List of exercises of the course "Geometry and Dynamics of Singular Symplectic manifolds"

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The problems of this course mainly follow the articles by Victor Guillemin, Eva Miranda, Ana Rita Pires, Geoffrey Scott, Jonathan Weitsman and others. We would refer the reader to [GMP11], [GMP14], [GMPS15], [GMPS17], [GMW18b], [GMW18a], [GMW19], [GMW21] and [KMS16] to cite a few places to check the basics and not-so-basics of b-symplectic geometry.

1 Problem session 1. Introduction to symplectic geometry, b-symplectic geometry, Poisson manifolds, b-forms, the path method

Exercise 1.1. *Check that the Circular Planar Restricted 3 Body Problem provides a b^3-symplectic structure. To do this, consider the symplectic form on $T^*\mathbb{R}^2$ in polar coordinates,*

$$\omega = dr \wedge dP_r + d\alpha \wedge dP_\alpha,$$

and apply to it the non-canonical McGehee change of coordinates, given by $r = \frac{2}{x^2}$, but without altering the momentum associated to r.

Solution: The discussion that follows is extracted from Section 2.1 in [DKdlRS19].

Recall that, after the change to polar coordinates, the Hamiltonian associated to the restricted circular three body problem is

$$H(r, \alpha, P_r, P_\alpha) = \frac{P_r^2}{2} + \frac{P_\alpha^2}{2r^2} - U(r\cos\alpha, r\sin\alpha).$$

Under the new coordinates, the Hamiltonian has the expression

$$H(x, \alpha, P_r, P_\alpha) = \frac{P_r^2}{2} + \frac{x^4 P_\alpha}{8} - U\left(\frac{2\cos\alpha}{x^2}, \frac{2\sin\alpha}{x^2}\right).$$

Furthermore, if we consider that $r = \frac{2}{x^2}$, then $dr = -\frac{4}{x^3}dx$, and this means that

$$\omega = -\frac{4}{x^3} dx \wedge dP_r + d\alpha \wedge dP_\alpha.$$

Thus, the non-canonical change of coordinates transforms the symplectic form into a b^3-symplectic form. The resulting dynamical system is nevertheless well defined, and provides information about the original problem.

Moreover, modeling this problem has the added benefit of providing a description of the dynamics within the critical set $Z = \{x = 0\}$. Granted, the dynamics within of Z carry no physical meaning, but their interplay with the dynamics outside and close to them (and, even more interesting, *towards* them) is a way to study the behaviour of the escape orbits in this context.

Exercise 1.2. *Let $Z = \{z_1, ..., z_k\} \subset S^1$ a finite collection of points within the circle. If we consider (S^1, Z) as a b-manifold, is it true that $TS^1 \cong {}^bTS^1$? This means, are the vector bundles isomorphic? Does this depend on the number of points k?*

Solution: First of all, let us recall Serre-Swan's Theorem, which is our tool to define the b^k-tangent bundles of a given manifold:

Theorem ([Swa62]): A $\mathcal{C}^\infty(M)$-module P is isomorphic to the module of sections of a vector bundle E (denoted as $\Gamma(E)$) if and only if P is finitely generated and projective.

In our case, the module of b^k-vector fields, locally expressible as $\left\langle x_1^k \frac{\partial}{\partial x_1}, \frac{\partial}{\partial x_2}, ..., \frac{\partial}{\partial x_n} \right\rangle_{\mathcal{C}^\infty(M)}$, always satisfies these conditions. However, this does not provide a characterization of the constructed vector bundle. All the information that we may obtain may come, then, from the study of the set of b^k-vector fields itself.

In our particular case, we know that TS^1 is trivial, so that $TS^1 \cong S^1 \times \mathbb{R}$. In the language of sections, we know that its module of sections is freely generated, and thus $\mathfrak{X}(S^1) = \langle 1 \rangle_{\mathcal{C}^\infty(S^1)} \cong \mathcal{C}^\infty(S^1)$.

This means that, in order to answer the question of whether ${}^bTS^1$ is isomorphic to TS^1 or not, we can simply study the generators of the module of sections of ${}^b\mathfrak{X}(S^1)$.

Let us assume that ${}^bTS^1$ is isomorphic to TS^1 and therefore trivial. Then, we may pick a single generator $X \in {}^b\mathfrak{X}(S^1)$. Seen as a section of ${}^bTS^1$ it cannot vanish anywhere, which, in our one-dimensional case, means that X vanishes precisely at $Z = \{z_1, ..., z_k\}$, and that it vanishes transversally at these points. It can be seen that this means that X must change sign at each point of Z. Notice that ${}^bTS^1$ has rank 1, so the section X can be regarded as a smooth function $X : S^1 \to \mathbb{R}$.

Let us look at the sign of X in $M\backslash Z$. Let $I_1, ..., I_k$ the connected components of $M\backslash Z$. By our observations in the last paragraph, the sign of X must differ in adjacent components.

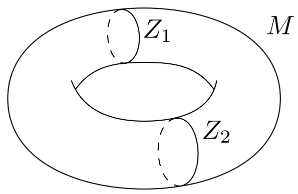
This solves the exercise:

a) If k is even, then we can construct such a section, and ${}^bTS^1$ is parallelizable, and thus isomorphic to TS^1.

   For instance, if we take $z_1 = 0$ and $z_2 = \pi$ (thinking of $S^1 \cong [0, 2\pi]/\{0 = 2\pi\}$), the section $\cos z$ generates ${}^bTS^1$.

b) If k is odd, then we reach a contradiction: it is not possible to find a single global generator of ${}^b\mathfrak{X}(S^1)$, so the b-tangent bundle cannot be parallelizable (actually, it is not even orientable!).

Exercise 1.3. *Consider $M = \mathbb{T}^2$ the 2-torus. Let Z_1, Z_2 be two disjoint non-contractible circles embedded in M as in the picture:*



Does the b-manifold (M, Z_1) admit a b-symplectic structure? Does $(M, Z_1 \cup Z_2)$?
Hint: Use the results of the last exercise.

Solution:

1. The b-manifold (M, Z_1) does not admit a b-symplectic structure, because, as we saw in the last exercise, its b-tangent bundle is not orientable.

2. The b-tangent bundle of $(M, Z_1 \cup Z_2)$ is isomorphic to $T\mathbb{T}^2$, and thus parallelizable. This means that it is orientable, so we may take any area $b-$form to be the b-symplectic form.

Exercise 1.4. *Let (M, ω) be a symplectic manifold and let Π be the corresponding bivector (that is, $\omega(X_f, X_g) = \{f, g\} = \iota_{X_f}(dg) = \Pi(df, dg)$ for any smooth functions f, g). Suppose ω is locally given as*

$$\omega = \sum_{i<j} \omega_{ij} dx_i \wedge dx_j.$$

Prove that the coefficients of Π satisfy $\pi_{i<j} = (\omega_{ij})^{-1}$.

Proof. Consider the vector bundle homomorphism

$$\begin{array}{cccc} \pi^\# : & T^*M & \longrightarrow & TM \\ & \alpha & \longmapsto & \pi^\#(\alpha) \end{array},$$

with $\pi^\#$ satisfying $\langle \beta, (\pi^\#(\alpha)) \rangle = \Pi(\alpha, \beta)$ for any α, β 1-forms.

This map is the inverse of the map

$$\begin{array}{cccc} \omega^\# : & TM & \longrightarrow & T^*M \\ & X & \longmapsto & -\iota_X\omega \end{array},$$

which satisfies

$$\langle \omega^\#(X), Y \rangle = -\langle \iota_X\omega, Y \rangle = \omega(Y, X)$$

Take α and β 1-forms and X, Y vector fields in M such that $\iota_X\omega = \alpha$ and $\iota_Y\omega = \beta$. Then, we have that

$$\omega(X, Y) = \omega((\omega^\#)^{-1}(\iota_X\omega), (\omega^\#)^{-1}(\iota_Y\omega)) = \omega((\omega^\#)^{-1}(\alpha), (\omega^\#)^{-1}(\beta)) = \Pi(\alpha, \beta).$$

On the other hand,

$$\Pi(\alpha,\beta) = -\Pi(\beta,\alpha) = -\langle\alpha,\pi^{\#}(\beta)\rangle = -\langle\iota_X\omega,\pi^{\#}(\iota_Y\omega)\rangle = \omega(X,\pi^{\#}(\iota_Y\omega)) = \omega(X,\pi^{\#}(\omega^{\#}(Y))).$$

Then,

$$\omega(X,\pi^{\#}(\omega^{\#}(Y))) = \omega(X,Y),$$

and by the non-degeneracy of ω, it follows that $(\pi^{\#})^{-1} = \omega^{\#}$, and $\pi_{ij} = (\omega_{ij})^{-1}$ for the coefficients. □

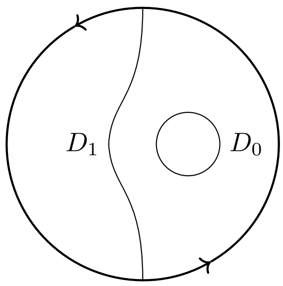
Exercise 1.5. *Prove that the cubic polynomial $g(x) = x(x-1)(x-t)$, $0 < t < 1$, defines a Poisson structure on $\mathbb{R}^2$ given by*

$$\Pi = (g(x) - y^2)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}.$$

Check that it extends smoothly to a b-symplectic structure on $\mathbb{R}P^2$ with critical set Z given by the real elliptic curve $y^2 = g(x)$. [GL14]

Proof. In order to check that Π defines a Poisson structure, one has to see that $[\Pi,\Pi] = 0$, which in this case is inmediate because $[\Pi,\Pi]$ would be a trivector field and they are all 0 in $\mathbb{R}^2$. In fact, any bivector $\Pi = f(x,y)\frac{\partial}{\partial x}\wedge\frac{\partial}{\partial y}$, with f smooth, defines a Poisson structure in $\mathbb{R}^2$.

We observe that the Poisson bracket of the coordinates $\{x,y\} = \pi_{xy}$ is defined as $\{g(x) - y^2 = x(x-1)(x-t) - y^2\}$. And we see that Π defines a b-symplectic structure, because $\omega = (\pi_{xy})^{-1}\, dx\wedge dy = \frac{1}{g(x)-y^2} dx\wedge dy$, which the factor $g(x) - y^2$ precisely vanishing on $Z = \{y^2 = g(x)\}$, which is made of two connected components D_0 and D_1 as in the picture. □



The critical set has two connected components: D_0, containing $\{(0,0),(t,0)\}$ and with trivial normal bundle, and D_1, containing $\{(1,0),(\infty,0)\}$ and with nontrivial normal bundle.

Exercise 1.6. *Prove that S^4 does not admit a b-symplectic structure.*

Proof. Let us recall the Marcut-Osorno theorem ([MT14]):

Theorem: Let (M^{2n}, Z, ω) a compact b-symplectic manifold. Then, there exists an element $c \in H^2(M)$ such that $c^{n-1} \neq 0$.

In this case, $n = 2$, so the condition is just that there is some non-vanishing element $c \in H^2(S^4)$. However, $H^2(S^4) = 0$, so such an element cannot exist, which shows that S^4 cannot admit a b-symplectic structure. □

Exercise 1.7. *Prove the following result from [Cav17]. If a compact oriented manifold M^{2n}, with $n > 1$, admits a b-symplectic structure, then there are classes $a, b \in H^2(M;\mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2 = 0$.*

Proof. Suppose that M has a b-symplectic structure with a non-empty singular set Z. Then, we may assume that the b-symplectic structure is proper, this means that Z is a symplectic fibration over the circle with fiber (F,ω), a $2(n-1)$-dimensional symplectic manifold. On the one hand, due to Marcut–Osorno-Torres's Theorem there is a globally defined closed 2-form $\tilde{\omega} \in \Omega^2(M)$ which restricts to the symplectic form on F. Let $a = [\tilde{\omega}]$.

As $(F,\tilde{\omega})$ is symplectic, the pairing of $[F] \in H_{2n-2}(M)$ with $a^{n-1} \in H^{2n-2}(M)$ is nonzero.

Let $b \in H^2(M)$ be the Poincaré dual of $[F]$. As F appears as a fiber of a fibration, we conclude that $b^2 = 0$. Moreover, by definition of Poincaré dual,

$$\langle a^{n-1}b,[M]\rangle = \langle a^{n-1},F\rangle \neq 0.$$

□

Exercise 1.8. *Prove the following corollaries*

1. *An orientable, compact, b-symplectic manifold M of dimension $2n$ has $b_{2i}(M) \geq 2$ for $0 < i < n$.*

   Proof. It follows directly from the relations $a^{n-1}b \neq 0$ and $b^2 = 0$ that the classes a^i and $a^{i-1}b$ are linearly independent for $0 < i < n$. □

2. *For $n > 1$, $\mathbb{C}P^n$ has no b-symplectic structure and, for $n > 2$, the blow-up of $\mathbb{C}P^n$ along a symplectic submanifold of real codimension greater than 4 also does not carry b-symplectic structures.*

   Proof. The only generator $\eta \in H^2(\mathbb{C}P^n)$ satisfies that $\eta^k \neq 0$ for all $0 \leq k \leq n$. Rephrasing, there exists no element $b \in H^2(\mathbb{C}P^n)$ such that $b^2 = 0$. □

Exercise 1.9. *Compute the b-cohomology class of the b-torus of Radko with $2n$ connected components*

Solution: Recall that the b-cohomology of the b-manifold (M, Z) is given by

$$^bH^k(M) \cong \begin{cases} H^0(M) & \text{for } k = 0 \\ H^k(M) \oplus H^{k-1}(Z) & \text{for } 0 < k \leq n \end{cases}.$$

The b-torus of Radko is $(\mathbb{T}^2, Z)$, a 2-torus where Z is the union of $2n$ disjoint non-contractible circles. From this we deduce that

$$H^k(\mathbb{T}^2) \cong \begin{cases} \mathbb{R} & \text{for } k = 0, 2 \\ \mathbb{R}^2 & \text{for } k = 1 \\ 0 & \text{for } k > 2 \end{cases}, \quad H^k(Z) \cong \begin{cases} \mathbb{R}^{2n} & \text{for } k = 0, 1 \\ 0 & \text{for } k \geq 2 \end{cases}.$$

Applying the formula,

$$^bH^k(\mathbb{T}^2) \cong \begin{cases} \mathbb{R} & \text{for } k = 0 \\ \mathbb{R}^{2(n+1)} & \text{for } k = 1 \\ \mathbb{R}^{2n+1} & \text{for } k = 2 \\ 0 & \text{for } k > 2 \end{cases}.$$

Exercise 1.10. *Let (R, π_R) be a Radko compact surface and (S, π_S) be a compact symplectic surface. Show that $(R \times S, \pi_R + \pi_S)$ is a b-Poisson manifold of dimension 4.*

Proof. We have to check first that $[\pi_R + \pi_S, \pi_R + \pi_S] = 0$.

$$\begin{aligned} [\pi_R + \pi_S, \pi_R + \pi_S] &= [\pi_R, \pi_R] + [\pi_R, \pi_S] + [\pi_S, \pi_R] + [\pi_S, \pi_S] && (1) \\ &= 0 + [\pi_R, \pi_S] + [\pi_S, \pi_R] + 0 && (2) \\ &= 2 \cdot [\pi_R, \pi_S], && (3) \end{aligned}$$

because π_R and π_S are proper Poisson structures and, hence, $[\pi_R, \pi_R] = [\pi_S, \pi_S] = 0$.

Since π_R is defined on the Radko surface and π_S on the sphere, it is clear that $[\pi_R, \pi_S] = [\pi_S, \pi_R] = 0$. It can be checked either putting any charts on R and R and working in coordinates or considering bivector fields as bi-derivations which naturally commute since they act on each component of the product manifold.

To check transversality, we compute $(\pi_R + \pi_S) \wedge (\pi_R + \pi_S)$.

$$\begin{aligned} (\pi_R + \pi_S) \wedge (\pi_R + \pi_S) &= \pi_R \wedge \pi_R + \pi_R \wedge \pi_S + \pi_S \wedge \pi_R + \pi_S \wedge \pi_S && (4) \\ &= 0 + \pi_R \wedge \pi_S + \pi_S \wedge \pi_R + 0 && (5) \\ &= 2\pi_R \wedge \pi_S, && (6) \end{aligned}$$

Since π_R is a b-Poisson structure, it is transverse to the zero section (in R). Then, $\pi_R \wedge \pi_S$ is transverse to the zero section (in $R \times S$), because it inherits the same transversality to the product of the singular locus in R with S. □

Exercise 1.11. *Take S^2 with the b-Poisson structure $\Pi_1 = h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}$ and the symplectic torus $\mathbb{T}^2$ with dual Poisson structure $\Pi_2 = \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}$. Prove that*

$$\hat{\Pi} = h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2$$

is a b-Poisson structure on $S^2 \times \mathbb{T}^2$.

Proof. We first check that the Schouten bracket $[\hat{\Pi}, \hat{\Pi}]$ is zero.

$$\begin{aligned}
[\hat{\Pi}, \hat{\Pi}] =&[h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2, h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2] && (7)\\
=&2 \cdot [h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}), \Pi_2] && (8)\\
=&2 \cdot [h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}), \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}] && (9)\\
=&2 \cdot [h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta}, \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}] + 2 \cdot [h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}] && (10)\\
=&2 \cdot [\Pi_1, \Pi_2] + 2 \cdot [h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}] && (11)\\
=&0 + 2 \cdot [h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}] && (12)\\
=&2 \cdot \Big(+ [h\frac{\partial}{\partial h}, \frac{\partial}{\partial \theta_1}] \wedge \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2} - [h\frac{\partial}{\partial h}, \frac{\partial}{\partial \theta_2}] \wedge \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_1} && (13)\\
&- [\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_1}] \wedge h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta_2} + [\frac{\partial}{\partial \theta_1}, \frac{\partial}{\partial \theta_2}] \wedge h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta_1}\Big) && (14)\\
=&2 \cdot \Big(+ 0 - 0 - 0 + 0\Big) = 0 && (15)
\end{aligned}$$

Now, we have to see that $\hat{\Pi} \wedge \hat{\Pi}$ is transverse to the zero section.

$$\begin{aligned}
\hat{\Pi} \wedge \hat{\Pi} =&(h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2) \wedge (h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) + \Pi_2) && (16)\\
=&2 \cdot (h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) \wedge \Pi_2) && (17)\\
=&2 \cdot (h\frac{\partial}{\partial h} \wedge (\frac{\partial}{\partial \theta} + \frac{\partial}{\partial \theta_1}) \wedge \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}) && (18)\\
=&2 \cdot (h\frac{\partial}{\partial h} \wedge \frac{\partial}{\partial \theta} \wedge \frac{\partial}{\partial \theta_1} \wedge \frac{\partial}{\partial \theta_2}), && (19)
\end{aligned}$$

which is clearly transverse to the zero section, since 0 is a regular value of the function h. □

Exercise 1.12. *Let (N^{2n+1}, π) be a regular corank-1 Poisson manifold, X be a Poisson vector field and $f : S^1 \to \mathbb{R}$ a smooth function. Prove that the bivector field*

$$\Pi = f(\theta)\frac{\partial}{\partial \theta} \wedge X + \pi$$

is a b-Poisson structure on $S^1 \times N$ if the function f vanishes linearly and the vector field X is transverse to the symplectic leaves of N.

Proof. First, we compute $[\Pi, \Pi]$.

$$[\Pi,\Pi] = [f(\theta)\frac{\partial}{\partial\theta}\wedge X+\pi, f(\theta)\frac{\partial}{\partial\theta}\wedge X+\pi] \tag{20}$$
$$= [f(\theta)\frac{\partial}{\partial\theta}\wedge X, f(\theta)\frac{\partial}{\partial\theta}\wedge X] + 2\cdot[f(\theta)\frac{\partial}{\partial\theta}\wedge X, \pi] \tag{21}$$
$$= 2\cdot[f(\theta)\frac{\partial}{\partial\theta}, X]\wedge f(\theta)\frac{\partial}{\partial\theta}\wedge X + 2\cdot[f(\theta)\frac{\partial}{\partial\theta}\wedge X, \pi] \tag{22}$$
$$= 0 + 2\cdot[f(\theta)\frac{\partial}{\partial\theta}\wedge X, \pi] \tag{23}$$
$$= 2\cdot f(\theta)\frac{\partial}{\partial\theta}\wedge[X,\pi] + 2\cdot[f(\theta)\frac{\partial}{\partial\theta}, \pi]\wedge X \tag{24}$$
$$= 0 + 0 = 0 \tag{25}$$

Now, since X is transverse to the symplectic leaves of N, we have that $X\wedge\pi \neq 0$ (it is non degenerate). And since the expression $\wedge^{n+1}\Pi = f(\theta)\frac{\partial}{\partial\theta}\wedge X\wedge\pi^n$ only vanishes transversally when $f(\theta)$ vanishes linearly, we also need this condition. □

Exercise 1.13. *Prove that the bracket $\{f,g\} = \omega(X_f, X_g)$ for $f, g \in C^\infty$ defines a Poisson structure on a symplectic manifold (M^{2n}, ω). Hint: To check the Jacobi identity, expand $d\omega(X_f, X_g, X_h)$.*

Proof. A Poisson manifold M is a differentiable manifold equipped with a Lie algebra structure on $C^\infty(M)$ defined by a Poisson Bracket

$$\{\cdot,\cdot\}: \quad C^\infty(M)\times C^\infty(M) \quad\longrightarrow\quad C^\infty(M),$$

which satisfies, for any $f, g, h \in C^\infty(M)$ the following properties:

- *Bi-linearity*: $\{\cdot,\cdot\}$ is a real-bilinear map.
- *Anti-symmetry* $\{f,g\} = -\{g,f\}$.
- *Leibniz rule* $\{f,gh\} = g\{f,h\} + h\{f,g\}$.
- *Jacobi identity* $\{\{f,g\},h\} + \{\{g,h\},f\} + \{\{h,f\},g\} = 0$.

To prove that $\{f,g\} = \omega(X_f, X_g)$ defines a Poisson structure on (M^{2n}, ω), we have to check the four properties.

By the bi-linearity of 2-forms, $\{f,g\} = \omega(X_f, X_g)$ is bi-linear. Also, since ω is an anti-symmetric 2-form, we have that

$$\{f,g\} = \omega(X_f,X_g) = -\omega(X_g,X_f) = -\{g,f\}.$$

To prove Leibniz Rule, observe that $\{f,g\} = \omega(X_f,X_g) = X_f(g) = dg(X_f)$.
Hence,

$$\{f,gh\} = X_f(gh) = d(gh)(X_f) = (gdh+hdg)(X_f) = gdh(X_f) + hdg(X_f) = g\{f,h\} + h\{f,g\}.$$

Finally, to prove Jacobi identity, recall that if $\omega \in \Omega^k(M)$, then $d\omega \in \Omega^{k+1}(M)$ and:

$$d\omega(X_0,\ldots,X_k) = \sum_{i=0}^{k}(-1)^i X_i\omega(X_0,\ldots,X_{i-1},X_{i+1},\ldots,X_k)+ \tag{26}$$
$$+\sum_{i<j}(-1)^{i+j}\omega([X_i,X_j],X_0,\ldots,X_{i-1},X_{i+1},\ldots,X_{j-1},X_{j+1},\ldots,X_k) \tag{27}$$

Hence:

$$d\omega(X_f,X_g,X_h) = \tag{28}$$
$$= X_f\omega(X_g,X_h) - X_g\omega(X_f,X_h) + X_h\omega(X_f,X_g) - \tag{29}$$
$$-\omega([X_f,X_g],X_h) + \omega([X_f,X_h],X_g) - \omega([X_g,X_h],X_f) = \tag{30}$$
$$= \{f,\{g,h\}\} - \{g,\{f,h\}\} + \{h,\{f,g\}\} - \tag{31}$$
$$-\{\{f,g\},h\} + \{\{f,h\},g\} - \{\{g,h\},f\} = \tag{32}$$
$$= 2\left(\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\}\right) \tag{33}$$

Since $d\omega = 0$, $\{f,\{g,h\}\} + \{g,\{h,f\}\} + \{h,\{f,g\}\} = 0$. □

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