

On large (Δ, D) -graphs*

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Abstract

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This paper deals with some new constructions of *large (Δ, D) -graphs*, i.e. graphs with (maximum) degree Δ , diameter D and number of vertices as large as possible. The constructions presented here are based on the compound graph technique and on expansions of Moore bipartite graphs, and they give rise to some of the largest known graphs.

1. Introduction

Let $G=(V, E)$ be a graph with set of vertices V and set of edges E . The adjacency between two vertices $x, y \in V$ will be represented by $x \sim y$ (x and y are then *endvertices* of the edge xy). The set of vertices adjacent to $x \in V$ is denoted by $\Gamma(x)$, the *degree* of x is $\delta(x) = |\Gamma(x)|$ and the (*maximum*) *degree* of G is $\Delta = \max_{x \in V} \delta(x)$. The graph G is *bipartite* when $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and any edge $xy \in E$ has one endvertex in V_1 and the other in V_2 . The *distance* between two vertices $x, y \in V$ is the length $d(x, y)$ of a shortest path between x and y . The *diameter* D of the graph is defined as $D = \max_{x, y \in V} d(x, y)$.

It is known that the *order*, i.e. the number of vertices, N of a graph with degree Δ (> 2) and diameter D must satisfy

$$N \leq 1 + \Delta + \Delta(\Delta - 1) + \dots + \Delta(\Delta - 1)^{D-1} = \frac{\Delta(\Delta - 1)^D - 2}{\Delta - 2},$$

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where the right-hand side is called the *Moore bound*. For $D \geq 2$ and $\Delta \geq 3$, this bound is attained only when $D=2$ and $\Delta=3, 7$, and (perhaps) 57; see, for instance, [5, Chapter 23]. In this context, the so-called (Δ, D) -problem consists in finding the maximum order $N(\Delta, D)$ of a graph with given degree Δ and diameter D . In this paper, we propose new constructions of large graphs which give rise to some largest known graphs, so that new lower bounds for $N(\Delta, D)$ are obtained (see Table 1).

Some of the constructions we propose are based on large bipartite graphs. It is known that, in the case of bipartite graphs, we have the inequality

$$N \leq 2 \frac{(\Delta-1)^D - 1}{\Delta-2} \quad (\Delta > 2).$$

The bipartite graphs attaining such a bound are called *bipartite Moore graphs*. They can exist only for $D=2$ (complete bipartite graphs $K_{\Delta, \Delta}$) or $D=3, 4, 6$; see [5, Chapter 23]. For these values of D , bipartite Moore graphs actually exist when $\Delta-1$ is a prime power, and they are based on incidence graphs of projective geometries; see [2, 5]. For $D=4$, they are called *generalized quadrangles*, Q_q , which have order $N=2(q^4-1)/(q-1)$ and degree $\Delta=q+1$. For $D=6$, they are called *generalized hexagons*, H_q , with parameters $N=2(q^6-1)/(q-1)$ and $\Delta=q+1$. In both Q_q and H_q , there exists a subset of q^2+1 vertices in the same stable set of the graph which are pairwise at distance exactly D ; see [8]. Moreover, if G is a (Δ, D) bipartite Moore graph and $d(x, y)=D$, $x, y \in V(G)$, there exist Δ disjoint paths between x and y of length exactly D . Finally, if $G=(V_1 \cup V_2, E)$ is any bipartite graph of even (odd) diameter D , the distance between $x \in V_1$ and any $y \in V_2$ ($y \in V_1$) is at most $D-1$. These results will be used in the next sections.

2. Compound graphs

The technique of compounding graphs to obtain large (Δ, D) -graphs was introduced by Bermond et al. [3] with the following construction. Given graphs $G_i=(V_i, E_i)$, on $N_i=|V_i|$ vertices and diameter D_i , $i=1, 2$, their *compound graph* $G_1[G_2]$, on $N_1 N_2$ vertices, is obtained by replacing each vertex $x \in V_2$ by a copy of G_1 , represented by G_1^x , and each edge $xy \in E_2$ by *one* edge that joins a vertex of G_1^x with a vertex of G_1^y . Clearly, the diameter D of $G_1[G_2]$ depends upon the way the copies of G_1 are joined, but in any case it satisfies the inequality

$$D \leq (D_2 + 1)D_1 + D_2;$$

see for instance [3]. This construction has been generalized by setting two or more edges, according to a given structure, between copies of G_1 , and by replacing each vertex of G_2 by more than one copy of G_1 ; see [9, 11, 13]. On the other hand, some of the best results have been obtained by taking G_1 and/or G_2 a bipartite graph, and considering those compound graphs in which any vertex belongs to one or more ‘intercopy’ edges; see [3, 10, 13]. In particular, when G_2 is bipartite, it seems natural to

Table 1

Largest known (A, D) graphs (January 1989) (the new data obtained in this paper are represented in boldface)

A	D	2	3	4	5	6	7	8	9	10
3	P	$C_5 * F_4$	YFA	YFA	H_2t	CR^*	CR^*	2cy	2cy	
	10	20	38	70	130	184	320	540	938	
4	$K_3 * C_5$	$P * F_4$	$C_5 * C_{19}$	H'_3	H_3s	CCD	CCD	CCD	CCD	
	15	40	95	364	734	1081	2943	7439	15 657	
5	$K_3 * X_8$	$Lente$	Q_4s	H'_3d	H_4s	8a	2cy	2cy	8a	
	24	70	182	532	2742	4368	11 200	33 600	123 120	
6	$K_4 * X_8$	$C_5 * C_{21}$	8a	8a	H_5s	8a	8a	8a	8a	
	32	105	355	1081	7832	13 104	50 616	202 464	682 080	
7	HS	$Allwr$	$(15) * m(32)$	8a	H_6s	8a	2cy	8a	2cy	
	50	128	480	2162	10 554	39 732	140 000	911 088	2 002 000	
8	P'_7	8a	Q_7s	8a	H_7s	8a	8a	8a	$Q_4 \Sigma_6 H_5$	
	57	203	842	2880	39 258	89 820	455 544	1 822 176	3 984 120	
9	P'_8d	Q'_8	$Q_5(T_4)$	8a	H_8s	$H_7 \wedge K_1$	2cy	$HS \wedge_p H_7$	$Q_7 \nabla_1 H_7$	
	74	585	1248	6072	74 954	215 688	910 000	3 019 632	15 686 400	
10	P'_9	Q'_8d	Q_9s	8a	H_9s	$H_8 \wedge K_1$	2cy	$HS \wedge_p H_8$	$Q_7 \Sigma_2 H_7$	
	91	650	1820	12 144	132 932	486 837	2 002 000	7 714 494	47 059 200	
11	P'_9d	Q'_8d	$Q_7(T_4)$	$K_1 \Sigma_8 Q_8$	$H_7(T_4)$	$K_{1,1} \Sigma'_7 H_8$	$K_{6,6} \Sigma_6 H_8$	$P_7 \Sigma_7 H_8$	$Q_7 \Sigma_6 H_8$	
	94	715	3200	14 625	156 864	898 776	4 044 492	21 345 930	179 755 200	
12	P'_{11}	Q'_8d	$Q'_8 * X_8$	8a	$H_{11}s$	$K_{1,1} \Sigma'_7 H_9$	$K_{7,7} \Sigma_6 H_9$	$P_8 \Sigma_7 H_9$	$Q_8 \Sigma_6 H_9$	
	133	780	4680	24 360	354 422	1 727 180	8 370 180	48 493 900	466 338 600	
13	$P'_{11}d$	Q'_8d	$Q_0(T_4)$	$Q'_8 * m(P'_7)$	$H_9(T_4)$	$H_{11} \wedge K_1$	$K_{1,3} \Pi' H_{11}$	$P_9 \Sigma_1 H_9$	$Q_9 \Sigma'_6 H_9$	
	136	845	6560	33 345	531 440	2 657 340	10 629 360	72 541 560	762 616 400	
14	P'_{13}	Q'_8d	$Q_0(T_5)$	$K_1 \Sigma_8 Q_{11}$	$H_{13}s$	$K_1 \Sigma_8 H_{11}$	$K_{7,7} \Sigma_6 H_{11}$	$P_9 \Sigma_7 H_{11}$	$Q_8 \Sigma_6 H_{11}$	
	183	910	8200	51 240	804 624	6 200 460	29 762 208	164 755 080	1 865 452 680	
15	$P'_{13}d$	$(\otimes Q_{2,4})'$	$Q_{11}(T_4)$	$K_1 \Sigma'_8 Q_{11}$	$H_{11}(T_4)$	$K_1 \Sigma'_8 H_{11}$	$K_{8,8} \Sigma_6 H_{11}d$	$P_{11} \Sigma_1 H_{11}$	$Q_{11} \Sigma'_6 H_{11}$	
	186	1215	11 712	58 560	1 417 248	7 086 240	35 947 392	282 740 976	3 630 989 376	
16	$P'_{13}d$	$(\otimes Q_3)'$	$Q_{11}(T_5)$	$(\otimes H_3)'$	$H_{11}(T_5)$	$K_1 \Sigma_8 H_{13}$	$K_{9,9} \Sigma_6 H_{13}$	$P_9 \Sigma_7 H_{11}$	$Q_{11}d \Sigma_6 H_{13}$	
	197	1600	14 640	132 496	1 771 560	14 882 658	86 882 544	585 652 704	7 394 669 856	

On large (A, D) -graphs

consider copies of two different graphs to replace vertices in different stable sets. All the above ideas are used in the new constructions proposed in this section. Since all of them share the same basic pattern, we describe only one of them in detail.

2.1. The graphs $B_0 \Sigma_6 B_1$

Let $B_i = (V_{i0} \cup V_{i1}, E_i)$, $i=0, 1$, be two bipartite graphs on $N_i = 2|V_{i0}| = 2|V_{i1}| = 2n_i$ vertices, degree Δ_i and even diameter D_i . The graph $B_0 \Sigma_6 B_1$ is obtained by replacing each vertex x of the complete bipartite graph $K_{p,q} = G_2 = (V \cup U, E)$, $p = |V| = n_1$ and $q = |U| = n_0$, by three copies of B_0 or B_1 , depending on $x \in U$ or V , respectively, and each edge of E by the structure depicted in Fig. 1. The graph $B_0 \Sigma_6 B_1$ so constructed has degree $\Delta = \max\{\Delta_0 + 3, \Delta_1 + 2\}$, order $N = 3n_1 N_0 + 3n_0 N_1 = 3N_0 N_1$ and, as shown later, diameter $D \leq D_0 + D_1$. More precisely, assume that the vertices of V_{ij} are identified with the integers modulo n_i . Then each vertex in $B_0 \Sigma_6 B_1$ is identified with a vector (s, i, j, k, t) , where $s \in \mathbb{Z}_{n_i}$, $j \in \mathbb{Z}_2$, $k \in \mathbb{Z}_3$ and $t \in \mathbb{Z}_{n_{i+1}}$, which represents the vertex s of the set V_{ij} belonging to the copy t in the set k . The adjacency rules are defined as shown in Fig. 1.

In order to show that its diameter satisfies the inequality $D \leq D_0 + D_1$, we can take advantage of the symmetry of the interconnection structure between copies. Then it suffices to compute the distances between any vertex and the vertices of the form $(s, 0, 0, 0, t)$, $(s, 0, 1, 0, t)$, $(t, 1, 0, 0, s)$ and $(t, 1, 1, 0, s)$. For instance, let us see paths with endvertex $u = (s, 0, 0, 0, t)$, the other cases being similar:

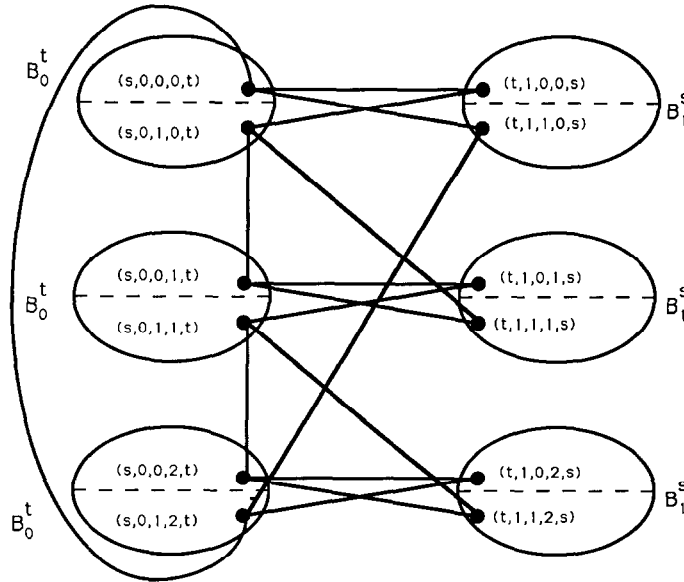


Fig. 1.

$$\begin{aligned}
& \stackrel{(\leq D_0)}{\mathbf{u}}, \dots, \stackrel{(1)}{(x, 0, 0, 0, t)}, \stackrel{(\leq D_1-1)}{(t, 1, j, 0, x)}, \dots, (y, 1, j+1, 0, x) \\
& \stackrel{(\leq D_0-1)}{, \dots, (x, 0, 1, 0, t)}, \stackrel{(1)}{(t, 1, 1, 1, x)}, \stackrel{(\leq D_1-1)}{\dots, (y, 1, 0, 1, x)}, \stackrel{(1)}{(x, 0, j, 1, y)} \\
& \stackrel{(\leq D_1)}{, \dots, (y, 1, 1, 1, x)} \\
& \stackrel{(\leq D_1-1)}{, (t, 1, 0, 0, x)}, \dots, \stackrel{(1)}{(y, 1, 1, 0, x)}, \stackrel{(1)}{(x, 0, 0, 0, y)} \\
& \stackrel{(1)}{, (x, 0, 1, 2, y)} \\
& \stackrel{(1)}{, (s, 0, 1, 2, t)}, \stackrel{(\leq D_0-1)}{\dots, (x, 0, 0, 2, t)}, \stackrel{(1)}{(t, 1, j, 2, x)}, \stackrel{(\leq D_1-1)}{\dots, (y, 1, j+1, 2, x)} \\
& \stackrel{(1)}{, (t, 1, 0, 0, s)}, \stackrel{(\leq D_1-1)}{\dots, (y, 1, 1, 0, s)}, \stackrel{(1)}{(s, 0, 0, 0, y)}, \stackrel{(\leq D_0-1)}{\dots, (x, 0, 1, 0, y)} \\
& \stackrel{(\leq D_0-1)}{, (s, 0, 1, 2, y)}, \dots, (x, 0, 0, 2, y).
\end{aligned}$$

The interconnection structure depicted in Fig. 1 is called the *basic configuration*. It can also be used as the basic pattern to construct large graphs, which we still denote by $B_0 \Sigma_6 B_1$, when Δ_0 or Δ_1 is sufficiently smaller than Δ . For instance, if this basic configuration is represented as shown in Fig. 2(a) and $\Delta_0 = \Delta - 2\alpha - 1$, $\alpha \in \mathbb{N}$, we actually can take 3α , instead of 3, copies of B_1 to replace each vertex of U in $K_{p,q}$. This is equivalent to using the ‘new’ basic configuration shown in Fig. 2(b), which we call the *induced configuration* of the one in Fig. 2(a). Analogous situations are described in detail in [13].

A slight modification of the above construction, denoted by $B_0 \Sigma'_6 B_1$, is obtained from the basic configuration of Fig. 1 by adding a copy of B_0 and the edges shown in Fig. 3.

This implies that we take $N_1/2$ additional copies of B_0 in $B_0 \Sigma_6 B_1$, so that the graph has now $N = (7/2)N_0 N_1$ vertices. Moreover, it has degree $\Delta = \max\{\Delta_0 + 3, \Delta_1 + 3\}$ and diameter $D \leq D_0 + D_1$.

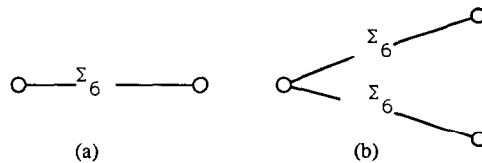


Fig. 2.

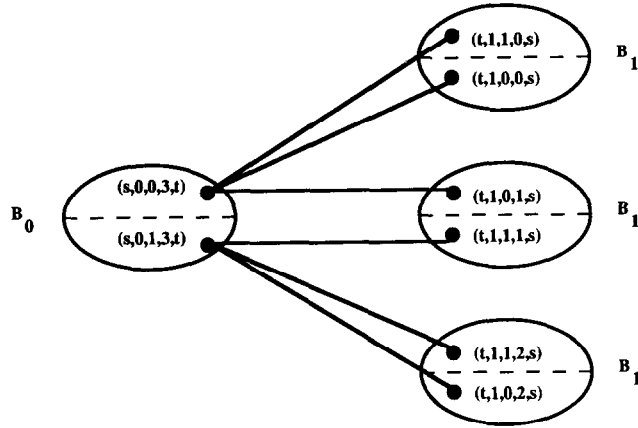


Fig. 3.

Similar constructions, based in different basic configurations, are described in the next paragraphs. Unless stated otherwise, we will use the same notation as before.

2.2. The graphs $B_0 \Sigma_7 B_1$

Let now B_0 and B_1 be two bipartite graphs with diameters D_0 odd and D_1 even. The graph $B_0 \Sigma_7 B_1$, built up from the basic configuration shown in Fig. 4, has order $N = (5/2)N_0N_1$, degree $\Delta = \max\{\Delta_0 + 3, \Delta_1 + 2\}$ and diameter $D \leq D_0 + D_1$.

When B_0 is the complete bipartite graph $K_{1,1}$ ($\Delta_0 = D_0 = 1$), and B_1 has degree Δ_1 such that $\Delta_1 + 2 = 1 + 3\alpha + \beta$, $0 \leq \beta \leq 2$, Δ_0 can be increased up to $1 + 3\alpha$ (as illustrated in Fig. 5), obtaining the graph $K_{1,1} \Sigma_7 B_1$, with parameters $N = (2 + 3\alpha)N_1$,

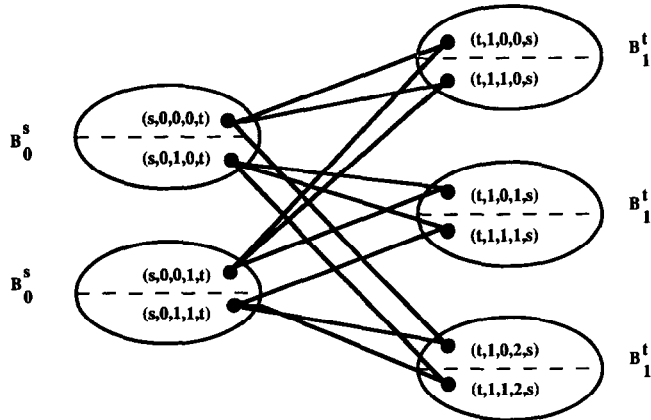


Fig. 4.

In situation (b) we construct another sequence of sets, $A_{10} = A_1, A_{11}, \dots$, for which $A_{1i} = A_{1(i-1)}/d_i = \{a \mid ad_i \in A_{1(i-1)}, a \in \mathbb{Z}\}$, where $d_i > 1$ is the integer for which $|A_{1i}| > |A_{1(i-1)}| - (l_i/\log l_i)^{1/2}$, where $l_i = \max\{a \mid a \in A_{1i}\}$. Clearly, $l_i \leq l_{i-1}/d_i$ and $|A_{1i}| > |A_1| - \sum_{j=0}^{i-1} (l_j/\log l_j)^{1/2} > |A_1| - 4(l/\log l)^{1/2} \gg m$. So, for some $p \ll \log l$, we have the set A_{1p} , elements of which are 'well distributed' among the residues of all integers. This fact confirms all assumptions of Section 3 for the set A_{1p} .

The above-mentioned reasoning allows us to obtain the set A_{1p} and a long arithmetic progression with difference $d = d_1 \cdots d_p$, the length of which can be increased by the other elements of A in the same way as was done in Section 2. Clearly, $d \leq 3l/m$ because the number of integers divisible by d is bounded by $\lfloor l/d \rfloor$ and we have $|A_{1p}| > \frac{1}{3}m$ such integers.

5. Applications and concluding remarks

We have proved that the set of subset-sums A^* contains an arithmetic progression under a rather wide range of conditions. This fact enables us to solve different problems connected with the structure of A^* .

For example, let $p(n, r)$, for $r \geq 2$, denote the maximal cardinality of a subset $A \subset [1, 2, \dots, n]$ such that there are no r -powers of an integer in A^* . In [1, 11], it was proved that, for $r \geq 10$ [11] and for $r \geq 6$ [1],

$$p(n, r) = (1 + o(1))2^{1/r+1}n^{(r-1)/(r+1)}. \quad (45)$$

From Theorem 1.1 we obtain that (45) is true for $r \geq 4$.

In [8, 9] it was shown how the fact of the existence of one arithmetic progression leads to a full description of the structure of A^* and gives a possibility of obtaining algorithms for solving dense SSP ($m \gg l^{2/3} \log^{1/3} l$). It is clear that the new result proving the existence of an arithmetic progression for $m \gg l^{1/2} \log^{1/2} l$ gives the main tool to solve SSP for a wider range of problems. A description of the algorithm, time bound and computational experience will be given in forthcoming papers.

The exponent $\frac{1}{2}$ in condition (2) is the best one to obtain a long arithmetic progression. If we take $m < l^\alpha$, $\alpha < \frac{1}{2}$, then there will be cases where a long arithmetic progression in A^* does not exist. Example: For $0 < \alpha < \frac{1}{2}$, let

$$A \{x \mid x = a + b \lfloor l^{1-\alpha/2} \rfloor, 0 \leq a, b < \lfloor l^{\alpha/2} \rfloor, a^2 + b^2 \neq 0\}.$$

For this set the set A^* has a 'two-dimensional' structure.

The idea to use maps $D \rightarrow \mathbb{R}^n$, where $D \subset \mathbb{R}$, was developed in [6]. An improved version of the main theorem and the method is given in [7]. In this paper the map $D \rightarrow \mathbb{R}^2$ is used. With the use of $D \rightarrow \mathbb{R}^n$, maps of the sets on a line into the spaces of larger dimensions, we hope to obtain a solution of SSP for $m > l^\varepsilon$ for each $\varepsilon > 0$.

Note. During the Israel–French conference on combinatorics (November 1988, Jerusalem), Professor Paul Erdős told me about similar results of Professor András

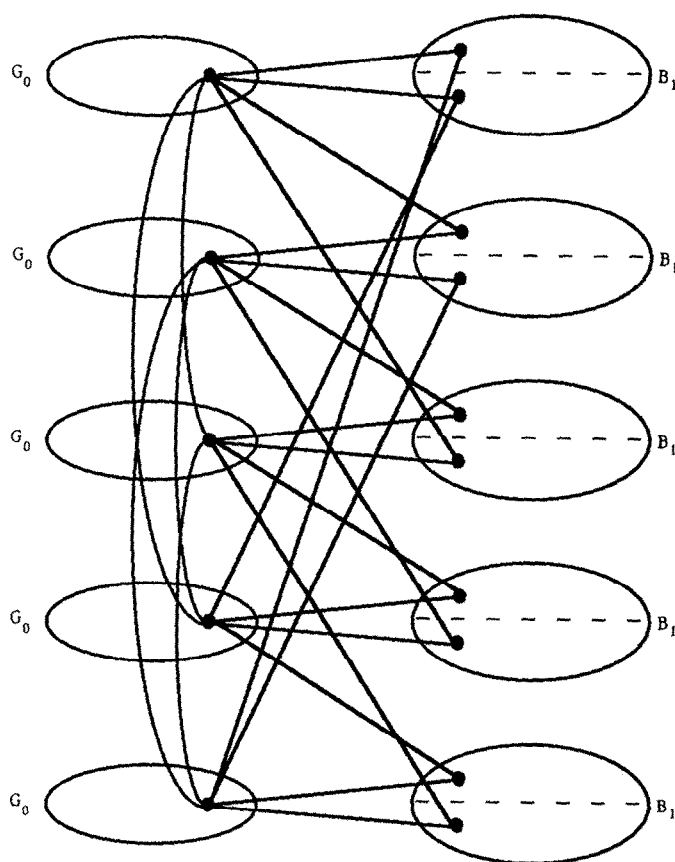


Fig. 6.

(that is, we have $n_1 = N_1/2$ copies of K_5), and those shown in Fig. 7, where $t \in \mathbb{Z}_{n_1}$ denotes a generic vertex of either V_{10} or V_{11} in any copy of B_1 . The $(15, 5)$ -graph $K_1 \Sigma'_8 Q_{11}$ is obtained from $K_1 \Sigma_8 Q_{11}$ in a similar way.

3. Some modifications to compound graphs

Let $B_0 = G_0$ be any graph of order N_0 and even diameter D_0 , and B_1 be a bipartite graph of order $N_1 = 2n_1$ and even diameter D_1 . A particular case of the compound graphs DQ_A proposed by Delorme and Quisquater [10] have the basic configuration shown in Fig. 8, where $V(B_0)$ is evenly partitioned into two sets V_{00} and V_{01} .

The parameters of the obtained graph are $N = \frac{1}{2}(n+1) N_0 N_1$, $\Delta = \max\{\Delta_0 + n, \Delta_1 + 1\}$ and $D \leq D_0 + D_1 + 1$. In the next paragraph we present a modification of these graphs when $D_0 = 2$, which leads to some new largest (up-to-date) known

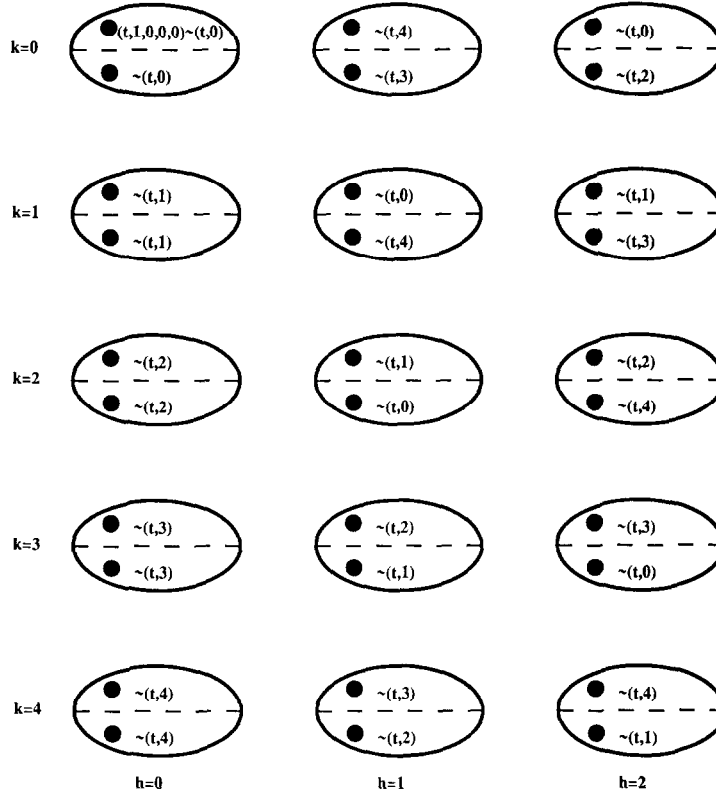


Fig. 7.

(Δ, D) -graphs. This technique was proposed in [12] together with similar constructions for $D_0 = 1, 3$.

Without loss of generality, we can assume that the vertices $(0, 0, 0, t)$ and $(0, 0, 1, t)$ are adjacent for any t . Then the n copies of B_1 linked to these two vertices in each copy of G_0 can be duplicated, modifying the corresponding intercopy edges as shown in Fig. 9.

In this way, we obtain the graph $G_0 \Delta_p B_1$ with order $N = \frac{1}{2}(n+1)N_0N_1 + nN_1$, degree $\Delta = \max\{\Delta_0 + n, \Delta_1 + 1\}$ and its diameter still satisfies $D \leq D_0 + D_1 + 1$.

Finally, let G_0 be the complete bipartite graph $K_{1,3}$ and $B_1 = (V_0 \cup V_1, E)$ a bipartite graph with $|V_0| = |V_1| = n$ and diameter even D_1 . The basic configuration depicted in Fig. 10 corresponds to a slight modification of $K_{1,3} \Delta B_1$ (only one edge from vertices $(0, x)$ to vertices $(x, 0, 0)$) and it gives rise to the graph $K_{1,3} \Pi B_1$, with parameters $N = 6N_1$, $\Delta = \max\{4, \Delta_1 + 1\}$ and $D = D_1 + 2$.

When $\Delta_1 = 2\alpha$, $\alpha \geq 2$, we can use the induced configuration of Fig. 10 (as done in the preceding cases). Moreover, since then the vertices $(0, x)$ have degree $\delta(0, x) = 3 + \alpha$, we can add to the basic configuration of the last graph $\alpha - 2$ copies of B_1 linked to these

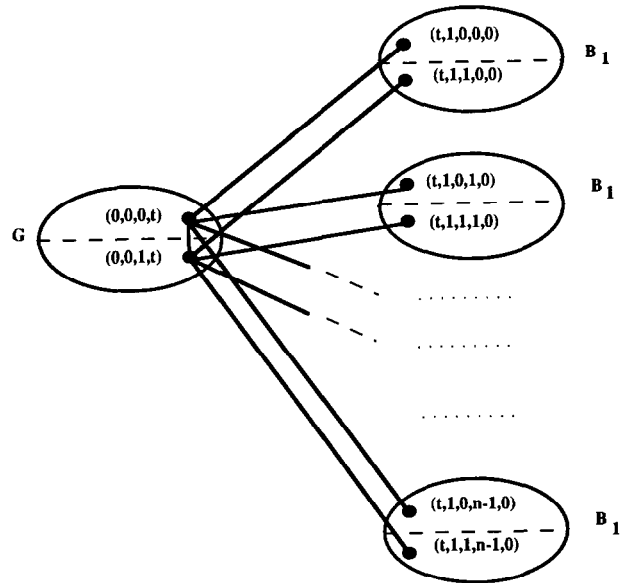


Fig. 8.

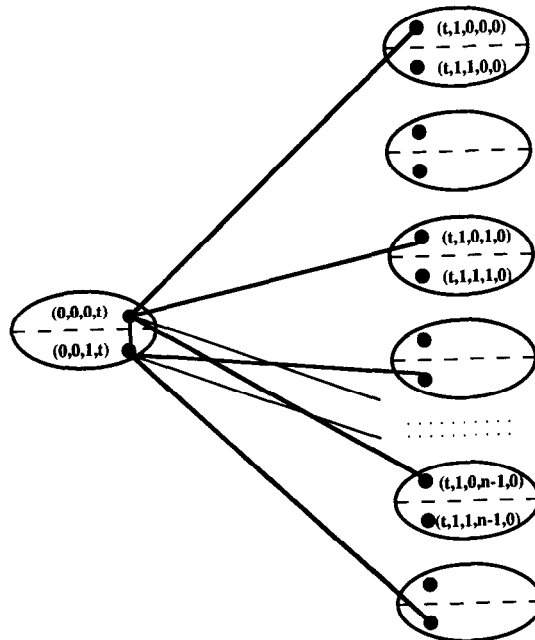


Fig. 9.

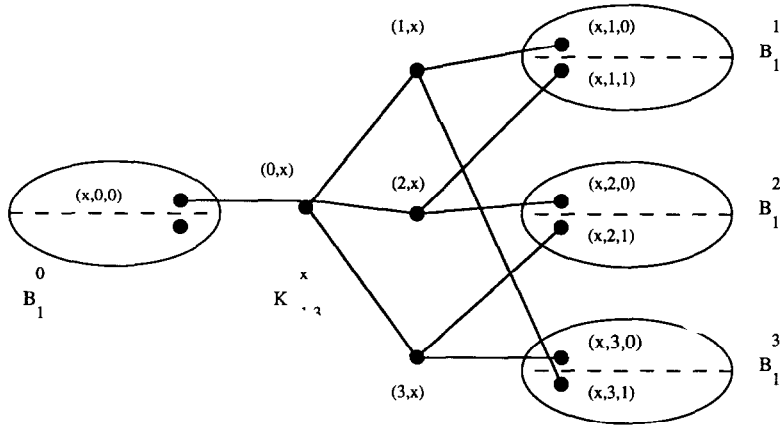


Fig. 10.

vertices as in the initial configuration. To be more precise, if the vertices of these copies are labeled with the vectors (x, i, j, p) , $p \in \mathbb{Z}_{\Delta-2}$, the vertices $(0, x) \in K_{1,3}^x$ are joined with the vertices $(x, 0, 0, p)$, $p \in \mathbb{Z}_{\Delta-2}$; see Fig. 11. The graph thus obtained, which we denote by $K_{1,3} \Pi' B_1$, has parameters $N = 5(\Delta - 1)N_1/2$, $\Delta = \Delta_1 + 1$ and $D = D_1 + 2$.

4. Large graphs from generalized quadrangles and hexagons

A simple technique that has been proved to be useful to obtain large graphs with $D = 4, 6$ consists in expanding the bipartite Moore graphs with the same diameters, by replacing a vertex by a complete graph; see [14]. In this section we present similar expansions of these graphs, as well as other constructions based on them, which give rise to some largest known (Δ, D) -graphs.

4.1. A cubic graph of diameter 6 on 130 vertices

We now construct a cubic graph of diameter $D = 6$ and order $N = 130$. To illustrate the technique used, let us first note that the Petersen graph P can be obtained from $K_{3,3}$ by taking a cycle of length 4 in $K_{3,3}$, say v_1, v_2, v_3, v_4 , and adding 4 vertices and 2 edges as shown in Fig. 12.

To obtain the new graph, which we denote by $H_2 t$, let us now consider a cycle of length 12 in the bipartite Moore graph H_2 ($D = 6$, $N = 126$), say $v_1, v_2, \dots, v_{12}$. Then $H_2 t$ is obtained from H_2 by adding the 4 vertices $u_1, \dots, u_4$, and 2 edges, as depicted in Fig. 13. According to the general properties of H_q , it can be checked that its diameter is still 6. Indeed, with $C = \{v_i, 1 \leq i \leq 12\}$ and $A = \{u_i, 1 \leq i \leq 4\}$, the cases to be considered are the following:

- (a) $x, y \in C \cup A$.

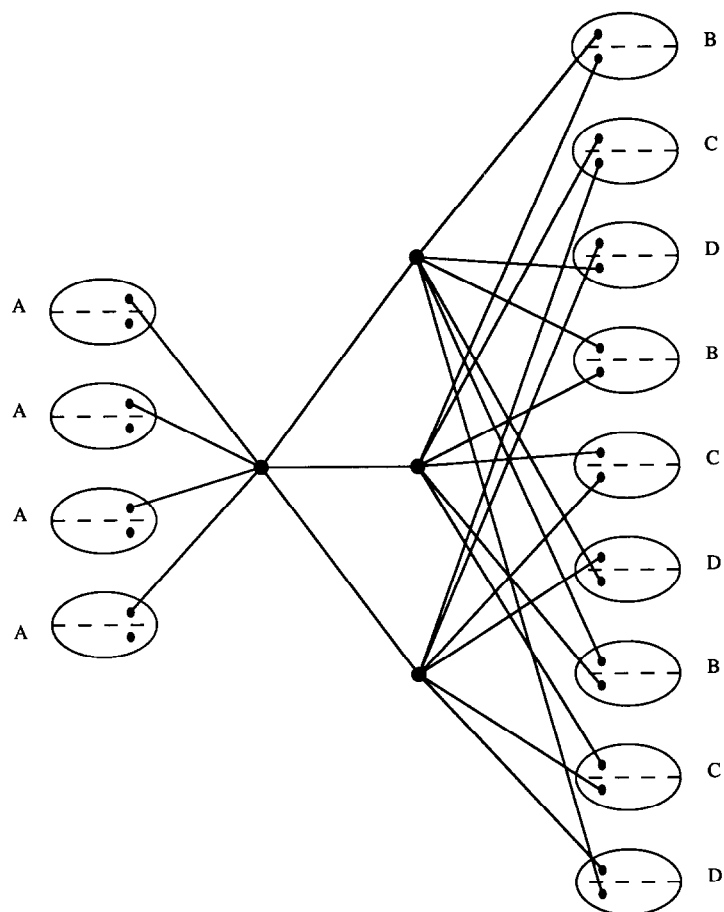


Fig. 11.

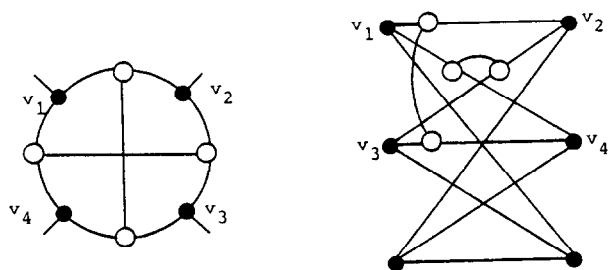


Fig. 12.

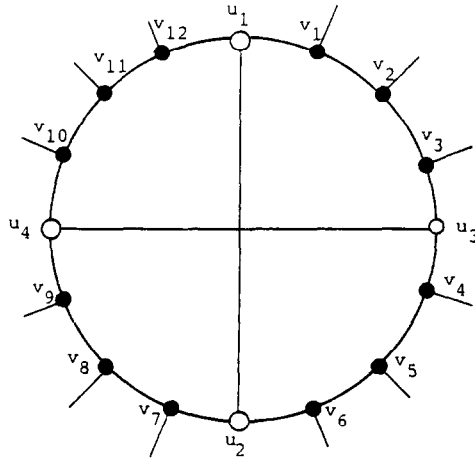


Fig. 13.

Trivially, $d(x, y) \leq 5$.

(b) $x \notin A, y \notin C \cup A$.

(b.1) $x \in U, y \in V$ (U and V denote the stable sets of H_2).

The distance between x and y is increased by, at most, one unit.

(b.2) $x, y \in U$ (the case $x, y \in V$ being analogous).

(b.2.1) If $d(x, y) = 6$ in H_2 , then there exist three disjoint paths of length 6 between them, and only one of them can contain vertices of C . Otherwise, there would exist cycles of length ≤ 12 in H_2 .

(b.2.2) If $d(x, y) \leq 4$ in H_2 , then their distance in $H_2 t$ increases by, at most, two units.

(c) $x \in A, y \notin C \cup A$.

Without loss of generality, let $x = u_1$. Then, there exists a path between y and v_1 or v_{12} of length ≤ 5 . If such a path contains neither v_4 nor v_9 , then $d(u_1, y) \leq 6$ (check in Fig. 13). Otherwise, the vertex u_3 , which is adjacent to u_1 , satisfies $d(u_3, y) \leq 5$.

4.2. The graphs H_q s and Q_q s

The graphs H_q s and Q_q s are obtained from H_q and Q_q , respectively, by replacing some of their vertices by complete graphs and creating some new adjacencies. To be more precise, let U_q, V_q be the stable sets of either H_q or Q_q , $u \in U_q$, and $\Gamma(u) = \{v_0, v_1, \dots, v_q\} \subset V_q$. Set $\Gamma(v_i) = \{u\} \cup \{u_{ij}, j=0, \dots, q-1\}$, $i=0, 1, \dots, q$. Then remove the edges adjacent to the vertices $v_0, v_1, \dots, v_{q-1}$, and replace each v_i , $i=0, 1, \dots, q-1$, by a complete graph K_q (of degree $q-1$) with vertices $v_{i0}, v_{i1}, \dots, v_{iq-1}$. Finally, set the new adjacencies

$$v_{ij} \sim \begin{cases} v_{ji} & \text{if } i \neq j, \\ u & \text{if } i = j, \end{cases}$$

and

$$v_{ij} \sim u_{ij}.$$

Note that there exists an edge between each pair of copies of K_q , and each old edge $u_{ij}v_j$ or uv_j is replaced by an edge $u_{ij}v_{ij}$ or uv_{ii} , as shown in Fig. 14 for $q=3$.

Thus, the proposed graph H_q s is regular of degree $\Delta = q + 1$ and has order

$$N = 2 \frac{(\Delta - 1)^6 - 1}{\Delta - 2} + (\Delta - 1)(\Delta - 2).$$

As before, using the basic properties of H_q , we can check that its diameter is still 6. To this end, let us consider the following cases, where A denotes the set of vertices u_{ij} :

(a) $x, y \in A$.

Trivially, $d(x, y) \leq 3$.

(b) $x, y \notin A$.

(b.1) $x \in U_q, y \in V_q$.

The distance between x and y is increased by, at most, one unit.

(b.2) $x, y \in U_q$ (the case $x, y \in V_q$ being analogous).

(b.2.1) If $d(x, y) = 6$, then there exist $\Delta = q + 1$ disjoint paths of length 6 between them, and at least one of these paths contains no vertex of A .

(b.2.2) If $d(x, y) \leq 4$ in H_q , then their distance in H_q s increases by, at most, one unit.

(c) $x \in A, y \notin A$.

(c.1) $y \in V_q$.

(c.1.1) If $d(x, y) = 6$ in H_q , then there exist a path of length 6, $x, u_{ij}, \dots, y$, which contains no other vertex of A .

(c.1.2) If $d(x, y) \leq 4$ in H_q , then their distance in H_q s increases by, at most, one unit.

(c.2) $y \in U_q$.

As $d(v_i, y) \leq 5$ in H_q , $d(x, y) \leq 6$.

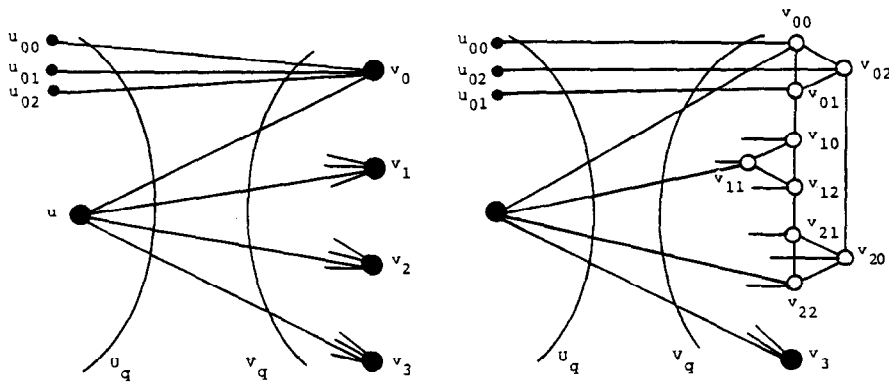


Fig. 14.

In a similar way, the Δ -regular graph $Q_q s$, $\Delta = q + 1$, has order

$$N = 2 \frac{(\Delta - 1)^4 - 1}{\Delta - 2} + (\Delta - 1)(\Delta - 2)$$

and, considering the same cases as in $H_q s$, it is readily checked that its diameter is 4.

4.3. A graph of degree 10 and diameter 4 on 1820 vertices

The graph here described is obtained from the graphs Q_8 ($\Delta = 9$, $D = 4$, $N = 1170$) and Q'_8 ($\Delta = 9$, $D = 3$, $N = 585$). According to the remarks of Section 1, there exists in $Q_8 = (V_8 \cup U_8, E)$ a subset $A \subset V_8$ of $q^2 + 1 = 65$ vertices such that the distance between each pair of them is exactly 4. Then, following the method of Delorme described in [9], each vertex $v \in A$ can be duplicated, giving rise to a couple $\{v, v'\}$ of vertices with the same neighborhood ($\Gamma(v) = \Gamma(v')$). In this way, we obtain a graph $Q_8 d$, with parameters $\Delta = 10$, $D = 4$ and $N = 1235$, which is still bipartite. The graph $Q_8 z$ proposed here is obtained from $Q_8 d$ just by linking each vertex $u \in V_8 \subset V(Q_8 d)$ to a vertex in Q'_8 . This can be done since V_8 has $585 + 65 = 650$ vertices and Q'_8 has 65 vertices of degree 8. Thus, it is possible to have 650 edges coming out of Q'_8 , with a degree not exceeding 10. The new graph is regular of degree $\Delta = 10$, has order $N = 1820$ and, as can be readily seen, its diameter is $D = 4$. As it was pointed out by one of the referees, such a construction can be shown better when applied to Q_2 instead of Q_8 , since it gives a graph on $30 + 5 + 15 = 50$ vertices, degree 4 and diameter 4. In this case, Q_2 is the Tutte–Coxeter graph and Q'_2 is obtained from Petersen graph by ‘adding’ a vertex to each edge of a perfect matching. Then Q'_2 and $Q_2 d$ are connected with 20 edges; see Fig. 15.

5. The table of largest known (Δ, D) graphs

Bermond, Delorme and Quisquater regularly publish the updated table of largest known (Δ, D) -graphs with degree $3 \leq \Delta \leq 15$ and diameter $2 \leq D \leq 10$. We include here the last version of this table (Table 1), with the new data (in bold) obtained in this paper. Explanations concerning the other constructions can be found in [4] or in the following list.

*CR**: Generalized chordal rings of Quisquater [14].

2cy: Graphs obtained by Bar-Yehuda and Etzion [1] connecting two cycles.

CCD: Cayley graphs on Borel groups of Chudnovsky et al. [7].

8a: Cayley graphs obtained by Campbell et al. [6], June 1988 (improved Nov. 1988).

Allwr: Construction of J. Allwright (personal communication via J.-C. Bermond), Aug. 1988 (improved Nov. 1988).

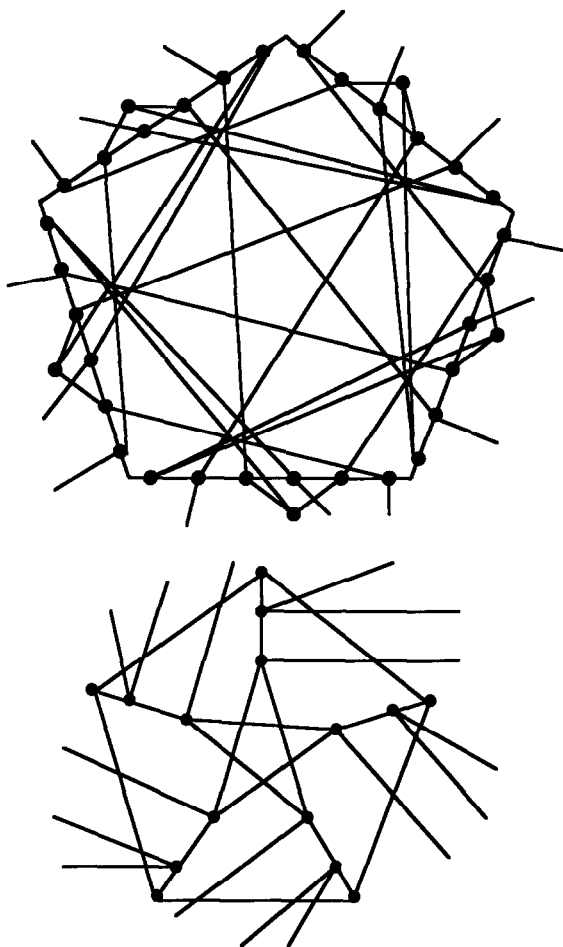


Fig. 15.

Acknowledgment

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