Bachelor Degree in Informatics Engineering

Facultat d'Informàtica de Barcelona

MATHEMATICS 1

Part I: Graph Theory

Answers to some exercises

Academic Year 2025-2026

This document contains the answers to some of the exercises of the second part of the course Mathematics 1.

Please let us know about any mistakes you spot.

Departament de Matemàtiques

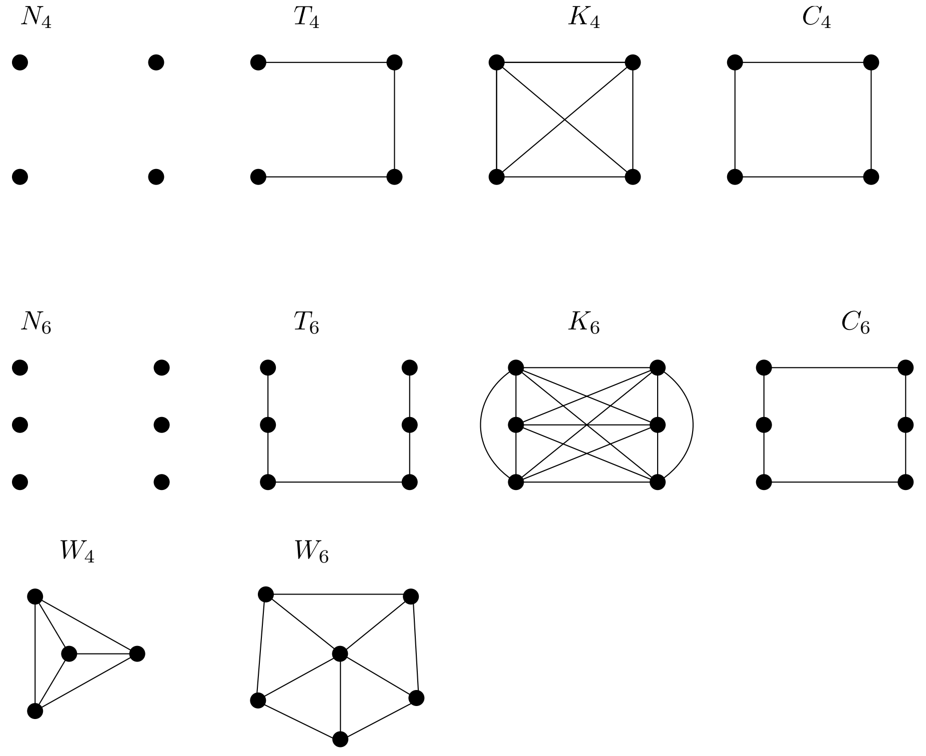
Universitat Politècnica de Catalunya

Answers

Graphs: basic concepts

1.1

1)



2)

$$M_A(N_5) = \begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \qquad M_A(K_5) = \begin{pmatrix} 0 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

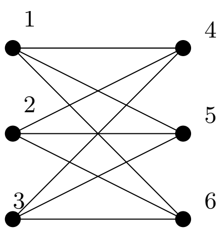
$$M_A(P_5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} \qquad M_A(C_5) = \begin{pmatrix} 0 & 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 1 & 0 \end{pmatrix}$$

$$M_A(W_5) = \begin{pmatrix} 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 & 0 \end{pmatrix}$$

1) For $n \geq 4$, if the graph is W_n, and for $n \geq 3$, in the remaining cases:
$N_n = (V, E): \ |V| = n, \ |E| = 0, \delta(N_n) = 0, \ \Delta(N_n) = 0$
$K_n = (V, E): \ |V| = n, \ |E| = \binom{n}{2}, \ \delta(K_n) = n - 1, \ \Delta(K_n) = n - 1$
$P_n = (V, E): \ |V| = n, \ |E| = n - 1, \ \delta(P_n) = 1, \ \Delta(P_n) = 2$
$C_n = (V, E): \ |V| = n, \ |E| = n, \ \delta(C_n) = 2, \ \Delta(C_n) = 2$
$W_n = (V, E): \ |V| = n, \ |E| = 2 \cdot n - 2, \ \delta(W_n) = 3, \ \Delta(W_n) = n - 1$

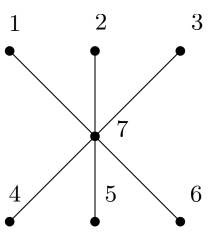
1.2

- Solution of 1. and 2.



1	2	3	4	5	6
4	4	4	1	1	1
5	5	5	2	2	2
6	6	6	3	3	3

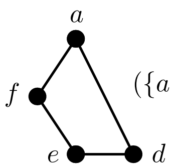
- Solution of 3. and 4.



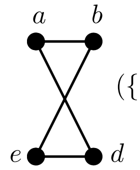
1	2	3	4	5	6	7
7	7	7	7	7	7	1
						2
						3
						4
						5
						6

1.4 1) $\frac{r \cdot n}{2}$; 2) $r \cdot s$;

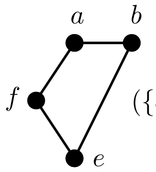
1.5



$(\{a, d, e, f\}, \{ad, de, ef, fa\})$



$(\{a, b, e, d\}, \{ab, be, ed, da\})$



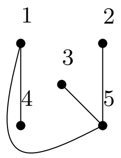
$(\{a, b, e, f\}, \{ab, be, ef, fa\})$

1.6

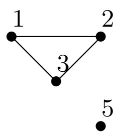
1) 5; 4. 2) 4; 2. 3) 5; 5. 4) 9; 8.

1.7

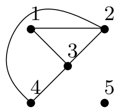
$$G^c: \quad A = \{14, 15, 25, 35\}; \quad M_A = \begin{pmatrix} 0 & 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \end{pmatrix};$$



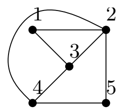
$$G - 4: \quad A = \{12, 13, 23\}; \quad M_A(G-4) = \begin{pmatrix} 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix};$$



$$G - 45\colon \quad A = \{12, 13, 23, 24, 34\}; \quad M_A(G-45) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix};$$



$$G + 25\colon \quad A = \{12, 13, 23, 24, 25, 34, 45\}; \quad M_A(G+25) = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 \end{pmatrix};$$

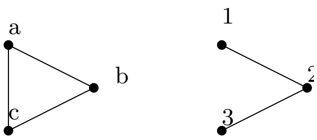


1.8

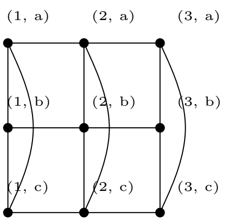
- $G^c = (V, E)$: $|V| = n$; $|E| = \binom{n}{2} - m$.
- $G - v = (V, E)$: $|V| = n - 1$; $|E| = m - d(u)$.
- $G - e = (V, E)$: $|V| = n$; $|E| = m - 1$.

1.10

1) $K_3 \cup P_3$ $\qquad E = \{ab, ac, bc, 12, 23\}$



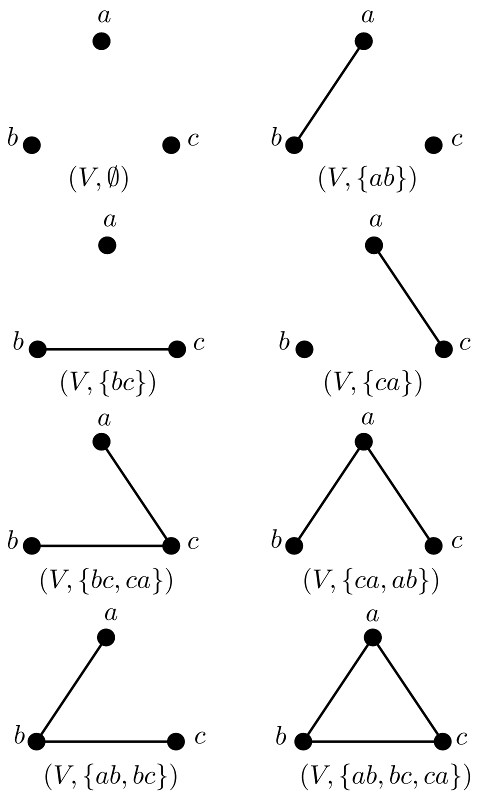
2) $P_3 \times K_3$



E = {(1,a)(1,b); (1,a)(1,c); (1,a)(2,a); (1,b)(1,c); (1, b)(2,b); (1,c)(2,c); (2,a)(2,b); (2,a)(2,c); (2,a)(3,a); (3,a)(3,b); (3,a)(3,c); (3,b)(3,c)}

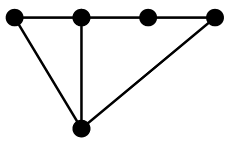
1.11 Order $|V_1||V_2|$, $d_{G_1 \times G_2}(u_1, u_2) = d_{G_1}(u_1) + d_{G_2}(u_2)$ and size $|V_1||E_2| + |V_2||E_1|$.

1.13

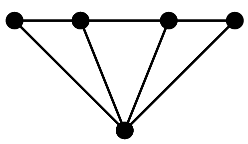


1.14 21; $2^{21} = 2097152$.

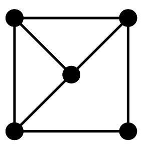
1.15

1) 

2) It does not exist.

3) 

4) It does not exist.

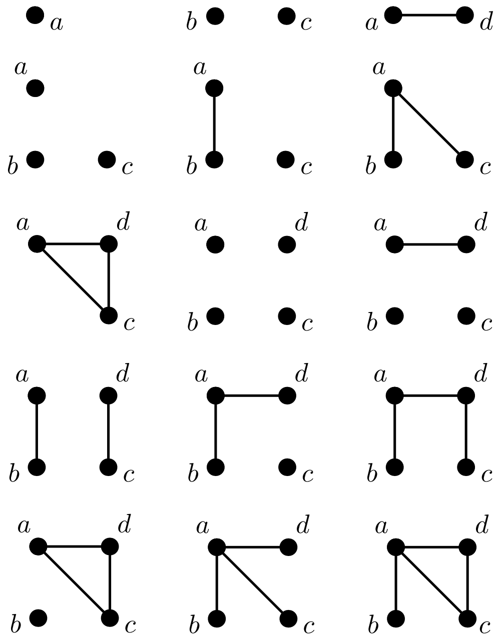
5) 

6) It does not exist.

1.20 Alex and Leo greeted 4 people each.

1.21

1.22



1.23

- $G_1 \cong G_2$
- $G_3 \cong G_4$
- $G_5 \cong G_6$
- G_7
- $G_8 \cong G_9 \cong G_{10}$
- G_{11}
- G_{12}
- G_{13}

1.25 2.

Walks, connectivity and distance

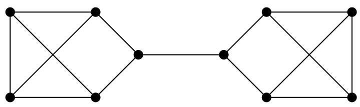
2.1 G_1: Path of length 9: 1 2 3 4 5 10 7 9 6 8. There are no paths of length 11 because G_1 has order 10. Cycles: 1 2 3 4 5 1; 1 2 3 8 10 5 1; 1 6 8 10 7 9 4 5 1; 1 2 3 4 9 7 10 8 6 1.
G_2: 1 2 3 4 5 10 6 7 8 9. There are no paths of length 11 because G_2 has order 11. Cycles: 1 2 3 4 5 1; 5 10 6 11 9 4 5; 2 3 4 5 10 9 8 7 2; 5 1 2 3 4 9 11 6 10 5.

2.4 1) $\langle\{a,b,d,e,f,g,i,j\}\rangle\bigcup\langle\{c,h\}\rangle$. 2) $\langle\{a,b,d,e,g,h,j,m\}\rangle\bigcup\langle\{c,f,i,k\}\rangle\bigcup\langle\{l\}\rangle$.

2.9

- G_1. Cut vertices: 4. Bridges: none.
- G_2. Cut vertices: 3, 6. Bridges: 36.
- G_3. Cut vertices: none. Bridges: none.

2.11 $n = 10$



2.14 1)

v	a	b	d	e	f	g	i	j
d(a,v)	0	2	1	1	1	3	3	3
d(b,v)	2	0	1	2	2	1	1	1

2)

v	a	b	d	e	g	h	j	m
d(a,v)	0	1	2	2	2	2	1	3
d(b,v)	1	0	1	1	1	1	1	2

2.15 1) 1. 2) $D(G_1) = 2$, $D(G_2) = 4$. 3) 2. 4) $\lfloor n/2 \rfloor$. 5) 2. 6) $n - 1$.

2.16 1) $G = W_6$ and u a vertex of degree 3. 2) $G = W_7$, u the vertex of degree 6. 3) $G = ([4], \{12, 13, 14, 23\})$, $u = 4$.

2.17 1) a) G_1: $e(v) = 2$, $1 \leq v \leq 10$; $r(G) = 2$; all the vertex are central, hence, the center is G_1. G_2: $e(1) = e(11) = 4$; $e(v) = 3$, if $2 \leq v \leq 10$; $r(G) = 3$; v central vertex if $2 \leq v \leq 10$. The center is ths subgraph induced $\{v : 2 \leq v \leq 10\}$, that is, $G - \{1, 11\}$. b) G: $e(4) = 2$, $e(v) = 3$, $v \neq 4$; $r(G) = 2$; the only central vertex is 4. The center is the subgraph induced by vertex 4 (the trivial graph). 2) C_6. 3) P_5.

Eulerian and Hamiltonian graphs

3.1 Only the graph G_4 is Eulerian

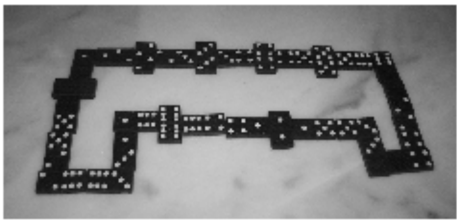
3.2 All of them, except the first one.

3.3 1) 5; 2) 4.

3.4 r and s even.

3.5 If the two components are complete, 4; otherwise, 3.

3.7



3.8 2) 2^n; $n2^{n-1}$; Q_n is n-regular. 3) n even.

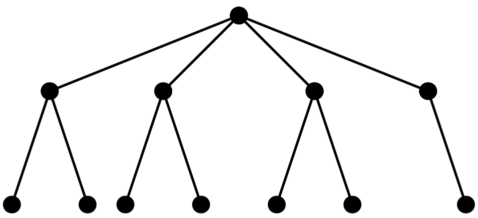
3.9 Only G_1 and G_2 are Hamiltonian graphs.

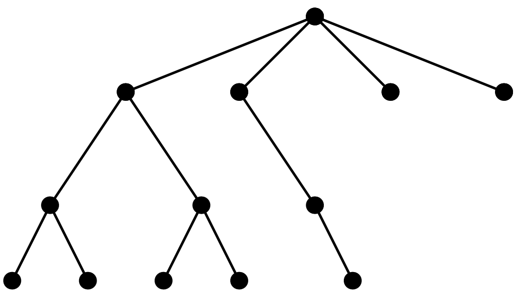
3.12 Two.

Trees

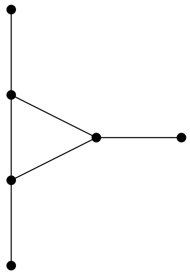
4.3 $n = 18$; order of T_2: 36; size of T_2: 35.

4.5 4,3,3,3,2,1,1,1,1,1,1,1.





4.6

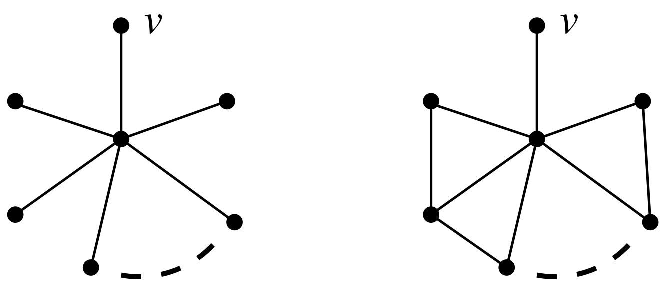


4.12

1) n; 1.

2) $r2^{r-1}$; $\lceil r/2 \rceil$.

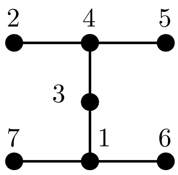
4.13



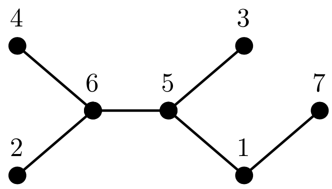
4.14 Two.

4.16 $(1,1,1,5)$; $(1,1,2,2,2,1)$; $(3,3,1,2,4,4,2,5,5)$.

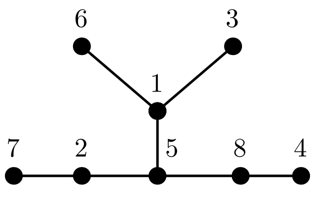
4.17 1)



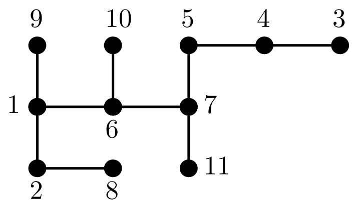
2)



3)



4)



4.18 The paths of order 3.

4.19

1) Star graphs.

2) “Bistar” graphs: two adjacent vertices u, v of degree at least 2, and the remaining vertices are leaves hanging from u or from v.

Review exercises

A.1

$$\begin{pmatrix} 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 \\ 1 & 1 & 0 & 1 & 1 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 \end{pmatrix}$$

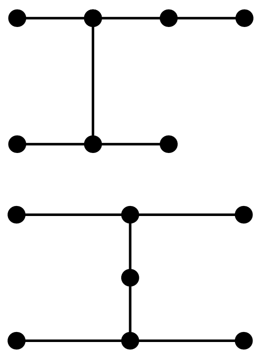
A.9 4 connected components. 7,14,2,4,6,8,10,12,3,9,15,5.

A.19 Yes; no.

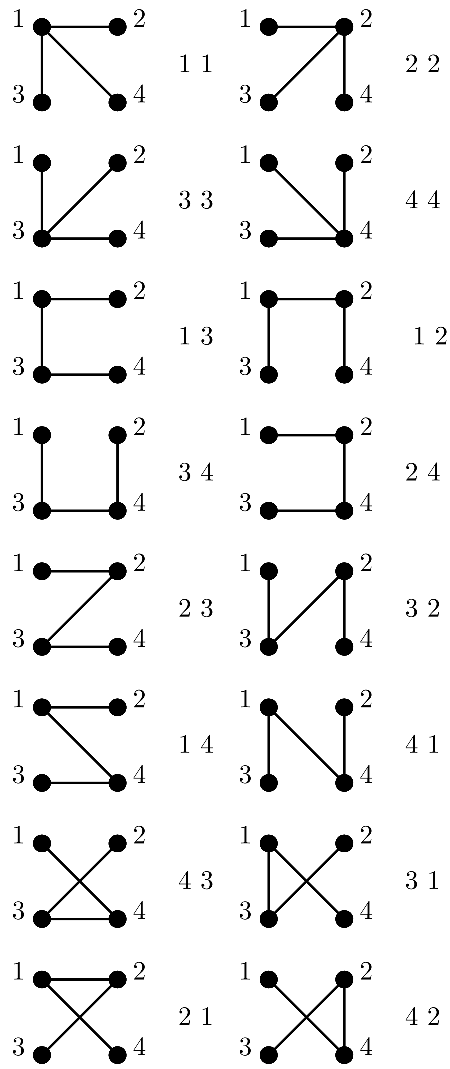
A.22 K_1 i P_4.

A.23 $k-1$.

A.24 3,3,2,1,1,1,1.



A.29



A.30 The paths of order $n \geq 4$.