^3) = (t, x, \bar{x}, \phi)$$

where F, H, S and T are independent real functions of ϕ and t. Taking the null tetrad as.

$$l_\mu = e^{H/2}(S + T)^{-1/2}(1, 0, 0, S)$$
$$n_\mu = e^{H/2}(S + T)^{-1/2}(1, 0, 0, -T)$$
$$m_\mu = e^{-F}(0, 1, 0, 0)$$

and using the Newman-Penrose equations [5], we find the projections of the Weyl tensor:

$$\Psi_0 = -\frac{e^{2F}}{S+T}(S_{xx} + 2F_x S_x + H_x S_x - \frac{2S_x^2}{S+T})$$
$$\Psi_4^* = -\frac{e^{2F}}{S+T}(T_{xx} + 2F_x T_x + H_x T_x - \frac{2T_x^2}{S+T}) \qquad (6)$$
$$\Psi_2 = -\frac{e^{2F}}{S+T}[H_{x\bar{x}} + 2F_{x\bar{x}} + \frac{1}{(S+T)^2}(S_{\bar{x}}T_x - 5S_x T_{\bar{x}})]$$

From where the Bel-Petrov type can be deduced for the metric (2). In general these solutions are of type I with the exception of flat space (q = 0); Van Stockum (q = -1/2, D = 0) which is type II [6] and q = 1 in its static limit ($D\to\infty$) which is type D. We shall study now the last case in some detail.

The functions (6) are particularly simple in the static limit:

$$\Psi_0 = \Psi_4^* = -\frac{q}{2B}\,\rho^{-2q^2-1}\,\frac{e^{-q\psi}}{Ch\psi/2}\,[-2 + Sh^2\psi/2 - 3qSh\psi/2Ch\psi/2$$

$$+ 3i(Sh\psi/2 - qCh\psi/2)] \qquad (7)$$

$$\Psi_2 = \frac{q}{2B}\,\rho^{-2q^2-1}e^{-q\psi}(Sh\psi/2 - qCh\psi/2)$$

Now, the Russell-Clark algorithm [7] can be easily applied and one finds that all solutions are type I with the flat exception $q = 0$, and $q = 1$ which turns out to be type D. This last metric can be written as:

$$ds^2 = \frac{B}{\sqrt{\rho^2 + z^2}}\,\rho^2 e^{\psi}(d\rho^2 + dz^2) + \frac{e^{-\psi/2}}{\rho}\,d\phi^2 - \rho^3 e^{\psi/2}dt^2 \qquad (8)$$

and it can be studied by considering their curvature scalars. The only independent curvature scalar is:

$$I = 12\Psi_2^2 = \frac{3}{B^2\,\mu^6} \qquad (9)$$

whose meaning can be better understood by using spheroidal coordinates:

$$\rho = \sigma(x^2 - 1)^{1/2}(1 - y^2)^{1/2}$$

$$z = \sigma xy \qquad \sigma = \text{constant} \qquad (10.a)$$

which are related to the Boyer-Lindquist coordinates (r,θ) by:

$$\sigma x = r - m$$

$$y = \cos\theta \qquad (10.b)$$

The curvature scalar (9) in the limit at large r can be written as:

$$I = \frac{3}{B^2\,r^6(1 - \cos\theta)^6} \qquad (11)$$

which one can compare with the Schwarzschild curvature escalar:

$$I_S = \frac{48\ m^2}{r^6}$$

suggesting a physical interpretation for the constant B.

The curvature scalar (11) is singular for $\theta = 0$ thus the solution (8) is not asimptotically flat. It would be interesting to study whether there is an asimptotically flat solution which can be aproximated by the solution (8) in some regions of the spacetime in a similar way as this is possible for the Kinnersley-Kelly [8] metrics.

We note that in general the curvature scalars behave in terms of r like $r^{-(2q^2+1)}$ therefore for large q the field vanishes quickly. The meaning of the parameter q becomes now clear.

3. RELATED ASYMPTOTICALLY FLAT SOLUTIONS.

The study of stationary axialsymmetric fields in terms of the Ernst [9] formulation has been most fruitful in the search for new solutions and for their physical interpretation. Here we will consider the Ernst formulation in order to relate the one-soliton solutions (2) to other known solutions. We will see also that the simplicity of the relevant Ernst potentials suggests new related families of asymptotically flat solutions. The Ernst potential for the two-parameter family (2) can be calculated to be:

$$\varepsilon = \frac{\rho^{1+2q}}{Ch(q\psi+D)} \left(Sh[(q+\tfrac{1}{2})\psi + D] - \frac{i}{\rho}\sqrt{\rho^2+z^2}\right) \qquad (12)$$

The "static" limit ($D \to \infty$) of (12) reduces to the following trivial family of Weyl (non flat) solutions:

$$\varepsilon = \mu\rho^{2q} \qquad (13)$$

which are obtained by combining the flat solutions $\varepsilon = \mu$ and $\varepsilon = \rho$ [7] .

The one-soliton solution (2) can thus be interpreted as the stationary generalization of the particular static Weyl family (13).

For $q = -1/2$ the Ernst potential is particularly interesting since in terms of prolate spheroidal coordinates (10), and taking the limit for x large, it reduces to.

$$\varepsilon = \frac{(1-y^2)^{1/2} + i\eta\gamma^{-1}y}{\gamma + y} \qquad (14)$$

with $\gamma = \mathrm{Coth}\, D$ and $\eta = (\gamma^2 - 1)^{1/2}$

This potential, which we have obtained as a limit of (12), is also a solution of the Ernst equation. This can be seen easily by defining another Ernst potential ξ

$$\xi = \frac{1-\varepsilon}{1+\varepsilon}$$

and checking that the Ernst equation [7],

$$(\xi\xi^*-1)\{[(x^2-1)\xi,_x],_x+[(1-y^2)\xi,_y],_y\} =$$
$$= 2\xi^*[(x^2-1)\xi,_x^2 + (1-y^2)\xi,_y^2] \qquad (15)$$

is verified by it.

The symmetry of equation (15) in the coordinates x and y has been exploited by Tomimatsu and Sato [10] to find new solutions. From any given solution $\xi(x,y)$ of (15) one can construct a new solution by commuting x and y. Therefore from (14) we can construct the new solution:

$$\varepsilon = i\,\frac{\sqrt{x^2-1}\;+\eta\gamma^{-1}x}{\gamma + x} \qquad (16)$$

and also, as it is easy to check, the static solution:

$$\varepsilon = \frac{\sqrt{x^2-1}\;-\;\eta\gamma^{-1}x}{\gamma + x} \qquad (17)$$

Both solutions are asymptotically flat. The last one (17) contains the Zipoy-Voorhees metric

$$\varepsilon = \left(\frac{x - 1}{x + 1}\right)^{\delta} \qquad (18)$$

with deformation parameter $\delta = 1/2$, when $\gamma = 1$, and a physical interpretation for it can be given.

In fact, Voorhees [11] gave a plausible physical interpretation for the family (18) by comparing the family with the member $\delta = 1$ which is the Schwarzschild metric. For the Schwarzschild solution the coordinates (r,θ) with $\sigma = m$ can be seen as spherical coordinates and (18) gives the field of a point particle of mass m. In general the coordinates adapted to the source have $\sigma = m/\delta$ and by expanding ε in terms of (r,θ) for large r, and comparing with the (spherical) Schwarzschild coordinates ($\sigma = m$) the metrics (18) can be interpreted as the external fields of rods (if $\delta<1$)

of mass m. For $\delta = 1/2$ the rod has length 4m.

For large x and finite γ (17) can be expanded as

$$\varepsilon = 1-\varepsilon\gamma^{-1} - (1-\varepsilon\gamma^{-1})\gamma\frac{1}{x} + [(1-\varepsilon\gamma^{-1})\gamma^2 - \frac{1}{2}]\frac{1}{x^2} + \ldots$$

this asymptotic form can de reduced to the usual

$$\varepsilon = 1-\frac{2m}{r} + (\text{Polynomial in } \cos\theta)\frac{1}{r^2} + \ldots$$

by means of an Ehlers transformation (Cosgrove,[1]) which will involve the parameter γ. Therefore the solution (17) can be seen, at least asymptotically, as an Ehlers transformation, depending on γ, of the Zipoy-Voorheés metric with deformation parameter $\delta = 1/2$.

The relation between the asymptotically flat solutions and the non asymptotically flat solution (14) is similar to that between the Zipoy-Voorhees metrics and the Kinnersley and Kelly [8] "extreme Kerr" solutions representing a region of the Tomimatsu and Sato [10] metrics near to its ergosphere. This seems to suggest that a similar interpretation might be found for the one-soliton solutions with $q = -1/2$ as describing some limited region of the external field of rods, at least in the $\gamma = 1$ limit.

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