Second list of exercises of the course ”Geometry and Dynamics of Singular Symplectic manifolds”

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The problems of this course mainly follow the articles by Victor Guillemin, Eva Miranda, Ana Rita Pires, Geoffrey Scott, Jonathan Weitsman and others. We would refer the reader to [GMP11], [GMP14], [GMPS15], [GMPS17], [GMW18b], [GMW18a], [GMW19], [GMW21] and [KMS16] to cite a few places to check the basics and not-so-basics of b-symplectic geometry.

1 Problem session 2. Classical problems for b^m-symplectic and b^m-contact manifolds. Toric actions, action-angle coordinates and integrable systems on b^m-symplectic manifolds. Perturbations of integrable systems and KAM theory

Exercise 1.1. *Let (M^n, Π) an orientable, connected Poisson manifold. Then, we know that $\Omega^n(M) \cong \mathcal{C}^\infty(M)$. We define the* **modular vector field** *as*

$$X_\Pi^\Omega : \begin{array}{ccc} \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ f & \longmapsto & \frac{\mathcal{L}_{u_f}\Omega}{\Omega} \end{array},$$

or, more formally, $X_\Pi^\Omega(f)$ is the only function such that $\mathcal{L}_{u_f}\Omega = X_\Pi^\Omega(f)\Omega$. Here, u_f denotes the Hamiltonian vector field of f, this means, such that $u_f(g) = \{f, g\}$.

a) Show that X_Π^Ω is a well defined derivation.

b) Show that, for any $H \in \mathcal{C}^\infty(M)$ nowhere vanishing,

$$X_\Pi^{H\Omega} = X_\Pi^\Omega - u_{\log|H|}.$$

c) Let (M^{2m}, ω) a symplectic manifold. Show that the modular vector field $X_{\omega^{-1}}^\Omega$ is a Hamiltonian vector field.

(Hint: Compute the modular vector field in local Darboux coordinates and use the previous part of the exercise to get the global result)

d) Compute the modular vector field for the b-Poisson manifold $(\mathbb{R}^2, \{\cdot,\cdot\})$, where $\{x, y\} = y$.

Solutions:

a) First of all, let us note that the map $u_\cdot : \mathcal{C}^\infty(M) \to \mathfrak{X}(M)$ that assigns the Hamiltonian vector field to a function is itself a derivation. Thus,

$$u_{\alpha f+\beta g} = \alpha u_f + \beta u_g \ ; \ u_{fg} = g u_f + f u_g.$$

Therefore, we can check that X_Π^Ω acts linearly on functions,

$$X_\Pi^\Omega(\alpha f + \beta g) = \frac{\mathcal{L}_{\alpha f+\beta g}\Omega}{\Omega} = \frac{\alpha\mathcal{L}_f\Omega + \beta\mathcal{L}_g\Omega}{\Omega} = \alpha\frac{\mathcal{L}_f\Omega}{\Omega} + \beta\frac{\mathcal{L}_g\Omega}{\Omega} = \alpha X_\Pi^\Omega(f) + \beta X_\Pi^\Omega(g).$$

Also, it satisfies the Leibniz rule:

$$X_\Pi^\Omega(fg) = \frac{\mathcal{L}_{fg}\Omega}{\Omega} = \frac{g\mathcal{L}_f\Omega + f\mathcal{L}_g\Omega}{\Omega} = g\frac{\mathcal{L}_f\Omega}{\Omega} + f\frac{\mathcal{L}_g\Omega}{\Omega} = gX_\Pi^\Omega(f) + fX_\Pi^\Omega(g).$$

b) First of all, let us note that

$$u_{\log|H|}(f) = \{\log|H|, f\} = -\{f, \log|H|\} = -u_f(\log|H|) = -\frac{1}{H}u_f(H).$$

Now, let us apply the definition of modular vector field to check that

$$X_\Pi^{H\Omega}(f)\Omega = \frac{1}{H}X_\Pi^{H\Omega}(f)H\Omega = \frac{1}{H}\mathcal{L}_{u_f}(H\Omega) = \mathcal{L}_{u_f}(\Omega) + \frac{1}{H}u_f(H)\Omega,$$

and combining the definition of modular vector field with the previous computation we deduce that

$$\mathcal{L}_{u_f}(\Omega) + \frac{1}{H}u_f(H)\Omega = \left(X_\Pi^\Omega(f) + u_{\log|H|}(f)\right)\Omega.$$

c) First of all, as ω^k is a volume form for M^{2k}, this means that $\Omega = F\omega^k$ for some non-vanishing function F. Therefore, for this proof we can assume that $\Omega = \omega^k$ without loss of generality.

Let us restrict to a Darboux open set with canonical coordinates $(x_1, y_1, ..., x_k, y_k)$. Thus, locally the symplectic form is given by

$$\omega = \sum_{i=1}^{k} dx_i \wedge dy_i.$$

We will compute the expression in local coordinates of $X^{\omega^k}_{\omega^{-1}}(f)$ for a given function $f \in \mathcal{C}^\infty(M)$.

The Hamiltonian vector field of f can be seen to be

$$u_f = \sum_{i=1}^{k} \left(\frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i} - \frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} \right),$$

and thus, using Cartan's formula,

$$\begin{aligned} \mathcal{L}_{u_f}\omega^k = d(i_{u_f}\omega^k) = d\left(i_{u_f}\left(dx_1 \wedge dy_1 \wedge \cdots \wedge dx_k \wedge dy_k\right)\right) = \\ = d\left(\sum_{i=1}^{k}\left(\frac{\partial f}{\partial y_i} dx_1 \wedge dy_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dy_k + \frac{\partial f}{\partial x_i} dx_1 \wedge dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dx_k \wedge dy_k\right)\right) = \\ = \left(\sum_{i=1}^{k}\left(\frac{\partial^2 f}{\partial x_i \partial y_i} - \frac{\partial^2 f}{\partial y_i x_i}\right)\right) dx_1 \wedge dy_1 \wedge \cdots \wedge dx_k \wedge dy_k = \\ = 0. \end{aligned}$$

This means that $X^{\omega^k}_{\omega^{-1}} = 0$, and thus the modular vector field of a symplectic manifold is always Hamiltonian, with the factor F as Hamiltonian function.

d) By the definition of the Poisson bracket, we deduce that

$$\{f, g\} = y\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - y\frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

so

$$u_f = y\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - y\frac{\partial f}{\partial y}\frac{\partial}{\partial x}.$$

As in the last section, we can work with the simplest volume form possible, this means, $\Omega = dx \wedge dy$.

Using Cartan's formula to compute $\mathcal{L}_{u_f}\Omega$,

$$\begin{aligned} di_{u_f}\Omega = d\left(y\left(-\frac{\partial f}{\partial x}dx - \frac{\partial f}{\partial y}dy\right)\right) = \\ = -dy \wedge \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = \\ = \frac{\partial f}{\partial x} dx \wedge dy. \end{aligned}$$

Therefore, $X^{dx\wedge dy}_{\pi} = \frac{\partial}{\partial x}$, which is not a Hamiltonian vector field.

In general, if $\Omega = F dx \wedge dy$, then $X^{\Omega}_{\pi} = \frac{\partial}{\partial x} - u_{\log|F|}$.

Exercise 1.2. *Consider $\Lambda^*(M)$ the algebra of multivector fields. Recall that the Schouten-Nijenhuis bracket is a bilinear map*

$$[\cdot, \cdot] : \Lambda^k(M) \times \Lambda^l(M) \longrightarrow \Lambda^{(k-1)(l-1)}(M)$$

such that

(i) $[a, b] = (-1)^{(|a|-1)(|b|-1)}[b, a]$

(ii) $[a, [b, c]] = [[a, b], c] + (-1)^{(|a|-1)(|b|-1)}[b, [a, c]]$

(iii) If $|a| = 1$*, then* $[a, b] = \mathcal{L}_a b$

(iv) If $f \in \mathcal{C}^\infty(M)$*, then* $[f, a] = -i_{df}(a)$*.*

Let $\Pi \in \Lambda^2(M)$ *a Poisson structure, and let*

$$\begin{array}{rccc} d_\Pi^k : & \Lambda^k(M) & \longrightarrow & \Lambda^{k+1}(M) \\ & V & \longmapsto & [\Pi, V] \end{array}.$$

a) Show that d_Π^* *is a cochain differential. This means, prove that* $d_\Pi^{k+1} \circ d_\Pi^k = 0$*.*
The resulting cohomology, denoted by $H_\Pi^*(M)$*, is called* **Poisson cohomology***.*

b) What do the classes of $H_\Pi^1(M)$ *represent?*

Solutions:

1. By abuse of notation, let us just write d_Π. Then,

$$d_\Pi^2(V) = [\Pi, [\Pi, V]] = [[\Pi, \Pi], V] + (-1)^{(2-1)(2-1)}[\Pi, [\Pi, V]] = [[\Pi, \Pi], V] - [\Pi, [\Pi, V]].$$

As Π is a Poisson structure, $[\Pi, \Pi] = 0$. Then,

$$[\Pi, [\Pi, V]] = -[\Pi, [\Pi, V]] = 0$$

2. The second cohomology group of the Poisson complex is the quotient of $\ker\left(d_\Pi^1\right)$ by $\operatorname{im}\left(d_\Pi^0\right)$, so we need to understand both spaces:

$$\ker\left(d_\Pi^1\right) = \{X \in \mathfrak{X}(M) \mid [\Pi, X] = 0\},$$

and

$$[\Pi, X] = [X, \Pi] = \mathcal{L}_X \Pi.$$

So the elements of $\ker\left(d_\Pi^1\right)$ are vector fields that preserve the Poisson structure, also called *Poisson vector fields.*

On the other hand, by the definition of the Schouten-Nijenhuis bracket,

$$\operatorname{im}\left(d_\Pi^0\right) = \{[\Pi, f] \mid f \in \mathcal{C}^\infty(M)\},$$

where

$$[\Pi, f] = -[f, \Pi] = i_{df}(\Pi) = \Pi(df, \cdot) = u_f,$$

the Hamiltonian vector field of f.

Thus, $\operatorname{im}\left(d_\Pi^0\right)$ is precisely the *space of Hamiltonian vector fields.*

Thus, $H_\Pi^1(M)$ is the space of Poisson vector fields quociented by the space of Hamiltonian vector fields. Rephrasing, for any Poisson vector field v we have a class

$$[v] = \{v + u_f \mid f \in \mathcal{C}^\infty(M)\}.$$

Exercise 1.3. *Consider the b-symplectic manifold* $(S^2, Z = \{h = 0\}, \omega = \frac{dh}{h} \wedge d\theta)$*, where the coordinates on the sphere are* $h \in [-1, 1]$ *and* $\theta \in [0, 2\pi]$*. Compute a moment map of the* S^1*-action given by the flow of* $-\frac{\partial}{\partial \theta}$ *and draw its image.*

Solution: A moment map on $M \backslash Z$ is $\mu(h, \theta) = \log|h|$.
We can find it by direct computation:

$$\iota_X \omega = -d\mu \iff \tag{1}$$

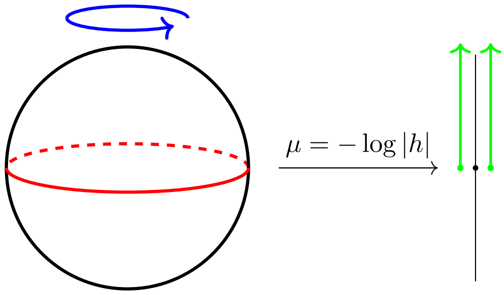
$$\iff \iota_{-\frac{\partial}{\partial\theta}} \frac{1}{h} dh \wedge d\theta = -d\mu \iff \tag{2}$$

$$\iff \frac{1}{h} dh = -d\mu \iff \tag{3}$$

$$\iff d\left(\log|h|\right) = -d\mu \iff \tag{4}$$

$$\iff -\log|h| = \mu \tag{5}$$

The image of $\mu(h,\theta) = -\log|h|$ is $\mathbb{R}^+$, the positive half-line in which each point has two connected components in its preimage: one in the northern hemisphere, and one in the southern hemisphere. By enlarging the codomain of our moment map to include points "at infinity", we can define moment maps for torus actions on a b-manifold that enjoy many of the same properties as classic moment maps: they will be everywhere defined and their image will be a parameter space for the orbits of the action.



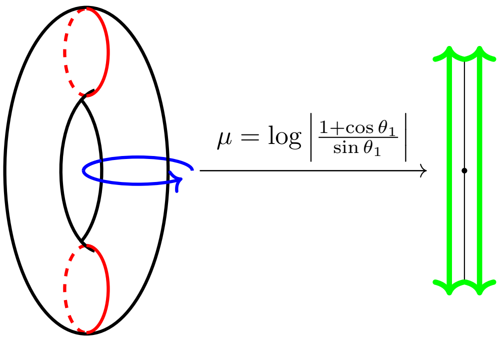
Exercise 1.4. *Consider the b-symplectic manifold*

$$(\mathbb{T}^2, Z = \{\theta_1 \in \{0,\pi\}\}, \omega = \frac{d\theta_1}{\sin\theta_1} \wedge d\theta_2),$$

where the coordinates on the torus are $\theta_1, \theta_2 \in [0, 2\pi]$. Find the ${}^bC^\infty$ Hamiltonian function of the circle action of rotation on the θ_2 coordinate and draw it.

Solution:

With the same procedure as in the previous exercise, we find that the ${}^bC^\infty$ Hamiltonian function $\log\left|\frac{1+\cos\theta_1}{\sin\theta_1}\right|$ gives the moment map of the circle action of rotation on the θ_2 coordinate.



In the following picture, courtesy of Pablo Nicolás, we can see with more precision the moment image of the distinct θ_1-circles.

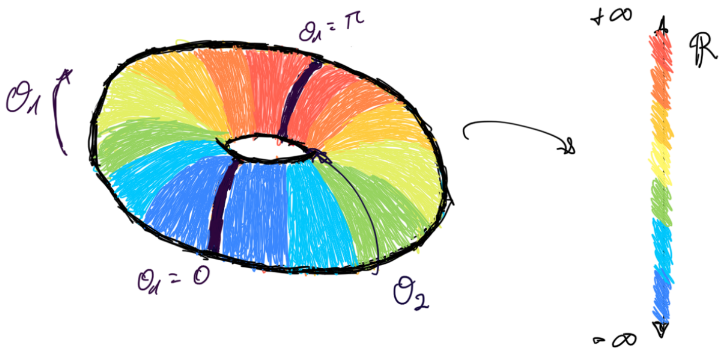


Figure 1: *
Moment map of the toric action of S^1 on ${}^bT^2$.

Exercise 1.5. *The moment image of a $2n$-dimensional b-symplectic toric manifold is represented by an n-dimensional polytope P, and the corresponding extremal polytope Δ_P is an $(n-1)$-dimensional Delzant polytope. Describe the extremal polytope for $n = 1$ and $n = 2$.*

Solution:

For $n = 1$, the extremal polytope is a point, and therefore a b-symplectic toric surface is equivariantly b-symplectomorphic to either a b-symplectic torus $\mathbb{T}^2$ or a b-symplectic sphere.

For $n = 2$, the extremal polytope is a line segment, corresponding to a symplectic toric sphere. As a consequence, a b-symplectic toric 4-manifold is equivariantly b-symplectomorphic to either a product $\mathbb{T}^2 \times S^2$ of a b-symplectic torus with a symplectic sphere, or to a manifold obtained from the product $S^2 \times S^2$ of two spheres, one b-symplectic and the other symplectic, by a series of symplectic cuts which avoid Z. In particular, $\mathbb{C}P^2 \# \overline{\mathbb{C}P^2}$ can be obtained from a b-symplectic $S^2 \times S^2$ with connected Z via symplectic cutting and therefore can be endowed with a b-symplectic toric structure. Because Z was connected (in fact, it would suffice for Z to have an odd number of connected components), there will be fixed points in both the portion of the manifold with positive orientation and in the one with negative orientation. Blowing up these fixed points (each such blow up destroys one fixed point and creates two new ones with the same orientation) corresponds to performing connect sum with either $\mathbb{C}P^2$ or $\overline{\mathbb{C}P^2}$, according to the orientation. Therefore, any $m\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $m, n \geq 1$ can be endowed with b-symplectic toric structures. Observe that $\mathbb{C}P^2 \# n\overline{\mathbb{C}P^2}$, with $n \geq 1$ admits both symplectic and b-symplectic toric structures.

Exercise 1.6. *Compute the moment map of the toric action $\mathbb{T}^2$ on $\mathbb{C}P^2$ given by $((\theta_1, \theta_2), [z_0 : z_1 : z_2]) \mapsto ([z_0 : e^{i\theta_1} z_1 : e^{i\theta_2} z_2])$. Then, construct a b-toric manifold applying symplectic blow-up and the Gompf sum on $\mathbb{C}P^2$ such that:*

- *it has 6 fixed points, or*
- *it has 12 fixed points.*

What you will obtain is a Hirzebruch surface.

Solution:

Let us start by computing the moment map of a simpler action, the one of S^1 on $\mathbb{C}$ given by

$$(t, z) \mapsto e^{it} z.$$

We can compute explicitly the infinitesimal generator of this action:

$$X = \frac{d}{dt}\Big|_{t=0} e^{it} z = iz$$

Then, in via the change $x + iy = z$ to real coordinates we arrive to

$$X_x \frac{\partial}{\partial x} + i X_y \frac{\partial}{\partial y} = X = i(x + iy) = ix - y$$

Hence, equaling complex and $X = -y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}$.

Now, we can compute

$$\iota_{-y\frac{\partial}{\partial x} + x\frac{\partial}{\partial y}} dx \wedge dy = -d\mu \iff \tag{6}$$

$$\iff -xdx - ydy = -d\mu \iff \tag{7}$$

$$\iff -d\left(\frac{x^2}{2} + \frac{y^2}{2}\right) = -d\mu \iff \tag{8}$$

$$\iff d\left(\frac{|z|^2}{2}\right) = d\mu \iff \tag{9}$$

$$\iff \frac{|z|^2}{2} = \mu \tag{10}$$

Then, it is clear that for each rotation of S^1 in a component of coordinate z, we obtain a moment map of $\mu = \frac{|z|^2}{2}$.

In consequence, in the case of the toric action $\mathbb{T}^2$ on $\mathbb{C}^3$ given by $((\theta_1, \theta_2), (z_0, z_1, z_2)) \mapsto (z_0, e^{i\theta_1} z_1, e^{i\theta_2} z_2])$, we would obtain the moment map $\mu = \left(\frac{|z_1|^2}{2}, \frac{|z_2|^2}{2}\right)$.

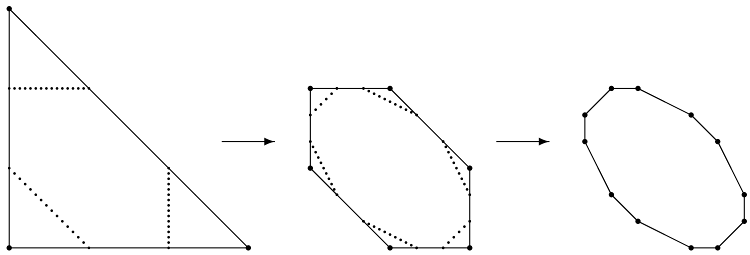
Finally, in the case of the toric action $\mathbb{T}^2$ on $\mathbb{C}P^2$ given by $((\theta_1,\theta_2),[z_0:z_1:z_2]) \mapsto ([z_0 : e^{i\theta_1}z_1 : e^{i\theta_2}z_2])$, we would obtain the same moment map but normalized, since in this case $[z_0 : z_1 : z_2]$ are homogeneous coordinates of $\mathbb{C}P^2$.

$\mu([z_0:z_1:z_2]) = \frac{1}{2}\left(\frac{|z_1|^2}{|z_0|^2+|z_1|^2+|z_2|^2}, \frac{|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}\right)$

To see that the normalization factor is $|z_0|^2+|z_1|^2+|z_2|^2$, think of the embedding of S^5 on $\mathbb{C}^3$ and also on the complex Hopf fibration by circle bundles $S^5 \to \mathbb{C}P^2$.

The fixed points of $\mu([z_0:z_1:z_2])$ are $[1:0:0]$, $[0:1:0]$ and $[0:0:1]$, and have as images, respectively, $(0,0)$, $(1/2,0)$ and $(0,1/2)$. Then, we know that the Delzant polytope is the triangle produced by the convex hull of these three points, which are its vertices.

Now, we can apply b-symplectic cutting to the moment map of this action on $\mathbb{C}P^2$ [KK07] to obtain, after each symplectic cut near a vertex, a copy of $\mathbb{C}P^2$ that is attached to the former $\mathbb{C}P^2$ via the symplectic or Gompf sum. Now, if we sum this manifold, at its turn, to a b-symplectic toric surface (i.e., the b-symplectic sphere or the b-symplectic torus) we obtain the b-symplectic manifold we were looking for.



From $\mathbb{C}P^2$ we can obtain $\mathbb{C}P^2\#3\mathbb{C}P^2$ after 3 symplectic cuts. Then, from $\mathbb{C}P^2\#3\mathbb{C}P^2$ we can obtain $\mathbb{C}P^2\#9\mathbb{C}P^2$ after 6 symplectic cuts.

The following exercises are related with the cotangent lift, which is an essential tool since Arnold-Liouville-Mineur Theorem can be restated in a cotangent lift version ([KM17]). These problems also show the connection of singularities with physical systems.

Exercise 1.7. *The coupling of two harmonic oscillators gives can be modelled in $T^*(\mathbb{R}^2)$. Check that, in this system, the energy function*

$$H = \frac{1}{2}(y_1^2+y_2^2) + \frac{1}{2}(x_1^2+x_2^2)$$

and the angular momentum function

$$L = x_1y_2 - x_2y_1$$

Poisson commute. I.e. check that $\{H,L\}=0$.

Solution: Recall that, in the particular case of a symplectic manifold with the canonical coordinates, the Poisson bracket has the form

$$\{f,g\} = \sum_{i=1}^{n}\left(\frac{\partial f}{\partial x_i}\frac{\partial g}{\partial y_i} - \frac{\partial f}{\partial y_i}\frac{\partial g}{\partial x_i}\right).$$

Then,

$$\{H,L\} = \frac{\partial H}{\partial x_1}\frac{\partial L}{\partial y_1} - \frac{\partial H}{\partial y_1}\frac{\partial L}{\partial x_1} + \frac{\partial H}{\partial x_2}\frac{\partial L}{\partial y_2} - \frac{\partial H}{\partial y_2}\frac{\partial L}{\partial x_2}.$$

Computing,

$$\{H,L\} = -x_1x_2 - y_1y_2 + x_2x_1 + y_2y_1 = 0.$$

Exercise 1.8. *Prove that the singularity at the top of the spherical pendulum is of focus-focus type. Hint: use local coordinates $(x,y,z) = (x,y,\sqrt{l^2-x^2-y^2})$.*

Solution:

The most basic physical example of a singularity of focus-focus type comes from the spherical pendulum. Consider a point of mass m attached to an end of a rigid massless rod of length l and assume that the other end of the rod is fixed at the origin and that the mass can move freely as long as it remains attached to the rod, as in Figure 2. The mass can move, then, on a sphere of radius l.

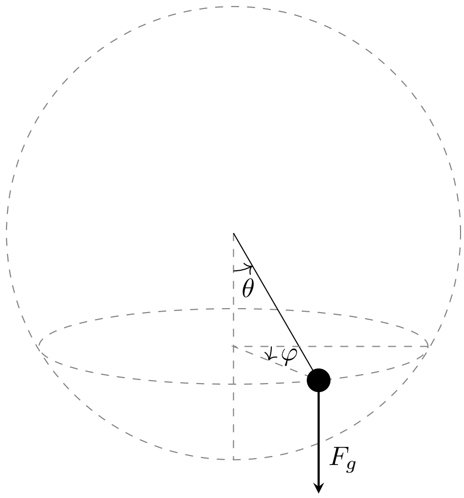


Figure 2: The spherical pendulum.

The natural phase space is the cotangent bundle T^*S^2 and, while spherical coordinates are the optimal setting to study the dynamics of the spherical pendulum, Cartesian coordinates are more appropriated to analyze the singularities of the system. In Cartesian, the position of the point of mass will be given by $\vec{r} = (x, y, z)$, with $\|\vec{r}\| = l$. The conjugate variable to $\vec{r}$ is the linear momentum of the point, $\vec{p} = (p_x, p_y, p_z) = m\dot{\vec{r}}$, which has to satisfy $\vec{r} \cdot \vec{p} = 0$ in order to be contained in the tangent space of the sphere.

The Hamiltonian of the system is the sum of kinetic and potential energies and in the symplectic setting $(\mathbb{R}^6, \omega = dx \wedge dp_x + dy \wedge dp_y + dz \wedge dp_z)$ writes as:

$$H(\vec{r}, \vec{p}) = \frac{\|\vec{p}\|^2}{2m} + mgl\frac{\vec{r} \cdot \hat{z}}{\|\vec{r}\|}, \tag{11}$$

where g accounts for the gravity acceleration and $\hat{z}$ is the unit vector in the z direction. There is another conserved quantity, the angular momentum in the z direction: $L := L_z = xp_y - yp_x$. H and L satisfy $\{H, L\} = 0$ and are independent almost everywhere. Hence, they form the Liouville integrable system corresponding the spherical pendulum.

There are two singularities in the pendulum system, one corresponding to $z = -l$ (or to $\vec{r}_- = (0, 0, -l)$) and the other one to $z = l$ (or to $\vec{r}_+ = (0, 0, l)$). We are interested in $\vec{r}_+$, the unstable equilibrium, where we are going to identify the focus-focus singularity.

To study the system near $z = l$, we use that $z = \sqrt{l^2 - x^2 - y^2}$ and take local coordinates $(x, y, z) = (x, y, \sqrt{l^2 - x^2 - y^2})$. The conjugate momentum $\vec{p} = (p_x, p_y, p_z)$ satisfies locally that $p_z = 0$. In these symplectic coordinates the symplectic form is $\omega = dx \wedge dp_x + dy \wedge dp_y$ and the Hamiltonian of the system writes as:

$$H = \frac{1}{2ml^2}\left(p_x^2(l^2 - x^2) + p_y^2(l^2 - y^2) - 2xyp_xp_y\right) + mg(\sqrt{l^2 - x^2 - y^2} - l). \tag{12}$$

At this point, it is convenient to apply a symplectic scaling in order to adimensionalize the Hamiltonian. We apply the following symplectic transformation:

$$\begin{cases} x = \frac{\xi}{\sqrt{m\nu}} \\ p_x = p_\xi\sqrt{m\nu} \\ y = \frac{\eta}{\sqrt{m\nu}} \\ p_y = p_\eta\sqrt{m\nu} \end{cases}, \tag{13}$$

where $\nu = \sqrt{g/l}$. In these local symplectic coordinates near the unstable equilibrium of the spherical pendulum, the symplecit form is rewritten as $\omega = d\xi \wedge dp_\xi + d\eta \wedge dp_\eta$ and the Hamiltonian becomes:

$$H = \nu\left(\frac{1}{2}(p_\xi^2 + p_\eta^2) - \frac{\kappa}{2}(\xi p_\xi + \eta p_\eta)^2 + \frac{1}{\kappa}(\sqrt{1 - \kappa\rho^2} - 1)\right), \tag{14}$$

where $\rho^2 = \xi^2 + \eta^2$, $\nu^2 = g/l$ and $1/\kappa = ml^2\nu = mgl/\nu$ and they are all constants.

Finally, a last symplectic transformation reveals that the Williamson normal form at the unstable equilibrium of the spherical pendulum corresponds to the focus-focus singularity. It is the following:

$$\sqrt{2}\xi = q_1 - p_1, \quad \sqrt{2}p_\xi = q_1 + p_1, \qquad \sqrt{2}\eta = q_2 - p_2, \quad \sqrt{2}p_\eta = q_2 + p_2\,. \tag{15}$$

In these coordinates, where $\omega = dp_1 \wedge dq_1 + dp_2 \wedge dq_2$, the Hamiltonian is:

$$H = \nu\left(p_1q_1 + p_2q_2 - \kappa\frac{1}{8}(q^2 - p^2)^2 + \frac{1}{\kappa}\sqrt{1-\kappa\rho^2} + \frac{\rho^2}{2} - \frac{1}{\kappa}\right), \tag{16}$$

where $q^2 = q_1^2 + q_2^2$, $p^2 = p_1^2 + p_2^2$ and $\rho^2 = p^2/2 + q^2/2 - (p_1q_1 + p_2q_2)$.

Observe that the quadratic part of the potential has been absorbed in the terms $H' = \nu(p_1q_2 + p_2q_2)$ and that the remaining terms of the potential are of order 4 and higher. The quadratic part of H is simply H' and the angular momentum in the p, q variables is $L = q_1p_2 - q_2p_1$. So, the system $F = (H', L)$ has a singularity of focus-focus type.

Exercise 1.9. *Compute the infinitesimal generator of the cotangent lift of the action given by:*

$$\begin{array}{rccccl} \rho: & (S^1\times\mathbb{R})\times\mathbb{R}^2 & \longrightarrow & & \mathbb{R}^2 & \\ & ((\theta,t),\begin{pmatrix}x_1\\x_2\end{pmatrix}) & \longmapsto & \rho_{\theta,t}\begin{pmatrix}x_1\\x_2\end{pmatrix} & = e^{-t}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} & , \end{array}$$

and see that it coincides with the vector field associated to the normal form of the focus-focus singularity.

Solution: To describe the basic singularity of focus-focus type in a manifold of dimension 4 we take coordinates (x_1, x_2, y_1, y_2). The symplectic form is $\omega = dx_1 \wedge dy_1 + dx_2 \wedge dy_2$ and the moment map associated to this singularity is $F = (f_1, f_2) = (x_1y_2 - x_2y_1, x_1y_1 + x_2y_2)$.

If we compute the Hamiltonian vector field associated to f_1 and f_2, we obtain

$$X_1 = -\frac{\partial f_1}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_1}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_1}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_1}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{17}$$

$$= x_2\frac{\partial}{\partial x_1} - x_1\frac{\partial}{\partial x_2} + y_2\frac{\partial}{\partial y_1} - y_1\frac{\partial}{\partial y_2} = (x_2, -x_1, y_2, -y_1), \tag{18}$$

and

$$X_2 = -\frac{\partial f_2}{\partial y_1}\left(\frac{\partial}{\partial x_1}\right) - \frac{\partial f_2}{\partial y_2}\left(\frac{\partial}{\partial x_2}\right) + \frac{\partial f_2}{\partial x_1}\left(\frac{\partial}{\partial y_1}\right) + \frac{\partial f_2}{\partial x_2}\left(\frac{\partial}{\partial y_2}\right) = \tag{19}$$

$$= -x_1\frac{\partial}{\partial x_1} - x_2\frac{\partial}{\partial x_2} + y_1\frac{\partial}{\partial y_1} + y_2\frac{\partial}{\partial y_2} = (-x_1, -x_2, y_1, y_2). \tag{20}$$

Now consider the action of a rotation and a radial dilation on $\mathbb{R}^2$ given by:

$$\begin{array}{rccccl} \rho: & (S^1\times\mathbb{R})\times\mathbb{R}^2 & \longrightarrow & & \mathbb{R}^2 & \\ & ((\theta,t),\begin{pmatrix}x_1\\x_2\end{pmatrix}) & \longmapsto & \rho_{\theta,t}\begin{pmatrix}x_1\\x_2\end{pmatrix} & = e^{-t}\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}x_1\\x_2\end{pmatrix} & . \end{array}$$

The differential of the induced action $\rho_{\theta,t}$ at a point $x = (x_1, x_2)$ is the following linear map:

$$\begin{array}{rccc} d\rho_{\theta,t}: & T_x\mathbb{R}^2 & \longrightarrow & T_x\mathbb{R}^2 \\ & \begin{pmatrix}y_1\\y_2\end{pmatrix} & \longmapsto & e^{-t}\begin{pmatrix}y_1\cos\theta + y_2\sin\theta\\ -y_1\sin\theta + y_2\cos\theta\end{pmatrix} \end{array}.$$

Then, $((d\rho_{\theta,t})^*)^{-1}$ acts as:

$$\begin{pmatrix}y_1\\y_2\end{pmatrix} \longmapsto e^t\begin{pmatrix}\cos\theta & \sin\theta\\ -\sin\theta & \cos\theta\end{pmatrix}\begin{pmatrix}y_1\\y_2\end{pmatrix}.$$

And the cotangent lift $\hat{\rho}_{\theta,t}$ associated to the group action $\rho_{\theta,t}$ is:

$$\begin{array}{cccc}\hat{\rho}_{\theta,t}: & T^*\mathbb{R}^2 & \longrightarrow & T^*\mathbb{R}^2 \\ & \begin{pmatrix} x_1 \\ x_2 \\ y_1 \\ y_2 \end{pmatrix} & \longmapsto & \begin{pmatrix} e^{-t}(x_1\cos\theta + x_2\sin\theta) \\ e^{-t}(-x_1\sin\theta + x_2\cos\theta) \\ e^{t}(y_1\cos\theta + y_2\sin\theta) \\ e^{t}(-y_1\sin\theta + y_2\cos\theta) \end{pmatrix} \end{array}.$$

Finally, deriving the last vector with respect to θ and evaluating at 0 and deriving the vector with respect to t and evaluating at 0 we obtain, respectively, $X_1 = (x_2, -x_1, y_2, -y_1)$ and $X_2 = (-x_1, -x_2, y_1, y_2)$, the vector fields associated with f_1 and f_2, the components of the moment map of the focus-focus singularity.

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