

DENSE COMPOUND GRAPHS

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ABSTRACT

This paper focuses on the study of the (Δ, D) problem that has deserved a special attention in the last years because of its implications in the design of interconnection networks. It consists in maximizing the number of vertices of a graph for given values of its (maximum) degree Δ and diameter D .

A basic method of constructing such dense graphs is the compounding of graphs. It consists in joining together several copies of a given graph according to the structure of another one. The diameter of the resulting graph depends on the choice of the edges connecting the copies.

After an overview of the more representative constructions of compound graphs known to date, we present in this paper some new ones. These proposals give rise to many of the largest known dense graphs.

1. INTRODUCTION

For the sake of completeness, we shall first recall the basic concepts used in this paper. Let $G = (V, E)$ be a graph with set of vertices V and set of edges E . The adjacency between two vertices $x, y \in V$, that is the existence of the edge xy ($=yx$) $\in E$, will be represented by $x \sim y$ (x and y are then *endvertices* of the edge xy). The *degree* of x , $\delta(x)$, is the number of vertices adjacent to it, and the *degree* of G , Δ_G or Δ , is defined as $\Delta \equiv \max_{x \in V} \delta(x)$. The graph G is (δ) -*regular* if all its vertices have the same degree (δ). G is *bipartite* si $V = V_1 \cup V_2$, $V_1 \cap V_2 = \emptyset$, and any edge $xy \in E$ has an *endvertex*,

say x , in V_1 and the other one, y , in V_2 . The *distance* $d(x,y)$ between two vertices x and y is the number of edges of a shortest path joining them.

Similar concepts can be defined for a digraph $G^* = (X, A)$ with set of vertices X and set of *arcs* (ordered pairs of vertices) A . When there exists the arc $xy (\neq yx) \in A$, we say that x is *adjacent to* y or, also, that y is *adjacent from* x . Then, the outdegree [indegree] of a vertex x , $\delta^+(x)$ [$\delta^-(x)$] is the number of vertices adjacent from [to] x . Distances in a digraph are concerned with oriented paths, so, in general, $d(x,y) \neq d(y,x)$. Other concepts about digraphs will be defined in Section 10.

The so-called (Δ, D) *problem* is related with the design of large interconnection or telecommunication networks, and it consists in finding the maximum *order*, i.e. number of vertices, $n(\Delta, D)$ of a graph G with degree Δ and diameter D . These three parameters are related themselves by the *Moore bound*:

$$\begin{aligned} n(2, D) &\leq 2D + 1 \\ n(\Delta, D) &\leq \frac{\Delta(\Delta-1)^D - 2}{\Delta - 2}, \quad \Delta \geq 3 \end{aligned} \quad (1)$$

For $D \geq 2$ and $\Delta \geq 3$, this bound is only attained when $D = 2$ and $\Delta = 3, 7$ and, perhaps, 57. See, for instance, [3, Chp. 23].

The graphs whose order is "as close as possible" to $n(\Delta, D)$ will be called *dense* graphs. In other words, dense graphs have large order for given diameter and degree.

2. BASIC COMPOUND GRAPHS

In [1, Th.1], Bermond, Delorme and Quisquater propose a construction based on a simple but very effective idea to obtain large dense graphs. This construction, that results in the so-called compound graphs, consists basically in considering several copies of a given graph G_1 and join them together by additional edges according to the structure of other graph G_2 . More precisely, given the graphs $G_i = (V_i, E_i)$, $i = 1, 2$, with $N_i = |V_i|$ vertices and diameter D_i , their *compound graph* $G_1[G_2]$, on $N = N_1 N_2$ vertices, is obtained by re-

placing each vertex $x \in V_2$ by one copy of G_1 , which we represent by G_1^x , and each edge $xy \in E_2$ by *one* edge that joins one vertex of G_1^x with another *one* of G_1^y . Thus we will be able to construct $G_1[G_2]$ in such a way that it has degree Δ if and only if $\Delta \geq \Delta_1$ and

$$\Delta - \Delta_1 \leq \Delta_2 \leq \Delta N_1 - \sum_{x \in V_1} \delta(x) \quad (2)$$

As it is pointed out in [2], the diameter D of $G_1[G_2]$ depends upon the way we join the copies of G_1 (that is, on the endvertices that are selected for the edges that join the copies). But it is easy to see that it must verify

$$D \leq (D_2 + 1)D_1 + D_2 \quad (3)$$

Indeed, to go from a vertex of the copy G_1^x to other one in G_1^y , we need to go through a maximum of D_2 edges "inter-copy" (maximum distance between x and y in G_2) and, hence, through a maximum of $D_2 + 1$ copies. Then, as in the worst case we need D_1 steps to go through each copy, bound (3) holds.

3. BIPARTITE COMPOUND GRAPHS

In [4], Delorme proposes a construction to obtain bipartite graphs that is similar to the preceding one. In it $G_1 = (V_1 \cup U_1, E_1)$ is a bipartite graph and, as before, each vertex $x \in V_2$ is replaced by a copy of G_1 , G_1^x . But, unlike the case above, each edge $xy \in E_2$ gives rise to *two* edges joining the copies G_1^x and G_1^y . One of this edges has an endvertex in V_1^x and the one in U_1^y , and vice versa for the other edge. Thus, given $\Delta \geq \Delta_1$ such that

$$\Delta - \Delta_1 \leq \Delta_2 \leq \min\left\{\sum_{x \in V_1} \delta(x), \sum_{x \in U_1} \delta(x)\right\} \quad (4)$$

we can obtain a bipartite compound graph on $N = N_1 N_2$ vertices with degree Δ and diameter

$$D \leq D_2(D_1 - 1) + D_1 + D_2 = D_1(D_2 + 1) \quad (5)$$

The proof of this bound is based on the following fact: since G_1 is bipartite, if its diameter is even [odd], the distance between two vertices of different sets (U_1 and V_1) [the same

set] does not exceed $D_1 - 1$.

4. COMPOUND GRAPHS FF

The two constructions of compound graphs proposed by the second author and Fábrega in [7] are closely related to the preceding ones. They can both be unified in the following way:

Like in Section 3, $G_1 = (V_1 \cup U_1, E_1)$ is a bipartite graph but the resulting compound graph is not. Moreover, each edge of G_2 produces *four* edges whose arrangement remind us of the complete bipartite graph $K_{2,2}$. So, if there exists the edge xy in G_2 , we join together the copies G_1^x and G_1^y by choosing the vertices $a, a' \in V_1^x$, $b, b' \in U_1^x$, $c, c' \in V_1^y$, $d, d' \in U_1^y$, and then joining them as shown in Figure 1.

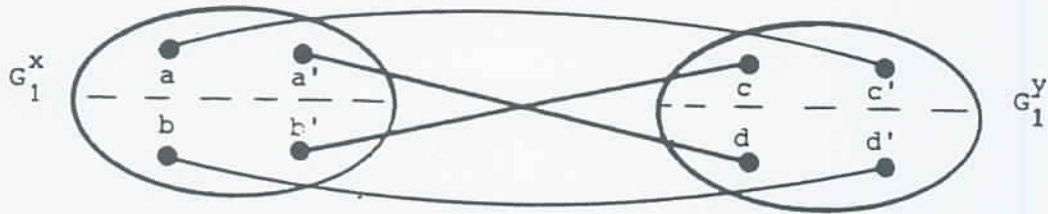


Figure 1

The vertices of the same set i and i' , $i \in \{a, b, c, d\}$, need *not* be different. The choice of such vertices can be so done that the resulting graph have degree Δ iff $\Delta \geq \Delta_1$ and

$$\Delta - \Delta_1 \leq 2\Delta_2 \leq \min\left\{ |V_1| \Delta - \sum_{x \in V_1} \delta_1(x), |U_1| \Delta - \sum_{x \in U_1} \delta_1(x) \right\} \quad (6)$$

Its order is $N = N_1 N_2$ and its diameter satisfies

$$D \leq D_1(D_2 + 1) - 1 \quad (7)$$

The proof of this bound is analogous to that of (5).

5. COMPOUND GRAPHS BDQ₂

In [1, Th.2], the authors also propose a somewhat more involved construction than the preceding ones. In it, copies of two (usually different) graphs $G_1 = (V_1, E_1)$ and $G'_1 = (V'_1, E'_1)$ are joined together according to the structure of a bipartite

graph $G_2 = (V_2 \cup V_2', E_2)$. Like in the basic construction of Section 2, each edge $xy \in E_2$ is replaced by another one whose endvertices are in G_1^x and $G_1'^y$, but with the additional constraint of choosing the vertices so that all of them are finally endvertices of one or more edges inter-copy. Then we can build the compound graph $(G_1, G_1')[G_2]$ so that it has degree Δ iff $\Delta > \Delta_1$, $\Delta > \Delta_1'$ and

$$\Delta - \Delta_1 \leq \Delta_2 \leq \Delta N_1 - \sum_{x \in V_1} \delta(x) \quad (8.a)$$

$$\Delta - \Delta_1' \leq \Delta_2' \leq \Delta N_1' - \sum_{y \in V_1'} \delta(y) \quad (8.b)$$

where Δ_2 and Δ_2' denote the maximum degree of the vertices in V_2 and V_2' respectively.

The resulting graph has $N = N_1|V_2| + N_1'|V_2'|$ vertices and, assuming $D_1 \geq D_1'$, its diameter satisfies

$$D \leq D_2 + \left\lceil \frac{D_2}{2} \right\rceil D_1 + \left\lfloor \frac{D_2}{2} \right\rfloor D_1' \quad (9)$$

(The proof of (9) can be found in [1]). If D_2 is odd and $D_1 \geq D_1' + 2$, the reasoning of the above reference can be slightly modified to improve the bound (9). Then, we obtain

$$D \leq D_2 + \left\lfloor \frac{D_2}{2} \right\rfloor D_1 + \left\lceil \frac{D_2}{2} \right\rceil D_1' + 1 \quad (10)$$

6. COMPOUND GRAPHS DQ_A

In [6], Delorme and Quisquater propose a construction in which some ideas of the preceding sections converge. This construction is similar to the one which produces the graphs $(G_1, G_1')[G_2]$, described in Section 5, since we use copies of two different graphs joined together according to the structure of a bipartite (and complete in this case) graph, and all the vertices are also at the end of some edge inter-copy.

Specifically, let $G_1 = B = (V_B \cup U_B, E_B)$ and $G_1' = G = (V_G, E_G)$ be two graphs, the first one bipartite with $|V_B| = |U_B|$, with degree and diameter Δ_B, D_B and Δ_G, D_G respectively. Let $G_2 = K_{1,m} = (V \cup U, E)$ be the complete bipartite graph on $1+m$ vertices (the parameters 1 and m will depend on the degree of the resulting graph). Then each vertex $x \in V$ [$y \in U$] is re-

placed by a copy of B , B^x , $[G, G^y]$ in order to have l copies of B and m copies of G . Like in Section 3, each edge $xy \in E$ gives rise to two edges that join the copies B^x and G^y . For that, we choose the vertices $a \in V_B$, $b \in U_B$ and $c, c' \in V_G$ (not necessarily different), and set up the edges ac and bc' as shown in Figure 2. As we have already said, the choice of these vertices must be done in such a way that, in the resulting graph, all of them are endvertices of some edge inter-copy.

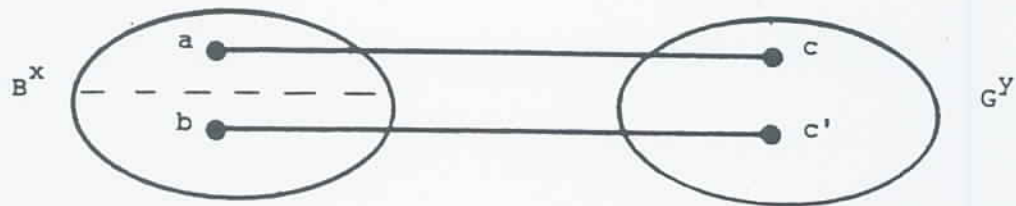


Figure 2

Then, we will be able to construct the compound graph $B \Delta G$ so that it has degree Δ (and maximum possible order) iff $\Delta > \Delta_B$, $\Delta > \Delta_G$ and l and m are given by the following formulas

$$l = \left\lfloor \frac{\Delta |V_G| - \sum_{x \in V_G} \delta(x)}{2} \right\rfloor \quad (11.a)$$

$$m = \Delta |V_B| - \sum_{x \in V_B} \delta(x) = \Delta |U_B| - \sum_{x \in U_B} \delta(x) \quad (11.b)$$

For example, if B and G are regular graphs with respective degrees Δ_B and Δ_G , we have

$$l = \left\lfloor \frac{|V_G|(\Delta - \Delta_G)}{2} \right\rfloor, \quad m = |V_B|(\Delta - \Delta_B) = |U_B|(\Delta - \Delta_B) \quad (12)$$

Let now us check that the diameter D of the graph $B \Delta G$ is bounded by

$$D \leq D_B + D_G + 1 \quad (13)$$

As each vertex in a copy of B is adjacent to a vertex in a copy of G , it suffices to show that the distance between the generic vertices $s \in B^x$ and $t \in G^y$ does not exceed $D_B + D_G$. Since B is bipartite, vertex s is at distance $\leq D_B - 1$ of a vertex $v \in B^x$ (in Figure 2 a or b) which, because

$G_2 = K_{1,m}$, is adjacent to a vertex $w \in G^y$ (c or c' in this figure) that is at distance $\leq D_G$ of vertex t .

7. COMPOUND GRAPHS $B_0 \nabla B_1$.

This new construction combines the main characteristics of the compound graphs FF and DQ_Λ . Like in the previous section, we use copies of two different graphs, G_1 and G'_1 , which now are both bipartite, joined together following the structure of a complete bipartite graph $K_{1,m} = (V \cup U, E)$ in such a way that all vertices are at the end of some edge inter-copy. Moreover, each edge of $K_{1,m}$ is replaced by *four* edges arranged like in the compound graphs FF , see Figure 1.

Thus, let $G_1 = B_0$ and $G'_1 = B_1$, $B_i = (V_{i0} \cup V_{i1}, E_i)$, be two bipartite graphs on $N_i = 2|V_{i0}| = 2|V_{i1}| = 2n_i$ vertices, degree Δ_i and diameter D_i . Then we can construct the compound graph $B_0 \nabla B_1$ with the maximum possible order $N = 1N_0 + mN_1$ and degree Δ if and only if $\Delta > \Delta_0$, $\Delta > \Delta_1$ and l, m satisfy

$$l = \left\lfloor \frac{\Delta |V_{10}| - \sum_{x \in V_{10}} \delta(x)}{2} \right\rfloor \quad (14.a)$$

$$m = \left\lfloor \frac{\Delta |V_{00}| - \sum_{x \in V_{00}} \delta(x)}{2} \right\rfloor \quad (14.b)$$

Theorem 1 : The compound graph $B_0 \nabla B_1$ has diameter $D \leq D_0 + D_1$.

Proof:

The proof of this result is similar to that of the preceding section for the graphs DQ_Λ and, as there, it suffices to show that the distance between a vertex of a copy of B_0 , $s \in V_{0i}^x$, and another one of a copy of B_1 , $t \in V_{1j}^y$, does not exceed $D_0 + D_1 - 1$.

Since B_0 is bipartite, s is at distance $\leq D_0 - 1$ of a vertex $v \in V_{0i'}^x$, $i' = i + (D_0 - 1) \bmod 2$, that, by construction of $B_0 \nabla B_1$, can be chosen to be adjacent to a vertex $w \in V_{1j'}^y$, $j' = j - (D_1 - 1) \bmod 2$, which is at distance $\leq D_1 - 1$ (B_1 is also bipartite) of vertex t .

Let now us see two particular cases in which a direct

construction can be made by joining together copies of B_0 and B_1 , that yield some of the largest known (Δ, D) graphs:

7.1 Using the same notation than before, let B_i , $i = 0, 1$, be two regular bipartite graphs of degree $\Delta_0 = \Delta_1$ on $N_i = 2|V_{i0}| = 2|V_{i1}| = 2n_i$ vertices; and assume that the vertices of V_{ij} are identified with the integers modulo n_i . We now take n_1 copies of B_0 and n_0 copies of B_1 labelled with the integers modulo n_1 and n_0 respectively, so that any vertex can be identified by a vector (s, i, j, t) , where $s \in V_{ij} \cong \mathbb{Z}/n_i\mathbb{Z}$ belongs to the copy $t \in \mathbb{Z}/n_i\mathbb{Z}$ ($\bar{i} = i+1 \bmod 2$) of the graph B_i . Then we join the copies of B_0 with those of B_1 increasing in two unities the degree of all vertices, by setting up the new adjacencies (see Figure 3):

$$(s, i, j, t) \sim \begin{cases} (t, \bar{i}, j, s) \\ (t, \bar{i}, \bar{j}, s) \end{cases} \quad (15)$$

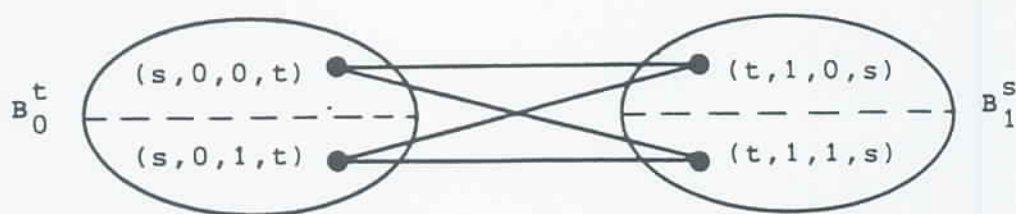


Figure 3

The graph $B_0 \nabla_0 B_1$ so constructed is regular of degree $\Delta = \Delta_0 + 2$ ($= \Delta_1 + 2$), and has order $N = n_0 N_1 + n_1 N_0 = N_0 N_1$ and diameter $D \leq D_0 + D_1$ (Theorem 1).

Note that, if we think of this construction as a particular case of the firstly described one, we have $K_{1,m} = K_{\frac{N_1}{2}, \frac{N_0}{2}}$ according to the formulas (14.a) and (14.b) with

$$\Delta = \Delta_0 + 2, \quad |V_{i0}| = n_i = \frac{N_i}{2} \quad \text{and} \quad \sum_{x \in V_{i0}} \delta(x) = \Delta_0 n_i = \Delta_0 \frac{N_i}{2}.$$

Since each vertex of a copy of B_i is adjacent to two vertices of a copy of $B_{\bar{i}}$, there exist a minimum of two shortest paths between vertices belonging to copies of the same graph. Hence, it seems possible to remove some edges without increasing the diameter. In fact, it is not difficult to

check that, from every set of four edges placed between two copies as shown in Figure 3, we can *remove one* of them obtaining a graph with diameter $D \leq D_0 + D_1$. In the following section we will use this idea to obtain patterns of interconnection which imply less path redundancy and, therefore, produce denser graphs.

7.2 If the sets of vertices of the graphs B_i satisfy

$$|V_{ij}| = n_i = 2n_i', \quad i = 0, 1 \quad (16)$$

we can do a construction in which the degree increases only in one unity. Indeed, according to (16) we can decompose each V_{ij} into two disjoint subsets V_{ijk} , $k \in \{0, 1\}$, each of them containing n_i' vertices labelled with the integers modulo n_i' . If we take n_1' copies of B_0 and n_0' copies of B_1 identified with the integers modulo n_1' and n_0' respectively, each vertex is now represented by the vector (s, i, j, k, t) , $s \in \mathbb{Z}/n_i'\mathbb{Z}$, $t \in \mathbb{Z}/n_i'\mathbb{Z}$, $i, j, k \in \{0, 1\}$, and the compound graph is obtained by considering the adjacency rule

$$(s, i, j, k, t) \sim (t, \bar{i}, k, j, s) \quad (17)$$

as it is shown in Figure 4 (compare with Figure 1).

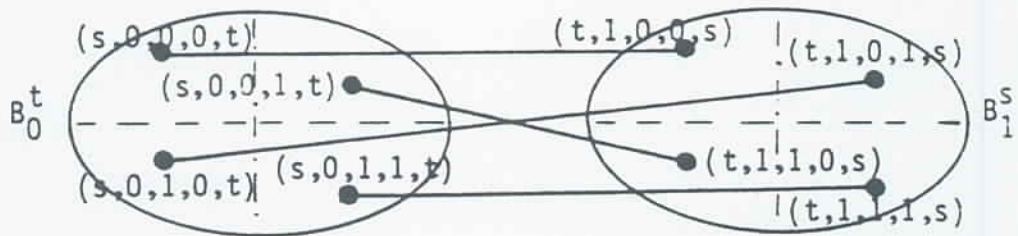


Figure 4

Thus, we obtain the graph $B_0 \nabla_1 B_1$ regular of degree $\Delta = \Delta_0 + 1$ ($= \Delta_1 + 1$), on $N = n_1' N_0 + n_0' N_1 = N_0 N_1 / 2$ vertices and with diameter $D \leq D_0 + D_1$ (according to the scheme of the general construction, we have now taken $K_{1,m} = K_{\frac{N_1}{4}, \frac{N_0}{4}}$).

For instance, we will show a path between the vertices $(x, 0, j, k, y)$ and $(s, 1, l, m, t)$ (all operations are modulo 2):

$$(x, 0, j, k, y), \dots, (t, 0, j+D_0-1, 1-(D_1-1), y), \\ (y, 1, 1-(D_1-1), j+D_0-1, t), \dots, (s, 1, 1, m, t)$$

Note that we first use a maximum of D_0-1 steps to "choose" the copy t of the destination vertex, then, with an additional step, we go to such a copy B_1^t and, finally, a number $\leq D_1 - 1$ of steps are used to "choose" such a vertex s .

A disadvantage of the above construction is that, according to (16), B_0 and B_1 must have order multiple of 4. To overcome it we can use a slightly modified construction in which it is assumed that

$$|V_{ij}| = n_i = \alpha n_i', \quad i = 0, 1, \quad \alpha \text{ odd} > 1 \quad (18)$$

Then, we first consider $(\alpha-1)n_i'$ vertices in each V_{ij} and, with them, we make the described construction. Moreover, to avoid increasing the diameter, every one of the remaining $2n_0'$ vertices of each copy of B_0 is joined to a similar vertex in a copy of B_1 .

For instance, if $\frac{N_i}{2} = 5n_i'$ (N_i multiple of 10), the edges joining each set of four copies can be as in Figure 5. The resulting graph, denoted by $B_0 \nabla_2 B_1$, has order $N = 2n_1'N_0 + 2n_0'N_1 = 2N_0N_1/5$ and the same degree and diameter than before.

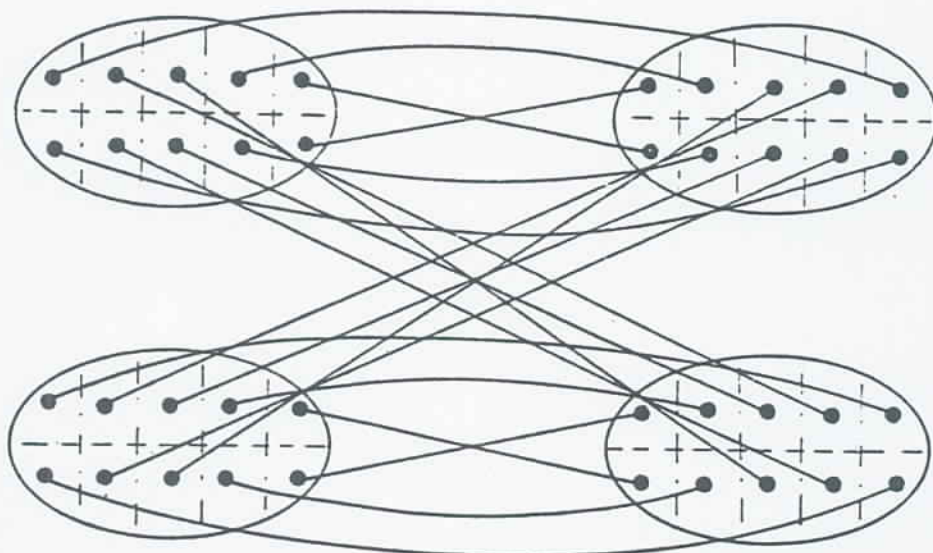


Figure 5

Obviously, the particular constructions described in 7.1 and 7.2 as well as the ones we are going to define in the next section, admit a generalization in the sense that, following the scheme of the construction studied at the beginning of this section (or the ones corresponding to the preceding sections):

- a) The graphs B_0 and B_1 cannot be regular
- b) The maximum degrees Δ_0 and Δ_1 can be different (then, if we use, for instance, the construction of 7.2, the degree of some vertices can be increased in more than one unity and, therefore, the number of copies is larger).

Of course, all these graphs will be represented using the same notation introduced for their corresponding "basic configurations".

8. COMPOUND GRAPHS $B_0 \Sigma B_1$

We have already seen that, in the graphs obtained by means of the particular construction in 7.1, there exists a duplicity of shortest paths between many pairs of vertices. For the same reason given there, this also holds for the general construction of Section 7, or for the graphs DQ_Λ described in Section 6, whenever some vertices in the copies of a graph are adjacent to two or more vertices in the copies of the other one. Moreover note that, in all these graphs, at distance $\leq D-1$ of any vertex there is a number of vertices which is of the order of $N/2$ (all of them belonging to the copies of the graph which is different to that containing the initial vertex). In order to overcome these difficulties and thus to obtain larger graphs, we propose in this section new constructions in which we try to avoid the presence of redundant shortest paths.

8.1 As in 7.1, let B_0 and B_1 be two bipartite graphs but with diameters D_0 even and D_1 odd. Let now us consider *three* sets of $n_1 = N_1/2$ copies of B_0 each and *three* sets of $n_0 = N_0/2$ copies of B_1 . Recalling the notation used in 7.1, the vector

$$(s, i, j, k, t)$$

will denote the vertex (s, i, j, t) of the set k , $k \in \mathbb{Z}/3\mathbb{Z}$. The

copies of B_0 and B_1 are then joined according to the adjacency conditions

$$(t, 1, j, k, s) \sim \begin{cases} (s, 0, j, k, t) \\ (s, 0, 0, k-1+2j, t) \\ (s, 0, 1, k-1+2j, t) \end{cases} \quad (19)$$

as it is shown in Figure 6.

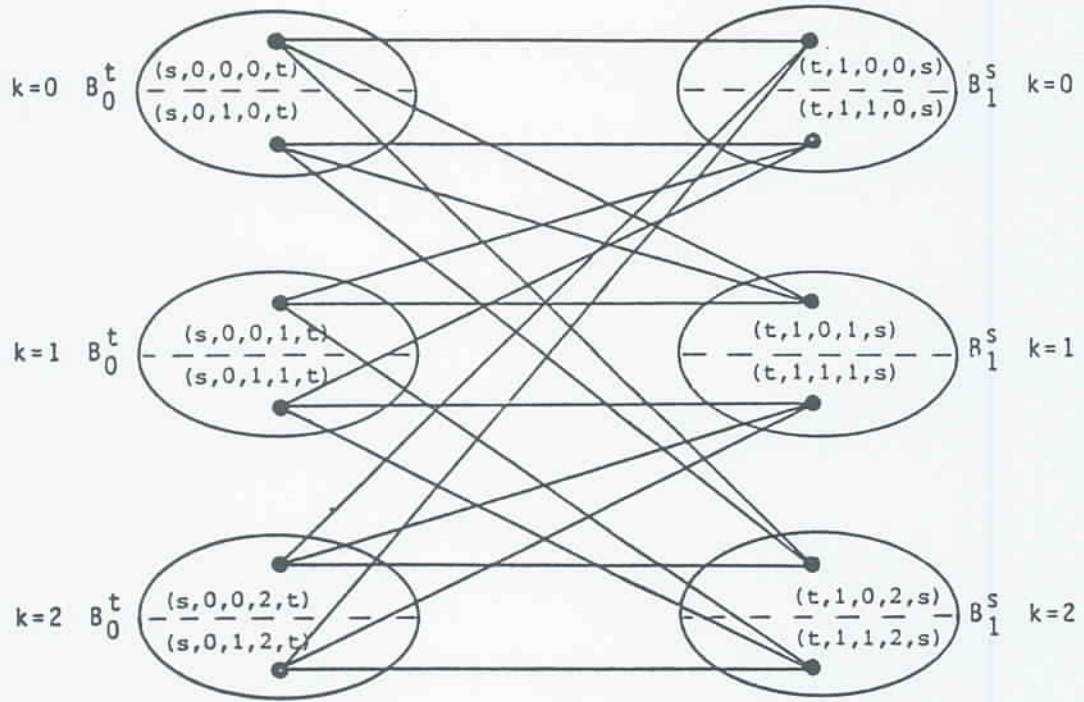


Figure 6

The resulting graph, which will be denoted by $B_0 \Sigma_1 B_1$, has degree $\Delta = \Delta_0 + 3$ and order $N = 3n_1 N_0 + 3n_0 N_1 = 3N_0 N_1$.

Theorem 2: The compound graph $B_0 \Sigma_1 B_1$ has diameter $D \leq D_0 + D_1$.

Proof:

The structure of the inter-copy interconnection shown in Figure 6 has the automorphisms

$$\begin{aligned} (s, i, j, k, t) &\mapsto (s, i, [j+k+1]_2, [k-1]_3, t) \\ (s, i, j, k, t) &\mapsto (s, i, [j+[2k]_3+1]_2, [-k]_3, t) \end{aligned} \quad (20)$$

—where $[]_n$ means to operate modulo n —which only leave the or bits respectively constituted by the left hand vertices ($i=0$) and the right hand vertices ($i=1$). Therefore it suffices to check the distances from the initial vertices $(s,0,0,0,t)$ and $(t,1,0,0,s)$. The different paths to be considered are then

a.1) from $(s,0,0,0,t)$ to $(x,1,1,m,y)$

a.2) from $(s,0,0,0,t)$ to $(x,0,1,m,y)$

b.1) from $(t,1,0,0,s)$ to $(x,0,1,m,y)$

b.2) from $(t,1,0,0,s)$ to $(x,1,1,m,y)$

we shall only analyse the cases a), since the proof for the cases b) is analogous. According to the possible destination vertices, the paths of length $\leq D_0 + D_1$ are the following:

$(\leq D_0 - 1) \quad (1) \quad (\leq D_1)$
-a.1 $(s,0,0,0,t), \dots, (y,0,1,1,t), (t,1,1,0,y), \dots, (x,1,0,0,y)$

$(\leq D_1 - 1)$
 $\dots, (x,1,1,0,y)$

$(1) \quad (\leq D_1 - 1)$
 $, (t,1,0,1,y), \dots, (x,1,0,1,y)$

$(\leq D_1)$
 $\dots, (x,1,1,1,y)$

$(1) \quad (\leq D_1 - 1)$
 $, (t,1,0,2,y), \dots, (x,1,0,2,y)$

$(\leq D_1)$
 $\dots, (x,1,1,2,y)$

-a.2

$(\leq D_0 - 1) \quad (1) \quad (\leq D_1 - 1) \quad (1)$
 $(s,0,0,0,t), \dots, (x,0,1,0,t), (t,1,0,1,x), \dots, (y,1,0,1,x), (x,0,0,0,y)$

(1)
 $, (x,0,1,0,y)$

$(\leq D_1 - 1) \quad (1)$
 $, (t,1,1,0,x), \dots, (y,1,1,0,x), (x,0,0,1,y)$

(1)
 $, (x,0,1,1,y)$

$(1) \quad (\leq D_1 - 1) \quad (1) \quad (\leq D_0 - 1)$
 $, (t,1,0,0,s), \dots, (y,1,0,0,s), (s,0,0,2,y), \dots, (x,0,1,2,y)$

$(1) \quad (\leq D_0 - 1)$
 $, (s,0,1,2,y), \dots, (x,0,0,2,y)$

Note that in the cases a.1 —both vertices are in copies of different graphs— the paths are obtained as in the example of 7.2 that is, we first choose the copy and then the vertex. However, some of these vertices can be now at distance

$D_0 + D_1$, so cases a.2 —both vertices in copies of the same graph— must be also considered.

8.2 The next constructions that we are going to define are similar to the preceding one. Therefore, we will limit ourselves to give the main characteristics of their basic configurations, as the initial graphs used, the inter-copy interconnection structure, the order of the resulting graph, etc. In fact the proofs, based in the same ideas used till now, are long and tedious but not difficult.

Considering the bipartite graphs B_0 and B_1 with (any) diameters D_0 and D_1 respectively, the compound graph $B_0 \Sigma_2 B_1$ is constructed by using the interconnection structure pictured in Figure 7, and it has $N = \frac{3}{2}N_0N_1$ vertices, degree $\Delta = \Delta_0 + 2$ ($=\Delta_1 + 2$) and diameter $D \leq D_0 + D_1$.

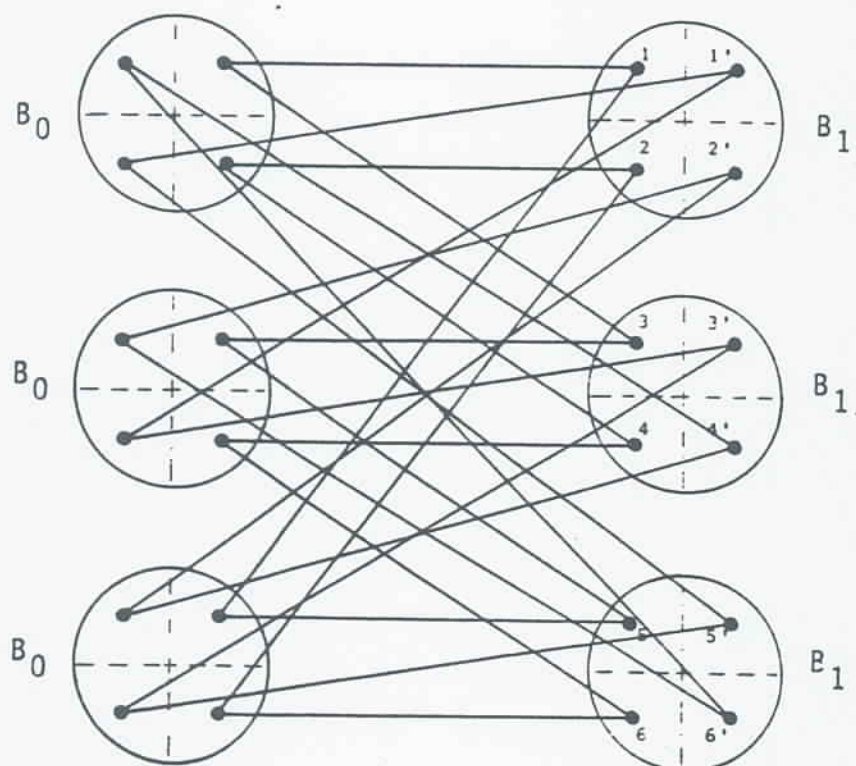


Figure 7

A construction derived of the preceding one (and hence denoted in the same way) assumes $\Delta_0 = \Delta_1 + 2$. It consists in identifying the vertices i and i' , $i = 1, 2, \dots, 6$, of the copies of B_1 , see Figure 7, and results in graphs with order $N = 9N_0N_1/4$, degree $\Delta = \Delta_0 + 2$ ($=\Delta_1 + 4$) and diameter $D \leq D_0 + D_1$.

8.3 As in the graphs DQ_Λ , to make this construction we use copies of a graph G_0 together with a bipartite graph B_1 . Moreover, the diameter of B_1 , D_1 , must now be *even*. The interconnection structure is shown in Figure 8.

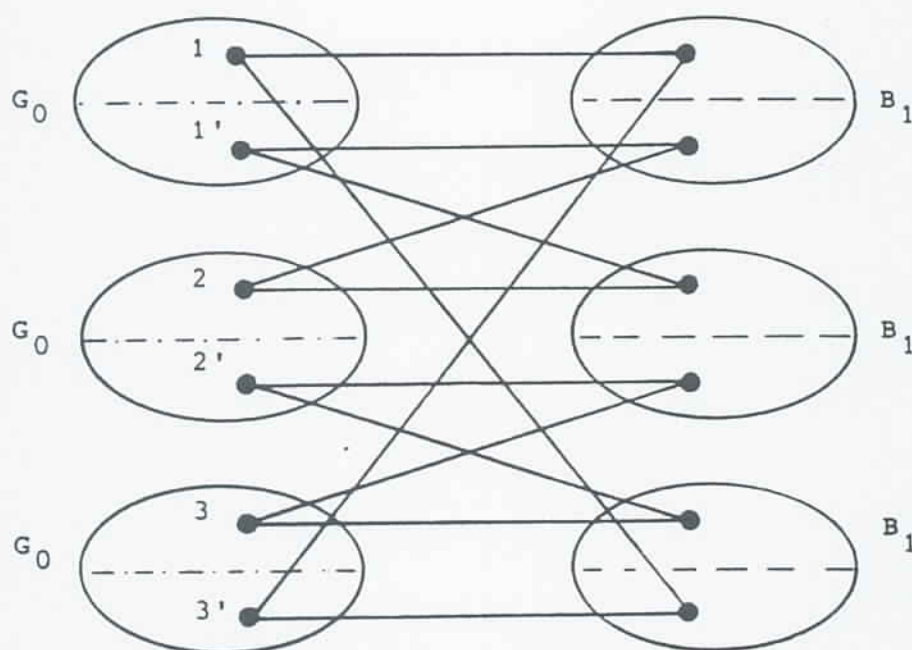


Figure 8

The resulting compound graph $G_0 \Sigma_3 B_1$ has $N = 3N_0N_1$ vertices, degree $\Delta = \Delta_0 + 2$ ($=\Delta_1 + 2$) and diameter $D \leq D_0 + D_1 + 1$. (as before, these parameters correspond to the so-called "basic configuration").

If $\Delta_0 + 2 = \Delta_1$, the vertices i and i' , $i = 1, 2, 3$, can be identified thus obtaining $N = \frac{9}{2}N_0N_1$ vertices, degree $\Delta = \Delta_0 + 4$ ($=\Delta_1 + 2$) and diameter $D \leq D_0 + D_1 + 1$. Furthermore, in this case D_1 can take any value (i.e. it does not need to be even).

8.4 Let us consider the preceding construction when G_0 is the trivial graph K_1 . Then the interconnection structure for the basic configuración ($\Delta_1 = 2$) is the one shown in Figure 9,

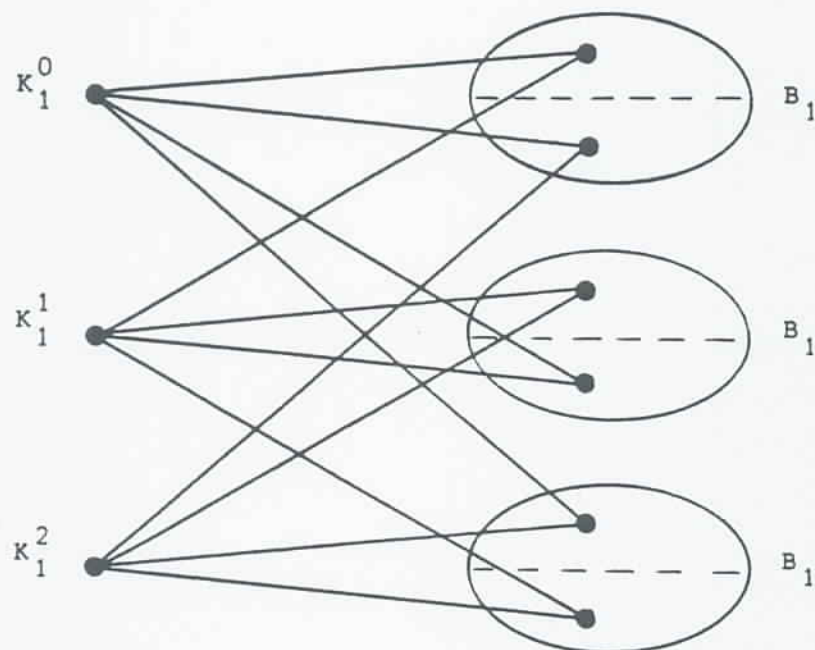


Figure 9

In general, if the degree of B_1 satisfies $\Delta_1 + 2 = 4\alpha + \beta$, $0 \leq \beta \leq 3$, it is possible to further increase the degree of the vertices K_1 up to 4α . Thus, the resulting graph $K_1 \Sigma_3 B_1$ has $N = 3(\frac{N_1}{2} + \alpha N_1) = \frac{3(2\alpha + 1)}{2} N_1$ vertices, degree $\Delta = \Delta_1 + 2$ and diameter $D \leq D_1 + 1$. Note that, if $\beta \neq 0$, the vertices K_1 will remain with a degree smaller than the vertices of the copies of B_1 . So, it would be interesting to modify the construction in order to take more copies of B_1 (increasing in β units the degree of such vertices K_1). Although there are many solutions, we will only define those that produce larger dense graphs than the ones previously known. In all cases it must be assumed that D_1 is *even*.

Let us see firstly a slight modification that permits to obtain an improvement even in the case $\beta = 0$. The added structure is shown in Figure 10. Note that, in it, there are also edges between the copies of B_1 .

Assuming $\Delta_1 + 2 = 4(\alpha - 1) + 4$, the resulting interconnection structure is the one obtained by identifying, for every $k = 0, 1, 2$, the vertices K_1^k of $\alpha - 1$ structures of Figure 9 and 1 structure of Figure 10. Thus we obtain a graph whose parameters are $N = \lceil 3N_1/2 \rceil + 3(\alpha - 1)N_1 + 4N_1 = \frac{6\alpha + 5}{2} N_1$, $\Delta = \Delta_1 + 2 = 4\alpha$

and $D \leq D_1 + 1$.

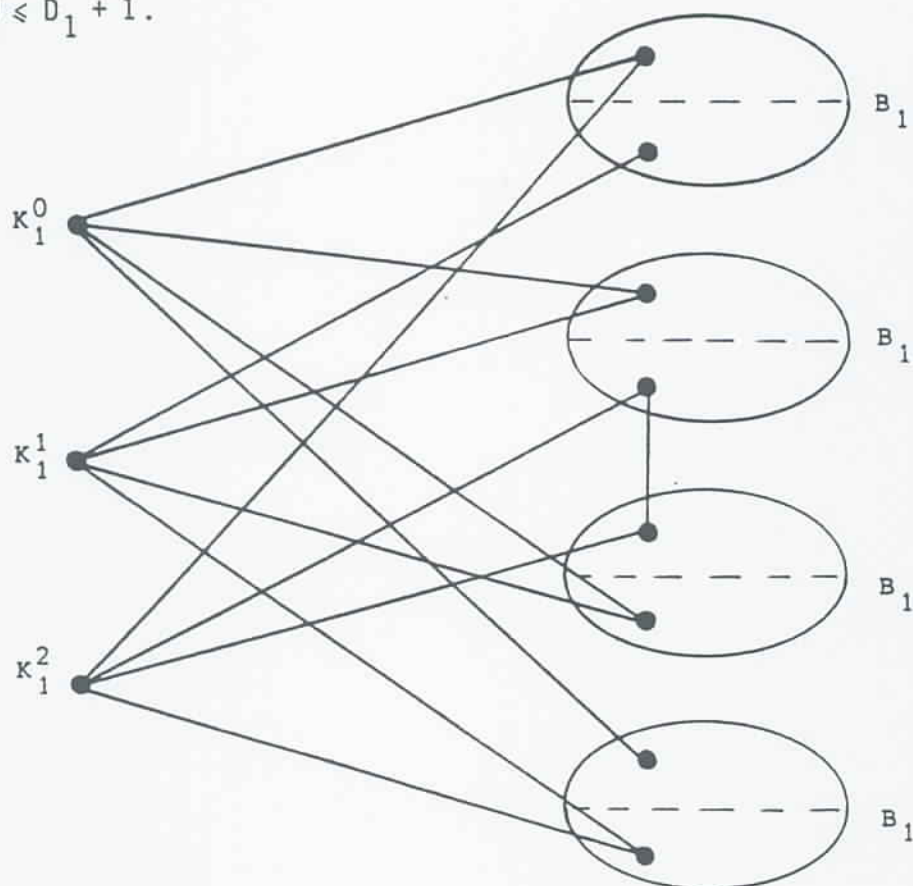


Figure 10

When $\beta = 2$, we can use the additional structure depicted in Figure 11. So, assuming $\Delta_1 + 2 = 4(\alpha - 1) + 6$ and as explained above, we obtain an interconnection structure that gives rise to compound graphs with parameters $N = \lceil 3N_1/2 \rceil + 3(\alpha - 1)N_1 + 6N_1 = \frac{6\alpha + 9}{2} N_1$, $\Delta = \Delta_1 + 2 = 4\alpha + 2$ and $D \leq D_1 + 1$.

Finally, when $\Delta_1 + 2 = 4\alpha + 3$ ($\beta = 3$), we can use the additional structure of Figure 12, from which we obtain graphs whose parameters are $N = \lceil 3N_1/2 \rceil + 3(\alpha + 1)N_1 = \frac{6\alpha + 9}{2} N_1$, $\Delta = \Delta_1 + 2 = 4\alpha + 3$ and $D \leq D_1 + 1$.

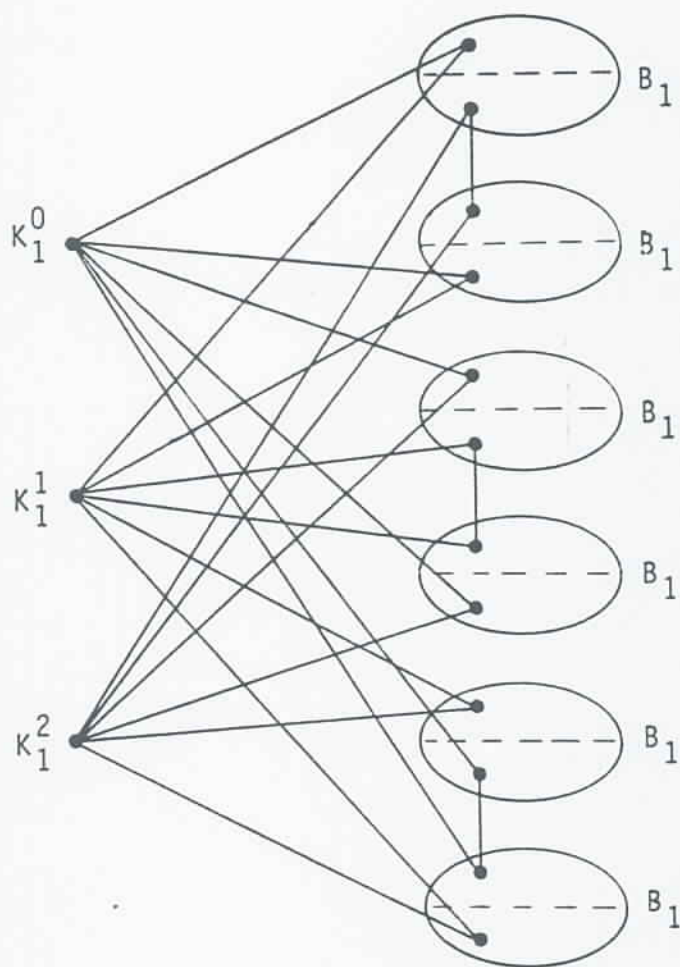


Figure 11

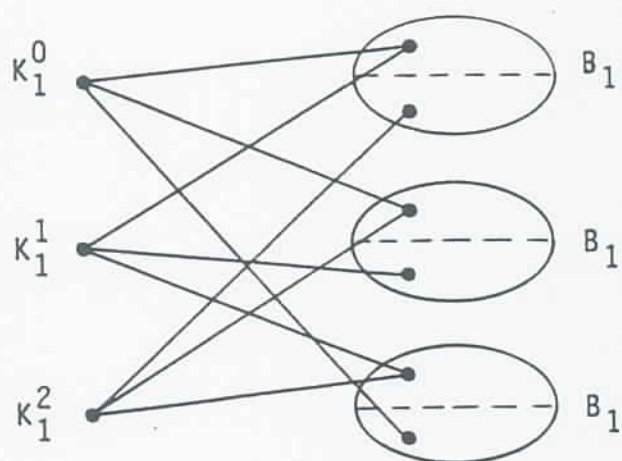


Figure 12

9. COMPOUND BIPARTITE GRAPHS $B_0 \Sigma B_1$

In this section we use the same ideas than in the preceding one to obtain dense bipartite graphs.

9.1 As in 8.1, let us consider *three* sets of copies of each of the bipartite graphs B_0 and B_1 . Then, we will join together these copies according to the interconnection structure shown in Figure 13 (the vertices belonging to each of the two sets of the resulting graph are denoted by A and B).

Assuming $\Delta_0 = \Delta_1$, we obtain in this way compound bipartite graphs, denoted by $B_0 \Sigma_4 B_1$, with $N = 3N_0N_1$ vertices, degree $\Delta = \Delta_0 + 2$ ($=\Delta_1 + 2$) and diameter $D \leq D_0 + D_1 + 1$.

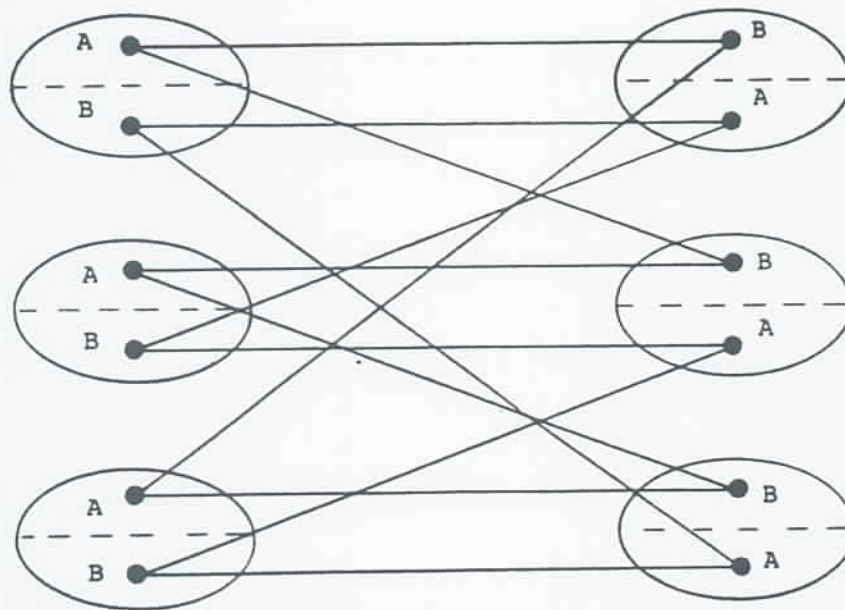


Figure 13

9.2 Other useful construction, in which the degree is increased a minimum of 3 units, is the one derived from the interconnection structure of Figure 14. The parameters of the resulting bipartite graph, $B_0 \Sigma_5 B_1$, are $N = 5N_0N_1$, $\Delta = \Delta_0 + 3$ ($=\Delta_1 + 3$) and $D \leq D_0 + D_1 + 1$.

9.3 Like in 8.4, we will now use copies of K_1 , i.e. isolated vertices. However the interconnection structure differs from the preceding ones and, in fact, the diameter obtained is not bounded by $D_0 + D_1 + 1 = D_1 + 1$ but by $D_1 + 2$.

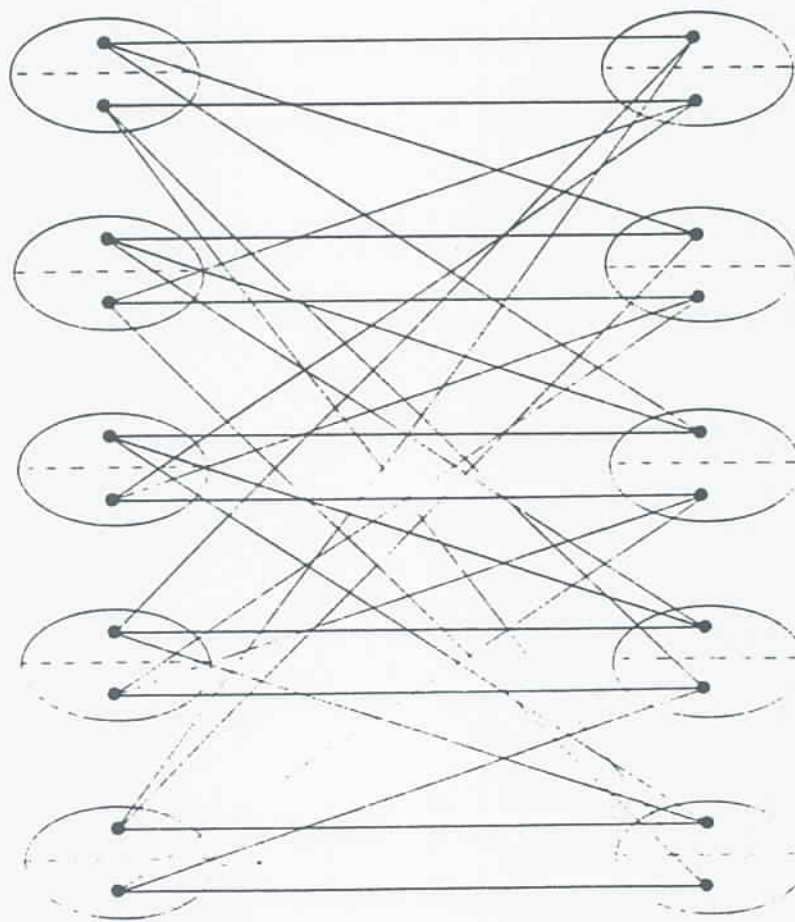


Figure 14

Let us take first *five* copies of a bipartite graph B_1 with diameter D_1 *even*, and *ten* sets of N_1 vertices each. Then every group of 10 vertices, one from each set, is joined to the copies of B_1 as shown in Figure 15.a. Now we increase till 4 the degree of such vertices by joining them to other 5 copies of B_1 as in Figure 15.b.

The bipartite graph so constructed has $N = 10 \frac{N_1}{2} + 10N_1 = 15N_1$ vertices, degree $\Delta = \max\{4, \Delta_1 + 2\}$ and diameter $D \leq D_1 + 2$. In general, if $\Delta_1 + 2 = 2\alpha + \beta$, $\alpha \geq 2$, $\beta = 0, 1$, we take $5(\alpha - 2)$ additional copies of B_1 in order to increase the degree of the isolated vertices till 2α by means of any of the two above structures. The resulting graph, which we denote by ΩB_1 , has the following parameters: $N = (10N_1/2) + 5\alpha N_1 = 5(\alpha + 1)N_1$, $\Delta = \Delta_1 + 2$ and $D \leq D_1 + 2$.

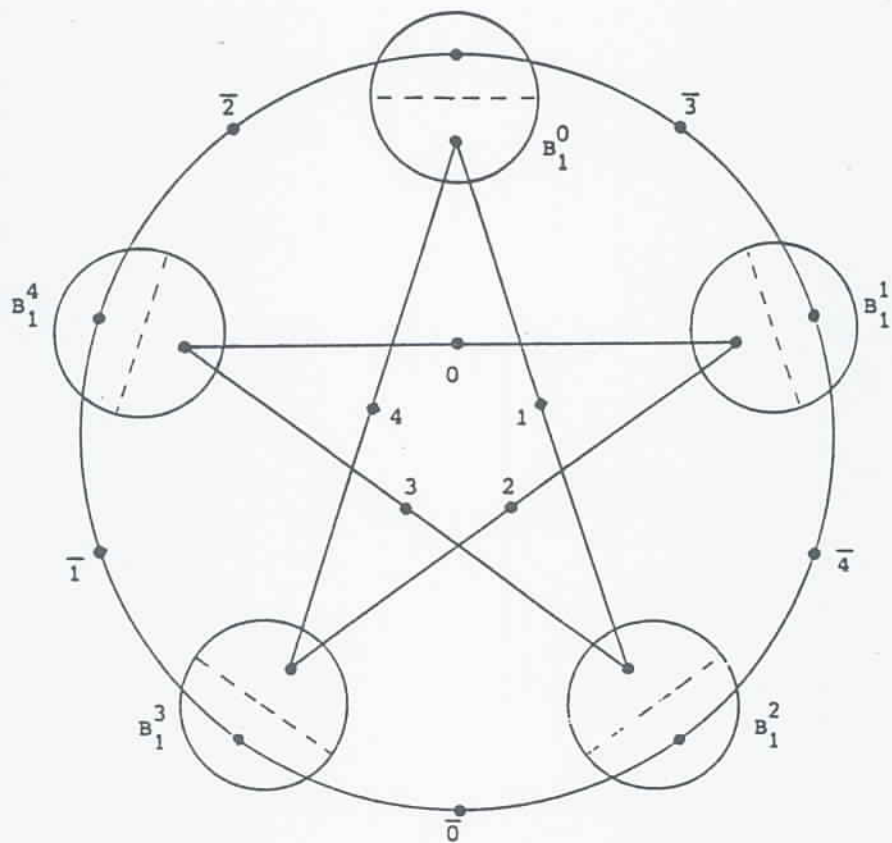


Figure 15.a

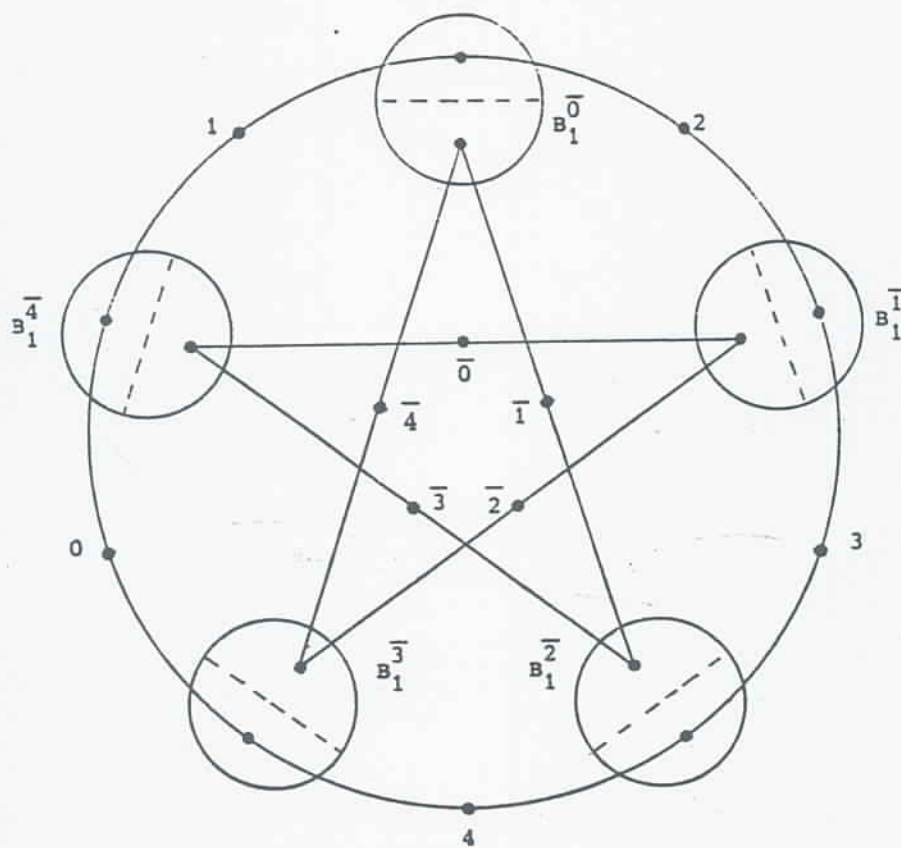


Figure 15.b

The new largest bipartite (Δ, D) graphs obtained with these constructions are the following (for the notation, see [1]):

$K_{6,6}\Sigma_4H_5$ (or $K_{4,4}\Sigma_4H_5$):	$N(8,9) \geq$	281 232
$K_{8,8}\Sigma_4H_9$:	$N(12,9) \geq$	9 565 920
$K_{8,8}\Sigma_4H_{11}$:	$N(14,9) \geq$	34 013 952
$K_{10,10}\Sigma_4H_{13}$ (or $K_{8,8}\Sigma_4H_{13}$):	$N(16,9) \geq$	96 536 160
$P_5\Sigma_4H_5$:	$N(8,10) \geq$	1 453 032
$P_9\Sigma_4H_9$:	$N(12,10) \geq$	72 541 560
$P_9\Sigma_4H_{11}$:	$N(14,10) \geq$	290 181 528
$P_{11}\Sigma_4H_{13}$:	$N(16,10) \geq$	962 948 196
$P_{11}\Sigma_5H_{11}$:	$N(15,10) \geq$	471 243 960
ΩP_3 :	$N(6,5) \geq$	520
ΩH_5 :	$N(8,8) \geq$	195 300
ΩH_7 :	$N(10,8) \geq$	1 176 480
ΩH_8 :	$N(11,8) \geq$	2 246 940
ΩH_9 :	$N(12,8) \geq$	4 650 100
ΩH_{11} :	$N(14,8) \geq$	14 172 480
ΩH_{11}^d :	$N(15,8) \geq$	14 978 080
ΩH_{13} :	$N(16,8) \geq$	36 201 060

10 GRAPHS $G < B$ AND $B(G^*)$

Let us consider the graphs $B = (V_0 \cup V_1, E_B)$ bipartite and $G = (V_G, E_G)$ with parameters N_B, Δ_B, D_B and N_G, Δ_G, D_G respectively. Assuming that $\min_{i=0,1} |V_i| \geq N_G$, we can join together these two graphs in the following way: for each vertex $a \in V_G$ we select two vertices $b \in V_0, c \in V_1$ (not selected before) and set up the new edges ab and ac , see Figure 16.

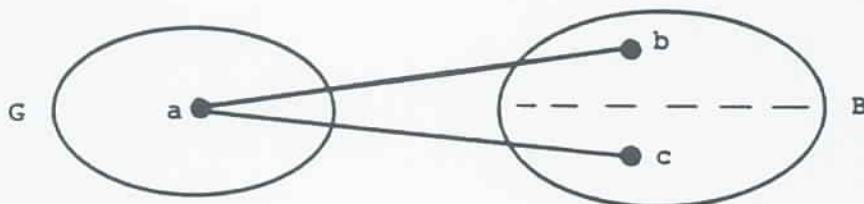


Figure 16

We obtain in this way a graph, denoted by $G < B$, with

$N = N_G + N_B$ vertices and degree $\Delta \leq \max\{\Delta_G + 2, \Delta_B + 1\}$. Note that, in it, there may be vertices of B non-adjacent to some vertex of G . To see that the diameter of $G \lt B$ is $D \leq \max\{D_G, D_B\}$, let us consider the vertices $s \in G$ and $t \in V_i$. By construction, s is adjacent to a vertex $u \in V_j$, $j = i + (D_B - 1) \bmod 2$, which is at distance $\leq D_B - 1$ of t .

If B is such that $|V_0| = |V_1| = N_B/2$ and D_B is even, we can take two copies of it, $G = B^0 = (V_0^0 \cup V_1^0, E_0)$ and $B^1 = (V_0^1 \cup V_1^1, E_1)$. Then, we make the above construction so that the vertices whose degree is increased in two units are all in V_0^0 (or in V_1^0), see Figure 17. In this case all vertices

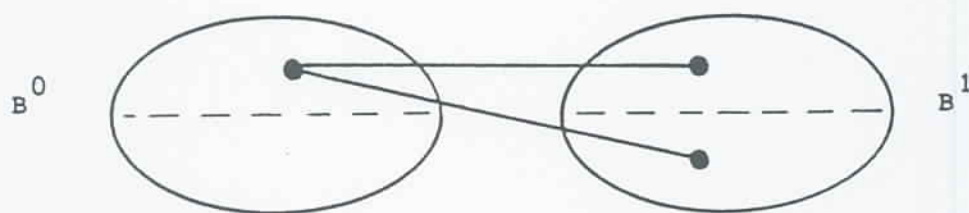


Figure 17

of B^1 are endvertices of an edge inter-copy and the resulting graph, $B(T_2)$, has order $N = 2N_B$, degree $\Delta_B + 1 \leq \Delta \leq \Delta_B + 2$ and, as we see next, diameter $D = D_B$.

Let us consider the vertices $s \in B^0$ and $t \in B^1$. It is necessary to distinguish two cases:

- a) If $s \in V_0^0$, it is adjacent to two vertices $t_0 \in V_0^1$ and $t_1 \in V_1^1$ one of which, at least, is at distance $\leq D_B - 1$ of t .
- b) If $s \in V_1^0$, it is at distance $\leq D_B - 1$ of a vertex s_0 that is adjacent to t .

This construction could be seen as a special kind of product of the graphs B and $K_{1,2}$. We can also see it as a compounding of B into the digraph T_2 (tournament of 2 vertices) which justifies the notation $B(T_2)$. This is the interpretation of Delorme and Quisquater in [6] for a similar construction that produces dense bipartite graphs denoted by $B\{G\}$. This lead us to the following generalization:

Let $G^* = (X, A)$ be a digraph on $N^* = |X|$ vertices, unilat-

erally connected (that is, for any $x, y \in X$, $d(x, y) < \infty$ or $d(y, x) < \infty$) and such that its unilateral diameter is D^* , i.e.

$$D^* = \max_{x, y \in V} \min\{d(x, y), d(y, x)\} \quad (21)$$

Then we construct the graph $B(G^*)$ in the following way: each vertex $x \in X$ of G^* is replaced by a copy of B , B^x , and each arc $xy \in A$ by N_B edges arranged as before, see Figure 17.

Thus, each arc from [to] x produces an increment of two [one] units in the degree of all vertices of V_0^x or V_1^x [V_0^x and V_1^x].

Therefore, we can do the construction so that the graph $B(G^*)$, on $N = N^*N_B$ vertices has degree

$$\Delta = \Delta_B + \max_{x \in V} \left\{ \delta^-(x) + 2 \left\lceil \frac{\delta^+(x)}{2} \right\rceil \right\} \quad (22)$$

Theorem 3: The graph $B(G^*)$ has diameter $D \leq D_B + D^* + 1$

Proof:

Let us consider two generic vertices $s \in B^x$ and $t \in B^y$ and assume that $d(x, y) = d \leq D^*$. Thus, in $B(G^*)$ there exist N_B paths of length d whose sets of endvertices are V_i^x ($i = 0$ or 1) and $V_0^y \cup V_1^y$. So we are in a similar situation than in the above case $G^* = T_2$ (in which we had $d = 1$) and, from here, the proof is the same that the one given there replacing paths of length 1 by paths of length d .

Two interesting particular cases of this construction, which produce the largest known (Δ, D) graphs, are obtained by taking $G^* = T_4, T_5$, the tournaments shown in Figure 18 (of course, the choice of them is not unique). The interconnection structure of the graph $B(T_4)$ is shown in Figure 19.

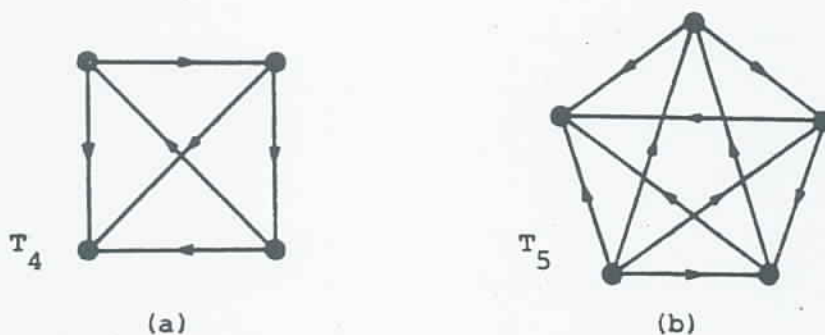


Figure 18

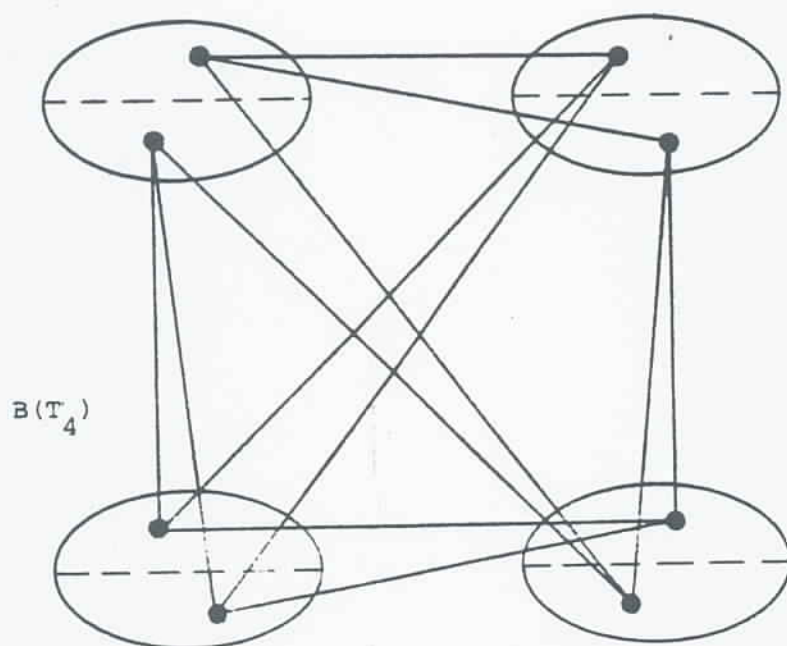


Figure 19

11 LARGEST KNOWN DENSE GRAPHS

In this section we give the table of largest (Δ, D) graphs known to date (July 1985) for $3 \leq \Delta \leq 16$ and $2 \leq D \leq 10$. Those marked with stars are compound graphs (the $(10, 4)$ one is the quotient graph by a polarity of a compound graph, see [6]). In particular, the graphs obtained by the new constructions proposed in this paper are indicated with two stars. Note that for $\Delta \geq 6$ and $D \geq 4$ almost all densest graphs are today obtained by compounding techniques.

Circled values are optimal. For example "Pet" is the Petersen's graph (Moore $(3, 2)$ graph) and "HS" is the Hoffman-Singleton's graph (Moore $(7, 2)$ graph). For further references about the notation used in the table see [1] and [6].

$\Delta \backslash D$	2	3	4	5	6	7	8	9	10
3	Pet (10)	$C_5^*F_4$ (20)	* YFA 38	* YFA 70	H_2r 128	CR 180	* YFA 286	CR 462	CR 708
4	$K_3^*C_5$ (15)	Pet $^*F_4$ 40	$C_5^*C_{19}$ 95	H_3^r 364	H_3r 731	CR 856	CR 1 872	CR 3 708	CR 7 100
5	$K_3^*X_8$ (24)	* YFA 66	Q_4r 174	H_3^rd 532	H_4r 2 734	H_4idr 2 988	CR 7 600	* BDQ 11 340	* BDQ 30 240
6	$K_4^*X_8$ 32	$C_5^*C_{21}$ 105	Pet $^*m(32)$ 320	CR 820	H_5r 7 817	* $H_4\Delta pt$ 12 285	* $H_4\Delta K_4$ 21 840	* $H_4\Delta(24)$ 65 520	** $Q_4^7H_4$ 185 640
7	HS (50)	CR 122	(15) $^*m(32)$ 480	CR 1 550	** $H_4r < H_5$ 10 546	* $H_5\Delta pt$ 35 154	* $H_5\Delta K_4$ 78 120	* $H_5\Delta(24)$ 281 232	** $Q_5^7H_5$ 1 218 672
8	P_7^r 57	HS $^*F_4$ 200	Q_7r 807	** $K_1\Gamma_3Q_5$ 2 652	H_7r 39 223	** $K_1\Gamma_3H_5$ 66 402	** $K_5\Gamma_3H_5$ 175 770	** $32\Gamma_3H_5$ 749 952	** $Q_5\Gamma_2H_5$ 3 656 016
9	P_8^rd 74	Q_8^rd 585	** $Q_5(T_4)$ 1 248	* HS $]C_{103}[$ 5 150	H_8r 74 906	* $H_4\Delta pt$ 215 688	* $H_7\Delta K_5$ 588 240	* $H_7\Delta HS$ 2 941 200	** $Q_7^7H_7$ 15 686 400
10	P_9^r 91	Q_8^rd 650	($Q_8(T_3)$) 1 755	** $K_1\Gamma_3Q_7$ 8 400	H_9r 132 869	* $H_8\Delta pt$ 486 837	* $H_8\Delta K_6$ 1 348 164	* $H_8\Delta HS$ 7 489 800	** $Q_7\Gamma_2H_7$ 47 059 200
11	P_9^rd 94	Q_8^rd 715	** $Q_7(T_4)$ 3 200	** $K_1\Gamma_3Q_8$ 12 285	** $H_7(T_4)$ 156 864	* $H_9\Delta pt$ 863 590	* $H_9\Delta K_6$ 2 790 060	** HS Γ_3H_8 16 852 050	* $H_9\Delta Q_8$ 120 969 030
12	P_{11}^r 133	Q_8^rd 780	$Q_8^*X_8$ 4 680	** $K_1\Gamma_3Q_9$ 18 860	$H_{11}r$ 354 323	** $K_1\Gamma_3H_9$ 1 527 890	** $K_{6,6}\Gamma_2H_9$ 4 782 960	** $P_9\Gamma_2H_9$ 36 270 780	** $Q_9\Gamma_2H_9$ 326 835 600
13	P_{11}^rd 136	Q_8^rd 845	** $Q_9(T_4)$ 6 560	$Q_8^*mP_7$ 33 345	** $H_9(T_4)$ 531 440	* $H_{11}\Delta pt$ 2 657 340	* $H_{11}\Delta K_7$ 9 920 736	** $P_9\Gamma_1H_9$ 72 541 560	** $Q_9^7H_{11}$ 581 071 680
14	P_{13}^r 183	Q_8^rd 910	** $Q_9(T_5)$ 8 200	$Q_8^*mP_8$ 42 705	$H_{13}r$ 804 481	** $K_1\Gamma_3H_{11}$ 4 783 212	** $K_7\Gamma_3H_{11}$ 18 601 380	** $P_{11}\Gamma_3H_{11}$ 145 090 764	** $Q_{11}\Gamma_2H_{11}$ 1 556 138 304
15	P_{13}^rd 186	($Q_{2,4}$) 1 215	** $Q_{11}(T_4)$ 11 712	* $P_{11}]C_{414}[$ 55 062	** $H_{11}(T_4)$ 1 417 248	* $H_{13}\Delta pt$ 6 837 978	* $H_{13}\Delta K_6$ 28 960 848	** $P_{11}\Gamma_1H_{11}$ 282 740 976	** $Q_{11}^7H_{13}$ 2 355 482 304
16	P_{13}^rd 197	(Q_3) 1 600	** $Q_{11}(T_5)$ 14 640	(Q_{H_3}) 132 496	** $H_{11}(T_5)$ 1 771 560	** $K_1\Gamma_3H_{13}$ 11 664 786	** $K_9\Gamma_3H_{13}$ 54 301 590	** $P_{11}\Gamma_2H_{13}$ 481 474 098	** $Q_{13}\Gamma_2H_{13}$ 5 743 901 520

Table of largest known (Δ, D) graphs (July 1985).

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REFERENCES

- [1] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Grands graphs de degré et diamètre fixés, Proc. Coll. C.N.R.S., Marseille, June 1981. *Ann. Discr. Math.*, North-Holland, vol. 17, 1982, pp. 65-73.
- [2] J.-C. Bermond, C. Delorme and J.-J. Quisquater, Strategies for interconnection networks: Some methods from graph theory, to be published.
- [3] N. Biggs, *Algebraic graph Theory*, Cambridge University Press, Cambridge, England, 1974.
- [4] C. Delorme, Large bipartite graphs with given degree and diameter, J. Graph Theory, to be published.
- [5] C. Delorme and G. Farhi, Large graphs with given degree and diameter -Part I, IEEE Trans. Comp. , vol. C-33, no. 9, Sep. 1984, pp. 857-860.
- [6] C. Delorme and J.-J. Quisquater, Manuscript about large graphs to be published (1984).
- [7] M. A. Fiol y J. Fàbrega, Algunos grafos compuestos, Stochastica, vol. VII, no. 2, 1983, pp. 137-143.