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ON LARGE VERTEX SYMMETRIC 2-REACHABLE DIGRAPHS *

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ABSTRACT

We are interested in the construction of the largest possible vertex symmetric digraphs with the property that between any two vertices there is a walk of length two (that is, they are 2-reachable). Other than a theoretical interest, these digraphs and their generalizations may be used as models of interconnection networks for implementing parallelism. In these systems many nodes are connected with relatively few links and short paths between them and each node may execute, without modifications, the same communication software. On the other hand, a message sent from any vertex reaches all vertices, including the sender, in exactly two steps.

In this work we present families of vertex symmetric 2-reachable digraphs with order attaining the upper theoretical bound for any odd degree. Some constructions for even degree are also given

Keywords: vertex symmetric digraphs, Cayley digraphs, k-reachable digraphs

1 Introduction

The design of topologies for highly symmetric interconnection networks, particularly for massive parallel computers leads to a question of special interest in graph theory: the construction of vertex symmetric digraphs with order as large as possible for a given maximum degree and diameter.

The search for large vertex symmetric digraphs has received special consideration in recent years. The interest comes from the fact that in the associated network each node is able to execute the same communication software without modifications. In this way these digraphs may be considered to obtain an easy implementation of parallelism.

This problem have been approached in some different ways. It is possible to give upper bounds for the order of the digraphs for a given maximum degree and

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diameter, but these theoretical bounds are very difficult to attain. As a consequence, most of the work deals with the construction of digraphs which, for given diameter and maximum degree, have a number of vertices as close as possible to the bounds. Most large vertex symmetric digraphs correspond to Cayley digraphs and have been found either by random computer search, special graph products or direct constructions.

Also most large vertex symmetric digraphs may be described as digraphs on alphabets. A digraph on an alphabet is constructed as follows: the vertices are labeled with words on the given alphabet and the arcs are defined according to a rule that relates two different words. This representation of the digraph usually facilitates the direct determination of the diameter. An interesting family of vertex symmetric digraphs, that was given by Faber and Moore [1] as Cayley coset digraphs may be seen as well as a family of digraphs on alphabets [2,3].

In this paper we are interested in the largest possible vertex symmetric digraphs with the property that between any two vertices there is a walk of length two (that is, they are 2-reachable). These digraphs may model networks with the additional advantage, over simple vertex symmetric digraphs, that a message sent from any vertex reaches all vertices in exactly two steps. Besides, they can be used in some constructions for large vertex symmetric digraphs [2,4].

Section 2 is devoted to notation and some previous results concerning vertex symmetric digraphs and k-reachable digraphs. In Section 3 we give a family of 2-reachable vertex symmetric digraphs that have the maximum possible order for odd degree. Finally, in Section 4, we present other constructions for even degree.

2 Notation and Preliminary Results

Let $G = (V, A)$ be a digraph with vertex set V and arc set A. The *out-degree* of a vertex $\delta^+(x)$ is the number of vertices adjacent from x, the *in-degree* of a vertex $\delta^-(x)$ is the number of vertices adjacent to x. A digraph is regular of degree Δ or Δ-*regular* if the in-degree and out-degree of all vertices equal Δ. A (Δ, D) *digraph* is a digraph with maximum degree Δ and diameter at most D. A digraph is *vertex symmetric* if its automorphism group acts transitively on its set of vertices. Vertex symmetric digraphs may be easily constructed from groups. A Cayley digraph $Cay(H, S)$ is the digraph generated from the finite group H with generating set S. The vertex set is H and there is an arc from g to gs for every $g \in H$ and $s \in S$. Dinneen [5] used a computer search to find large vertex symmetric (Δ, D) digraphs based on linear groups and semi-direct products of cyclic groups. A digraph is k-*reachable* if for every pair of (not necessarily different) vertices $x, y \in V$ there exists a path of exactly k arcs from x to y. See [6,7] for more details about k-reachable digraphs.

For $D \geq 3$ there exists a family of large vertex symmetric digraphs that are D-reachable. Faber and Moore [1] denoted them by $\Gamma_\Delta(D)$. These digraphs may be defined as digraphs on alphabets in the following way: The vertices are labeled with different words of length D, $x_1 x_2 \cdots x_D$, such that they form a D-permutation of an alphabet of $\Delta+1$ letters. The adjacencies are given by

$$x_1x_2\cdots x_D \quad\rightarrow\quad \begin{cases} x_2x_3x_4\cdots x_Dx_{D+1}, & x_{D+1}\neq x_1,x_2,\ldots,x_D \\ x_2x_3x_4\cdots x_Dx_1 \\ x_1x_3x_4\cdots x_Dx_2 \\ x_1x_2x_4\cdots x_Dx_3 \\ \cdots \\ x_1x_2x_3\cdots x_Dx_{D-1} \end{cases}$$

These digraphs have order $\frac{(\Delta+1)!}{(\Delta-D+1)!}$, diameter D and are Δ-regular ($\Delta \geq D$). The proof that for $D\geq 3$, the digraphs are $D-$reachable may be found in [2]. On the other hand, the digraphs $\Gamma_\Delta(2)$ are in fact the Kautz digraphs $K(\Delta,2)$ defined next.

The *Kautz digraph* $K(\Delta,D)$, $\Delta\geq 2$, has vertices labeled with words $x_1x_2\cdots x_D$ where x_i belongs to an alphabet of $\Delta+1$ letters and $x_i\neq x_{i+1}$ for $1\leq i\leq D-1$. A vertex $x_1x_2\cdots x_D$ is adjacent to the Δ vertices $x_2\cdots x_Dx_{D+1}$, where x_{D+1} can be any letter different from x_D. Hence, the digraph $K(\Delta,D)$ is Δ-regular, has $\Delta^D+\Delta^{D-1}$ vertices and diameter D. For $D=2$ the Kautz digraphs are vertex symmetric. The Kautz digraphs of diameter D are $(D+1)$-reachable. Figure 1 shows $K(2,2)$. In this figure a line represents two opposite arcs.

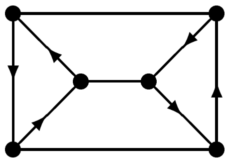

Fig. 1: *K(2,2), a 2-regular vertex symmetric 3-reachable digraph with diameter 2.*

For $D=2$ we may obtain a 2-reachable digraph from a $K(\Delta,2)$ by adding a loop to each vertex. Hence the new digraph has degree $\Delta+1$ and order $\Delta^2+\Delta=(\Delta+1)^2-(\Delta+1)$.

The problem considered in this article consists of finding large vertex symmetric 2-reachable digraphs.

The maximum order of a k-reachable $\Delta-$regular digraph is Δ^k, since this is the maximum number of different walks of length k. This bound, however, is not attained for vertex symmetric digraphs when $k>1$ and $\Delta>1$. Indeed, in order to attain it there should be just one walk of length k between any two vertices. It follows that $A^k=J$, where A is the adjacency matrix of the digraph and J is the matrix with all entries equal to 1. Besides, as the digraph is $\Delta-$regular the eigenvalues of A are 0 and Δ (Δ with multiplicity one). Thus the trace of A is Δ and the digraph has exactly Δ loops, which is incompatible with symmetry.

A similar study with the matrix equation $A^2=J+I$ shows that there is no incompatibility with a digraph being vertex symmetric and having maximum order Δ^2-1.

3 Largest Vertex Symmetric 2-Reachable Digraphs with Odd Degree

In this section we present, for any odd degree, a family of vertex symmetric 2-reachable digraphs with the maximum possible order.

Let γ_Δ be a digraph on alphabet defined as follows: For a given odd integer $\Delta \geq 3$ we consider the $\Delta + 1$ symbols $0_i, 1_i$, $i = 0, 1, \ldots, \frac{\Delta-1}{2}$. Vertices are labeled with different words, $x_i y_j$ with $x, y \in \mathbf{Z}_2$ and $i \neq j$.

The adjacencies are given by

$$x_i y_j \rightarrow \begin{cases} y_j z_k, & k \neq j \\ x_i (y+1)_j \end{cases}$$

Where the arithmetic is modulo 2. Then the digraph γ_Δ is Δ-regular and has order $(\Delta+1)(\Delta-1) = \Delta^2 - 1$, that is the maximum for a vertex symmetric 2-reachable digraph. Figure 2 shows the case $\Delta = 3$.

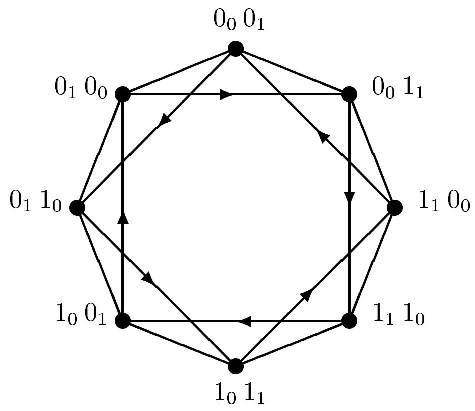

Fig. 2: A 2-reachable vertex symmetric digraph on 8 vertices.

Theorem 1 *The digraph γ_Δ is vertex symmetric and 2-reachable.*

Proof.

- The digraph is vertex symmetric: Let ψ_1 and ψ_2 be mappings on the vertex set of γ_Δ defined by $\psi_1(x_i y_j) = x_{\sigma(i)} y_{\sigma(j)}$, where σ is any given permutation of the elements of $\mathbf{Z}_m$, and $\psi_2(x_i y_j) = (x+i)_i (y+j)_j$ (arithmetic modulo 2). These mappings are automorphism of γ_Δ. Indeed,

$$\begin{aligned} \psi_1(x_i y_j) = x_{\sigma(i)} y_{\sigma(j)} &\rightarrow y_{\sigma(j)} z_{\sigma(k)} = \psi_1(y_j z_k); \\ \psi_1(x_i y_j) = x_{\sigma(i)} y_{\sigma(j)} &\rightarrow x_{\sigma(i)} (y+1)_{\sigma(j)} = \psi_1(x_i (y+1)_j); \end{aligned}$$

and

$$\begin{aligned} \psi_2(x_i y_j) = (x+i)_i (y+j)_j &\rightarrow (y+j)_j (z+k)_k = \psi_2(y_j z_k); \\ \psi_2(x_i y_j) = (x+i)_i (y+j)_j &\rightarrow (x+i)_i (y+j+1)_j = \psi_2(x_i (y+1)_j). \end{aligned}$$

Clearly, the group of automorphisms generated by ψ_1 and ψ_2 acts transitively on the set of vertices of γ_Δ.

- The digraph is 2-reachable: Let us see, that from a generic vertex x_iy_j, there is a walk of length 2 to any other vertex w_kz_l. We basically consider two cases.

  (i) $k \neq j$. The walk is $x_iy_j \to y_jw_k \to w_kz_l$;
  (ii) $k = j$. In this case there are two possibilities:
  $x_iy_k \to x_i(y+1)_k \to (y+1)_kz_l, \quad \text{if } w = y+1;$
  $x_iy_k \to y_k(z-1)_l \to y_kz_l, \quad \text{if } w = y.$ □

□

4 Other Constructions

It is not possible to give a general family attaining the theoretical upper bound for the order when the degree is even. Indeed, although for the trivial case $\Delta = 2$ the complete symmetric digraph K_3^* is 2-reachable, it can be proved (by an exhaustive study) that there is no vertex symmetric 2-reachable 4-regular digraph on 15 vertices. In this context it is worth to consider the following generalization of the digraph γ_Δ.

Let $n, m \geq 2$, n even, be integers. Let P be any given partition of $\mathbf{Z}_n$ in $n/2$ subsets of two elements (considered as ordered pairs). The digraph Υ_Δ has vertices labeled with different words, x_iy_j with $x, y \in \mathbf{Z}_n$ and $i, j \in \mathbf{Z}_m$, $i \neq j$.

The adjacencies are given by

$$x_iy_j \quad \to \quad \begin{cases} y_jz_k, & k \neq j \\ (x+\alpha)_i(y+\beta)_j, & (\alpha, \beta) \in P. \end{cases}$$

Then the digraph is Δ-regular with $\Delta = n(m-1) + \frac{n}{2}$ and it has order $N = n^2m(m-1) = \Delta^2 - \frac{n^2}{4}$.

Theorem 2 *The digraph Υ_Δ is vertex symmetric and 2-reachable.*

Proof.

- $\Upsilon_\Delta$is vertex symmetric: Consider mappings ψ_1 and ψ_2 as in the proof of Theorem 1.
- Υ_Δ is 2-reachable: Let us see that from a generic vertex x_iy_j there is a walk of length two to any other vertex w_kz_l. The case $k \neq j$ is as in Theorem 1 while for $k = j$ we consider now the following walks:

$$\begin{aligned} x_iy_k &\quad \to \quad (x+\alpha)_i(y+\beta)_k \to (y+\beta)_kz_l, \quad \text{if } w = y+\beta; \\ x_iy_k &\quad \to \quad y_k(z-\beta)_l \to (y+\alpha)_kz_l, \quad \text{if } w = y+\alpha; \end{aligned}$$

For each $(\alpha, \beta) \in P$. The condition on such pairs assure that any value of w is obtained.□

□

Notice that the case $n = 2$ corresponds to the digraph γ_Δ. For $n = 4$ we obtain a family of vertex symmetric 2-reachable digraphs with degree $\Delta = 4m - 2$ and order $\Delta^2 - 4$. Although these digraphs do not attain the upper bound, they are assimptotically optimal as Δ increases.

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