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## Large Graphs with Given Degree and Diameter — Part I

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**Abstract** — The following problem arises in the study of interconnection networks: find graphs of given diameter and degree having the maximum number of vertices. In this correspondence we give some constructions of graphs proving in particular that  $\lim_{\Delta \rightarrow \infty} \inf N(\Delta, D) \cdot \Delta^{-D} \geq 2^{-D}$ , where  $N(\Delta, D)$  is the maximum number of vertices of a graph with degree  $\Delta$  and diameter  $D$ .

If the diameter is 2 or 4 modulo 6, we give another construction ensuring the inequality  $\lim_{\Delta \rightarrow \infty} \inf N(\Delta, D) \Delta^{-D} \geq 3 \cdot 2^{-D}$ .

**Index Terms** —  $(d, k)$ -graph problem, interconnection networks, regular networks, undirected graphs.

### I. INTRODUCTION

We are interested in the  $(\Delta, D)$  graph problem, which arises in telecommunication networks, or microprocessor networks.

Let  $G = (X, E)$  be an undirected graph with vertex set  $X$  and edge set  $E$ . The distance  $\delta(x, y)$  between two vertices  $x, y$  is the length of a shortest path between  $x$  and  $y$ . The diameter  $D$  of  $G$  is  $D = \max_{(x, y) \in X^2} \delta(x, y)$ . The degree of a vertex  $x$  is the number of vertices adjacent to  $x$ ; the degree of  $G$  is the maximum degree of its vertices.

The  $(\Delta, D)$  problem consists of finding the maximum number of vertices  $N(\Delta, D)$  of a graph with given degree  $\Delta$  and diameter  $D$ .

This problem arises in the study of interconnection networks: the vertices represent the stations (or processors), the degree of a vertex is the number of links incident to it, the diameter represents the maximum number of links to be used to transmit a message.

It is known that  $N(\Delta, D)$  satisfies Moore inequalities

$$N(2, D) \leq 2D + 1 \quad \text{and for } \Delta > 2,$$

$$N(\Delta, D) \leq (\Delta(\Delta - 1)^D - 2)/(\Delta - 2).$$

There exist graphs with that number of vertices for  $\Delta = 2$  (namely the  $(2D + 1)$  cycles), for  $D = 1$  (namely the  $(\Delta + 1)$  clique), for  $D = 2$  and  $\Delta = 3$  (Petersen graph), for  $D = 2$  and  $\Delta = 7$  (Hoffman-Singleton graph), and perhaps for  $D = 2$  and  $\Delta = 57$ .

Many authors have described various graphs with given degree and diameter, having many vertices; for a survey, see [2], [7], or [20]. Some of these authors use the letters  $d$  for degree and  $k$  for diameter, but  $\Delta$  and  $D$  used here come from graph-theorists' customs.

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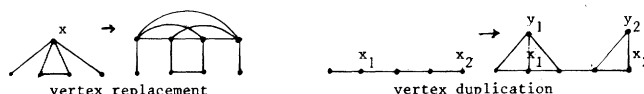


Fig. 1. Ascending behavior of  $N(\Delta, D)$ .

We give graphs with at least  $(\Delta/2)^D$  vertices (with  $\Delta$  even), and graphs with about  $3(\Delta/2)^D$  vertices if  $D \equiv 2$  or  $4 \pmod 6$  (with  $\Delta$  even).

Some tedious details are omitted in this correspondence, but they are written in [10].

### A. Some Inequalities

**Proposition:**  $N(\Delta, D + 1) \geq N(\Delta, D) + \Delta - 1$ .

**Proof:** We may replace a vertex  $x$  of degree  $\Delta$  by a complete graph on  $\Delta$  vertices, and connect each new vertex to a different neighbor of  $x$ .

**Proposition:**  $N(\Delta + 1, D) \geq N(\Delta, D) + \lceil N(\Delta, D)/(\Delta^2 + 1) \rceil$ .

**Proof:** Let  $t = \lceil N(\Delta, D)/(\Delta^2 + 1) \rceil$ . We can find at least  $t$  vertices  $x_i$  with mutual distances at least 3; we can then add  $t$  new vertices  $y_i$ , with  $y_i$  adjacent to  $x_i$  and to the neighbors of  $x_i$ .

Some improvements of these ideas can be found in [8] and in [19].

**Proposition:**  $N(\Delta + \Delta', D + D') \geq N(\Delta, D)N(\Delta', D')$ .

**Proof:** Consider the Cartesian sum of two graphs with diameter  $D$  (respectively,  $D'$ ) degree  $\Delta$  (respectively,  $\Delta'$ ) and  $N(\Delta, D)$  (respectively,  $N(\Delta', D')$ ) vertices. Its diameter is  $D + D'$ , its degree is  $\Delta + \Delta'$ , and it has  $N(\Delta, D)N(\Delta', D')$  vertices.

Some improvements of this idea can be found in [3], [4].

### B. Graphs of Type $C_s$

**Definition:** Let us call graphs of type  $C_s$  the graphs built in the following way. The vertex set is  $Z/mZ \times E^k$ , where  $Z/mZ$  is the set of residues modulo  $m$  in the integers (with  $m \geq 1$ ), and  $E$  is a set with  $n$  elements (with  $n \geq 2$ ). Thus, the graph has  $mn^k$  vertices. Each vertex is defined by its coordinates  $(a, e_1, e_2, \dots, e_k)$  where  $a \in Z/mZ$  and the  $e_i$ 's are in  $E$ . The vertices adjacent or equal to  $(a, e_1, \dots, e_k)$  are the  $n$  vertices  $(a + 1, e_2, \dots, e_k, x)$ , where  $x \in E$ , and the  $n$  vertices  $(a - 1, x, e_1, e_2, \dots, e_{k-1})$ , where  $x \in E$ .

Thus, the degree of each vertex is at most  $2n$  (it may be less than  $2n$  if  $m$  is 1 or 2) and the degree of the graph is at most  $2n$ . It is  $2n$  if  $m \geq 3$ , it is  $2n - 1$  if  $m = 1$  or  $2$  and  $k = 2$ , it is  $n$  if  $k = 1$  and  $m = 2$  (the graph is then the complete bipartite graph  $K_{n,n}$ ), it is  $n - 1$  if  $k = m = 1$  (the graph is then the complete graph  $K_n$ ).

The cases with  $k = m - 1$  were described in [18].

The case where  $m = 1$ ,  $|E| = 2$  was described by N. G. de Bruijn [21] with a quite different goal.

**Theorem:** The diameter of the graph of type  $C_s$  is  $k + \sup_{b \in Z/mZ} (\inf(|b - k|, |b + k|))$ , where  $|b|$  is the smallest integer which is  $\geq 0$  and such that  $b$  or  $-b$  is in the same residue class modulo  $m$  as  $|b|$  (in other words,  $|b - k|$  is the distance between  $b$  and  $k$  on the cycle on  $m$  vertices).

**Corollary:** If  $m = 1$ , then  $D = k$ ; we thus have graphs of diameter  $D$  and degree  $\Delta$  ( $\Delta$  even), with  $n = (\Delta/2)^D$  vertices. Thus,  $\liminf_{\Delta \rightarrow \infty} N(\Delta, D) \Delta^{-D} \geq 2^{-D}$ .

**Proof:** Since the map of the vertex set of the graph into itself, given by  $(a, e_1, e_2, \dots, e_k) \rightarrow (a + \alpha, e_1, \dots, e_k)$ , is an automorphism of the graph, it is sufficient to compute the distance of the vertices of the graph to the vertices with first coordinate equal to 0. Let us call  $X_a$  the part of the graph containing the vertices with first coordinate equal to  $a$ . It is clear that among the vertices at distance  $\leq d$  ( $d \leq k$ ) from  $(0, e_1, \dots, e_k)$  we find all vertices with coordinates  $(d, e_{d+1}, e_{d+2}, \dots, e_k, x_1, x_2, \dots, x_d)$ , where the  $x_i$ 's are arbitrary elements of  $E$ . Thus, there exists a path of length at most  $k$  between every vertex in  $X_0$  and every vertex in  $X_k$ . Also, every vertex

in  $X_0$  is at distance at most  $k$  from every vertex in  $X_{-k}$ . But every vertex in  $X_b$  is at distance  $|b - \pm k|$  of some vertex in  $X_{\pm k}$ . Thus, its distance to every vertex in  $X_0$  is at most  $\inf(|b - k|, |b + k|) + k$ , the diameter of the graph is at most  $\sup_{b \in \mathbb{Z}/m\mathbb{Z}} \inf(|b - k|, |b + k|) + k$ .

One can check that the diameter is indeed the number above, but the proof is rather tedious, and we do not give it here. (See [10] for more details.) However, if  $m = 1$  and thus  $k \geq D$ , the number of the vertices of the graphs exceeds the Moore bound corresponding to diameter  $k - 1$  if  $n$  is big enough; the diameter is then certainly  $k$ .

*Example:* To obtain diameter 4, we can use the following values of  $k$  and  $m$ :

$$\begin{aligned} k &= 1 \text{ or } 2, & m &= 9 \\ k &= 3, & m &= 4 \\ k &= 4, & m &= 1. \end{aligned}$$

If  $n$  is 5 or more, a best graph of type  $C_s$  is obtained with  $m = 1$  and  $k = D$ .

These graphs  $C_s$  with  $m = 1$  are described by Imase and Ito [14]. These authors use a clever arithmetical representation: the set  $E$  is  $\{0, 1, \dots, n - 1\}$ , the sequence  $e_1, e_2, \dots, e_k$  is interpreted as  $e_1 + e_2n + \dots + e_k n^{k-1}$ . They use these graphs in the oriented case. However, an improvement is possible even in the oriented case [12]: the construction is the same, but two consecutive coordinates  $e_i, e_{i+1}$  are different; the graph obtained by this method has diameter  $k$ , maximum degree  $2n - 2$  (if  $k \geq 3$  and  $n \geq 3$ ), and it has  $n(n - 1)^{k-1}$  vertices.

*Remark:* The graph  $C_s$  is vertex transitive if  $m \geq k$ , but it is not vertex transitive if  $m < k$  since vertices where coordinates verify  $e_{i+m} = e_i$  for  $1 \leq i \leq k - m$  are in more cycles of length  $m$  than the other vertices.

Similar remarks about graphs of type  $C'_s, C_p$  described later in this correspondence could be done.

### C. Graphs of Type $C'_s$ ( $m$ Even)

To obtain graphs with an odd degree, we slightly modify the construction given above.

The vertex set is  $\mathbb{Z}/m\mathbb{Z} \times E^k \times E'^{k'}$ , with  $m$  even  $\geq 2$ , with  $k' = k$  or  $k + 1$  and  $k \geq 1, E$  has  $n$  elements (with  $n \geq 2$ ),  $E'$  has  $n'$  elements (with  $n' \geq 1$ ) (and so the graph has  $m n^k n'^{k'}$  vertices).

If  $a$  is an odd element of  $\mathbb{Z}/m\mathbb{Z}$  (this has sense because  $m$  is even) then the vertex  $(a, e_1, \dots, e_k, e'_1, \dots, e'_{k'})$  is adjacent to the vertices  $(a + 1, e_2, \dots, e_k, x, e'_1, \dots, e'_{k'})$  and  $(a - 1, e_1, \dots, e_k, y, e'_1, \dots, e'_{k'-1})$ , where  $x$  is any element of  $E$  and  $y$  is any element of  $E'$ . Thus, its degree is at most  $n + n'$ . If  $a$  is even, then the vertex  $(a, e_1, \dots, e'_1, \dots)$  is adjacent to  $(a + 1, e_1, \dots, e_k, e'_2, \dots, e'_{k'}, y)$  and to  $(a - 1, x, e_1, \dots, e_{k-1}, e'_1, \dots)$  and its degree is at most  $n + n'$ . Thus, the degree of the graph is at most  $n + n'$ . Now we give the diameter.

*Theorem:* The diameter of the graph of type  $C'_s$  is  $D$ , with

$$\begin{aligned} \text{if } k' = k, \quad D &= k + k' \\ &+ \sup_{b \in \mathbb{Z}/m\mathbb{Z}} (\inf(|b - k - k'|, |b + k + k'|)) \\ \text{if } k' = k + 1, \\ D &= k + k' + \sup_{b \in \mathbb{Z}/m\mathbb{Z}} (\inf(|b - k - k'|, |b + k + k' + 1| + 1, \\ &|b + k + k' - 1| + 1)). \end{aligned}$$

*Proof:* Since the maps of the vertex set of the graph into itself, given by  $(a, e_1, \dots, e'_1, \dots) \rightarrow (a + 2\alpha, e_1, \dots, e'_1, \dots)$  and by  $(a, e_1, \dots, e'_1, \dots) \rightarrow (2\alpha + 1 - a, e_1, \dots, e'_1, \dots)$  are automorphisms of the graph, it is sufficient to compute the distances of the vertices of the graph to the vertices in  $x_0$  (whose first coordinate is 0). It is clear that among the vertices at distance  $2d \leq k + k'$  from  $(0, e_1, \dots, e'_1, \dots)$  we find all vertices with coordinates

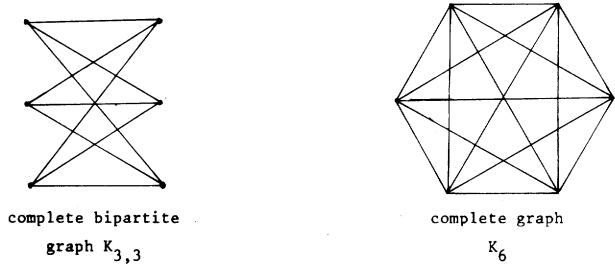
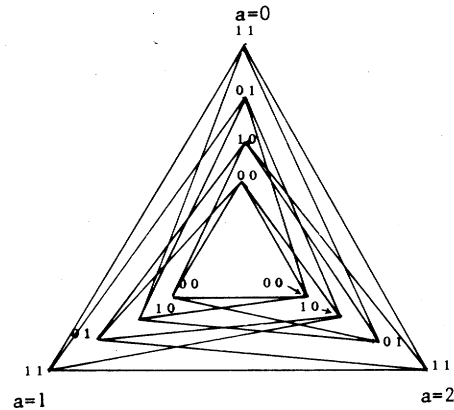


Fig. 2. Graphs  $C_s$  with small  $m$  and  $k$ .



the graph  $C_s$  with  
 $m=3, k=2, n=2$   
Fig. 3. A typical graph  $C_s$ .

$(2d, e_{d+1}, \dots, e_k, x_1, \dots, x_d, e'_{d+1}, \dots, e'_{k'}, x'_1, \dots, x'_d)$  and among the vertices at distance  $2d + 1 \leq k + k'$  from  $(0, e_1, \dots, e'_1, \dots)$  we find all vertices  $(2d + 1, e_{d+1}, \dots, e_k, x_1, \dots, x_d, e'_{d+2}, \dots, e'_{k'}, x'_1, \dots, x'_{d+1})$ , where the  $x_i$ 's are arbitrary elements of  $E$  and the  $x'_i$ 's are arbitrary elements of  $E'$ . Thus, every vertex in  $X_{k+k'}$  is at distance at most  $k + k'$  of every vertex in  $X_0$ .

If  $k = k'$ , we see that every vertex in  $X_{-k-k'}$  is also at distance at most  $k + k'$  of every vertex in  $X_0$ ; if  $k' = k + 1$ , every vertex in  $X_{-k-k'-1}$  is at distance at most  $k + k'$  of every vertex in  $X_{-1}$  and every vertex in  $X_{-k-k'+1}$  is at distance at most  $k + k'$  of every vertex in  $X_1$ . Since every vertex in  $X_b$  is at distance  $b - b'$  of some vertex in  $X_{b'}$ , we see that the diameter is at most the number given by the formula above.

Here again, we do not prove that the diameter is not less than the given number.

One can prove that the formula for  $k' = k + 1$  holds again with  $n = 1$  and  $n' \geq 2$ , and  $k \geq 0$  thus providing graphs of degree  $\geq 3$ .

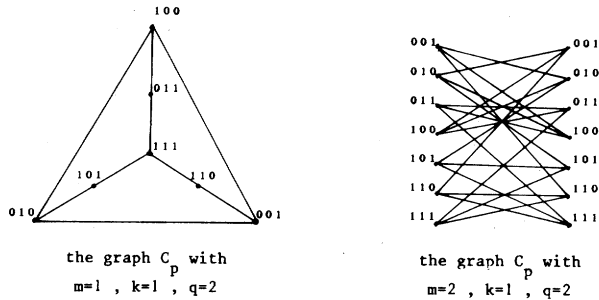
These  $C'_s$  graphs do not give best values in the  $(\Delta, D)$ -graph problem; however, they deserve some consideration because they give good results in the bipartite  $(\Delta, D)$ -graph problem, and they provide a basis for some constructions giving the best results [8], [13].

### D. Projective Planes with Polarity

If  $q$  is a power of a prime, there exists a field with  $q$  elements. We consider the set  $P$  of one-dimensional vector subspaces in the three-dimensional vector space  $K^3$ . This set  $P$  is called projective plane over  $K$ . It has  $q^2 + q + 1$  elements that can be represented by nonnull vectors, that is, triples of elements of  $K$ , not  $(0, 0, 0)$ . The two-dimensional subspaces of  $K^3$  are unions of  $q + 1$  one-dimensional subspaces of  $K^3$ . The corresponding subsets of  $P$  are the lines of  $P$ . Since every one-dimensional subspace of  $K^3$  is included in  $q + 1$  two-dimensional subspaces, every point of  $P$  is on  $q + 1$  lines of  $P$ . We now define a map on  $P$ : to a point  $M$  of  $P$ , represented by the triple  $(x_0, y_0, z_0)$ , we associate the line  $D(M)$  of  $P$ , the points of which are represented by the (nonnull) triples  $(x, y, z)$  such that  $xx_0 + yy_0 + zz_0 = 0$ . This is a polarity since it is

TABLE I  
ORDERS OF GRAPHS  $C_p$ 

| diameter D<br>degree $\Delta$ | 2  | 3   | 4    | 5    | 6    | 7     | 8      | 9       | 10       |
|-------------------------------|----|-----|------|------|------|-------|--------|---------|----------|
| 2                             | 3  | 6   |      |      |      |       |        |         |          |
| 3                             | 7  | 14  |      |      |      |       |        |         |          |
| 4                             | 13 | 26  | 27   | 63   | 117  | 243   | 297    | 513     | 1053     |
| 5                             | 21 | 42  |      |      |      |       |        |         |          |
| 6                             | 31 | 62  | 147  | 343  | 637  | 3087  | 7203   | 14406   | 50421    |
| 7                             |    |     |      |      |      |       |        |         |          |
| 8                             | 57 | 114 | 507  | 1183 | 2197 | 19773 | 85683  | 171366  | 1113879  |
| 9                             | 73 | 146 |      |      |      |       |        |         |          |
| 10                            | 91 | 182 | 1323 | 3087 | 9261 | 83349 | 583443 | 1166886 | 12252303 |

Fig. 4. Graphs  $C_p$  with small  $m$  and  $k$ .

a one-to-one map from the points to the lines of  $P$ , and that  $N \in D(M)$  holds if and only if  $M \in D(N)$  holds.

#### E. Graphs of Type $C_p$

The vertex set is  $Z/mZ \times P^k$ , where  $P$  is a projective plane. The vertex  $(a, p_1, p_2, \dots, p_k)$  is adjacent to the vertices  $(a+1, p_2, \dots, p_k, x_1)$ , where  $x_1$  is any element of  $D(p_1)$  and to the vertices  $(a-1, x_k, p_1, \dots, p_{k-1})$ , where  $x_k$  is any element of  $D(p_k)$ . Thus, the degree is  $2(q+1)$  if  $m \geq 3$  or  $k \geq 2$ , it is  $q+1$  if  $k=1$  and  $m=1$  or  $2$ . The number of vertices is  $m(q^2 + q + 1)^k$ .

**Theorem:** The diameter of a graph of type  $C_p$  is  $D$  with

$$D = 2k + \sup_{b \in Z/mZ} (\inf(|b|, |b-2k|, |b+2k|)).$$

**Proof:** Once again, the map  $(a, p_1, \dots, p_k) \mapsto (a+\alpha, p_1, \dots, p_k)$  is an automorphism, and we need only the distances to the vertices in  $X_0$ . Among the vertices at distance at most  $d, d \leq k$ , from  $(0, p_1, \dots, p_k)$ , we find the vertices  $(d, p_{d+1}, \dots, p_k, x_1, \dots, x_d)$ , and the vertices  $(-d, x_{k-d+1}, \dots, x_k, p_1, \dots, p_{k-d})$ , where  $x_i$  is any vertex of  $D(p_i)$ .

Now, among the vertices at distance at most  $k$  from  $(0, p_1, \dots, p_k)$  and  $(2k, p_1, \dots, p_k)$  we find  $(k, p'_1, \dots, p'_k)$ , where  $p'_i$  is a point common to the lines  $D(p_i)$  and  $D(p'_i)$  (there always exists such a point); thus, the distance between any vertex in  $X_0$  and  $X_{2k}$  is at most  $2k$ . The same holds with  $X_0$  and  $X_0$  and also with  $X_0$  and  $X_{-2k}$ . Since there exists a path of length at most  $b-b'$  between any vertex in  $X_b$  and some vertex in  $X_{b'}$ , we have shown that the diameter is at most the number given by the formula. We do not prove here the equality.

One can check that if  $k$  is not a multiple of 3, and  $m=3$ , then the diameter is  $2k$ . The type  $C_p$  thus gives a graph with  $3((\Delta^2/4) - (\Delta/2 + 1))^{D/2}$  vertices if  $\Delta$  is  $2q+2$ , where  $q$  is a power of a prime and  $K$  is 2 or 4 modulo 6 and  $D \geq 4$ . This improves the asymptotic behavior of the lower bound of  $N(\Delta, D)$ . If  $D=2$ , the graph has  $\Delta^2 - \Delta + 1$  vertices if  $\Delta$  is  $q+1$  and  $m=k=1$ ; this kind of graph is described in [1]. If  $D=3$ , the graph has  $2(\Delta^2 - \Delta + 1)$  vertices if  $\Delta$  is  $q+1$  and  $k=1, m=2$ ; this kind of graph is described in [15].

Table I gives the number of vertices of graphs of type  $C_p$ .

It is easy to define type  $C'_p$  like type  $C_p$  with two projective spaces. The degree is then  $q+q'+2$ ; the number of vertices is  $m(q^2 + q + 1)^k(q'^2 + q' + 1)^{k'}$ ; the diameter is  $D$  with

TABLE II  
ORDERS AND PARAMETERS OF GRAPHS  $C'_p$ 

| diameter D<br>degree $\Delta$ | 5       | 6        | 7                   | 8                    | 9                                  | 10                                  |
|-------------------------------|---------|----------|---------------------|----------------------|------------------------------------|-------------------------------------|
| 5                             | 126     | 252      | 315                 | 1470                 | 2646                               | 5292                                |
|                               | 6.3.7   | 12.3.7   | 15.3.7              | 10.3.7 <sup>2</sup>  | 6.3 <sup>2</sup> .7 <sup>2</sup>   | 12.3 <sup>2</sup> .7 <sup>2</sup>   |
| 7                             | 546     | 1092     | 2646                | 13230                | 49686                              | 99372                               |
|                               | 6.7.13  | 12.7.13  | 2.3.21 <sup>2</sup> | 10.3.21 <sup>2</sup> | 6.7 <sup>2</sup> .13 <sup>2</sup>  | 12.7 <sup>2</sup> .13 <sup>2</sup>  |
| 9                             | 1638    | 3276     | 13454               | 67270                | 447174                             | 894348                              |
|                               | 6.13.21 | 12.13.21 | 2.7.31 <sup>2</sup> | 10.7.31 <sup>2</sup> | 6.13 <sup>2</sup> .21 <sup>2</sup> | 12.13 <sup>2</sup> .21 <sup>2</sup> |

TABLE III  
ORDERS OF LARGEST KNOWN GRAPHS (DECEMBER 1983)

| diameter D<br>degree $\Delta$ | 2   | 3    | 4    | 5      | 6      | 7        | 8        | 9         | 10         |
|-------------------------------|-----|------|------|--------|--------|----------|----------|-----------|------------|
| 3                             | 10  | 20   | 38   | 70     | 128    | 180      | 286      | 462       | 708        |
| 4                             | 15  | 40   | 95   | 364    | 731    | 856      | 1872     | 3708      | 7000       |
| 5                             | 24  | 66   | 174  | 532    | 2734   | 2988     | 7000     | 11340     | 30240      |
| 6                             | 32  | 105  | 320  | 820    | 7817   | 10920    | 19138    | 43744     | 131232     |
| 7                             | 50  | 122  | 480  | 1550   | 8998   | 31248    | 62536    | 156340    | 562824     |
| 8                             | 57  | 200  | 807  | 2550   | 39223  | 48984    | 154800   | 524288    | 1310729    |
| 9                             | 74  | 585  | 1178 | 5050   | 74906  | 160800   | 480250   | 1176690   | 5883450    |
| 10                            | 91  | 650  | 1755 | 7550   | 132869 | 380835   | 1117550  | 3906250   | 14981200   |
| 11                            | 94  | 715  | 2925 | 11388  | 142474 | 723060   | 1990050  | 5580498   | 33217250   |
| 12                            | 133 | 780  | 4680 | 18720  | 354323 | 1346440  | 3778261  | 20155392  | 85887453   |
| 13                            | 136 | 845  | 5265 | 25844  | 374464 | 2146224  | 7211386  | 24340680  | 130691232  |
| 14                            | 183 | 910  | 5850 | 37440  | 804481 | 4289520  | 12694773 | 80707214  | 322828871  |
| 15                            | 186 | 1215 | 7605 | 54796  | 842062 | 6432816  | 22303302 | 81352872  | 550731776  |
| 16                            | 197 | 1600 | 9747 | 132496 | 884077 | 11333560 | 39070745 | 268435456 | 1805076171 |

$$D = 2(k + k') + \sup_{b \in Z/mZ} (\inf(|b|, |b - 2(k + k')|, |b + 2(k + k')|)) \quad \text{if } k = k' \text{ (and } m \text{ even)},$$

$$D = 2(k + k') + \sup_{b \in Z/mZ} (\inf(|b|, |b - 2(k + k')| + 1, |b + 2(k + k') - 2| + 2, |b + 2(k + k')| + 2, |b + 2(k + k') + 2| + 2)) \quad \text{if } k \neq k'.$$

Table II presents some results with graphs of type  $C'_p$ .

Under the number of vertices, we write  $m \cdot (q^2 + q + 1)^k \cdot (q'^2 + q' + 1)^{k'}$  to indicate the way we obtain it.

Most of these values are, in fact, not the best known because many authors, using various techniques, give larger (and sometimes far larger) graphs [8], [9], [16], [17].

In Table III we give a table of best known values; the methods used to obtain them are recalled in [7]. It contains some  $C_p$  graphs.

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