Rigidity of Poisson group actions

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In this talk we first review some classical results of rigidity for group actions of compact Lie groups on smooth manifolds and then we prove some rigidity results in the case the group action preserves a Poisson structure. The details of these proofs can be found in the paper [11] and the preprint [10].

In the general case of actions of compact Lie groups on smooth manifold there are two well-known results that entail rigidity. The first one is the theorem of Bochner [1] that says that actions of compact Lie groups can be linearized in a neighbourhood of a fixed point for the action. The second one is the theorem of Palais [12], that establishes that C^1-close actions of compact Lie groups are conjugated via a diffeomorphism close to the identity.

1. The Poisson case. Local rigidity in a neighbourhood of a fixed point

Let (P, Π) be a Poisson manifold and let ρ stand for a Poisson action of a compact Lie group G. Ginzburg proved in [6] that Poisson actions are rigid by deformations. In fact using the proof of rigidity by deformations provided in the book [7], we can prove rigidity by deformations for actions preserving additional structures.

In the case when we are not given a path of actions connecting both actions and preserving the Poisson structure, the first attempt is to try to use Moser's path method as can be done in the symplectic case (see [14] and [2]).

Unlike the symplectic case, the path method does not seem to work so well for Poisson structures. Locally, we can construct paths using the smooth geometric data given by the theorem of Vorobjev [13] associated to the Poisson structure.

In order to guarantee that the geometric data are smooth we need to assume an additional hypothesis on the Poisson structure at a point, which we call tameness [11].

In [11] this tameness condition is studied and several examples of tameness and non-tameness are given. In particular, all 2 and 3-dimensional Lie algebras are tame Poisson structures and all semisimple Lie algebras of compact type are tame.

For this class of Poisson structures, we can find Weinstein's splitted coordinates [15] for the Poisson structure such that the group action is locally linear as it is proven in [11]. Namely,

Theorem 1.1. *Let (P^n, Π) be a smooth Poisson manifold, p a point of P, $2k =$ rank $\Pi(\mathrm{p})$, and G a compact Lie group which acts on P in such a way that the action preserves Π and fixes the point p. Assume that the Poisson structure Π is tame at p. Then there is a smooth canonical local coordinate system $(x_1, y_1, \ldots, x_k, y_k,$*

$z_1, \ldots, z_{n-2k})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{k} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \sum_{ij} f_{ij}(z) \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1.1}$$

and in which the action of G *is linear and preserves the subspaces* $\{x_1 = y_1 = \ldots x_k = y_k = 0\}$ *and* $\{z_1 = \ldots = z_{n-2k} = 0\}$.

We can combine this result with Conn's linearization theorem [5] for semisimple Lie algebras of compact type, to obtain the following equivariant linearization result (also contained in [11]).

Theorem 1.2. *Let* (P^n, Π) *be a smooth Poisson manifold,* p *a point of* P*,* $2r =$ rank Π(p)*, and* G *a compact Lie group which acts on* P *in such a way that the action preserves* Π *and fixes the point* p*. Assume that the linear part of transverse Poisson structure of* Π *at* p *corresponds to a semisimple compact Lie algebra* $\mathfrak{k}$*. Then there is a smooth canonical local coordinate system* $(x_1, y_1, \ldots, x_r, y_r, z_1, \ldots, z_{n-2r})$ *near* p*, in which the Poisson structure* Π *can be written as*

$$\Pi = \sum_{i=1}^{r} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i} + \frac{1}{2} \sum_{i,j,k} c_{ij}^k z_k \frac{\partial}{\partial z_i} \wedge \frac{\partial}{\partial z_j}, \tag{1.2}$$

where c_{ij}^k *are structural constants of* $\mathfrak{k}$*, and in which the action of* G *is linear and preserves the subspaces* $\{x_1 = y_1 = \ldots x_r = y_r = 0\}$ *and* $\{z_1 = \ldots = z_{n-2r} = 0\}$.

2. The Poisson case. Rigidity for close smooth actions.

In the case that we consider C^1-close Hamiltonian actions with moment maps $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$ on a compact Poisson manifold (M, Π), we can prove a rigidity result for the case the Lie algebra $\mathfrak{g}$ is semisimple of compact type. Two moment maps $\mu_0 : M \longrightarrow \mathfrak{g}^*$ and $\mu_1 : M \longrightarrow \mathfrak{g}^*$ are C^1-close if they are close in the C^1-topology.

More precisely we can prove [10],

Theorem 2.1. *Let* $\mu_0 : M \longrightarrow \mathfrak{g}^*$ *and* $\mu_1 : M \longrightarrow \mathfrak{g}^*$ *be* C^1*-close moment maps with* M *compact and* $\mathfrak{g}$ *semisimple of compact type, then there exists a Poisson diffeomorphism* Φ *such that* $\mu_1 = \mu_0 \circ \Phi$.

The proof uses the inverse theorem of Nash and Moser [8] via the statement provided by Hamilton in [9] which uses exact sequences.

Roughly speaking, the result of Hamilton says that if a linear-complex defined on graded Frchet spaces is locally exact via tame homotopy operators, then a non-linear complex which has this associated linear-complex is also locally exact via tame homotopy operators (for details and definitions about graded Frchet spaces and this tameness condition see [8]).

In our case a Hamiltonian action induces on the set of smooth functions $\mathcal{C}^\infty(M)$ the structure of a a $\mathfrak{g}$-module. We can associate a Chevalley-Eilenberg complex to this $\mathfrak{g}$-module as explained in [3]. The space of cochains is given by multilinear

alternating functions from $\mathfrak{g}$ to $\mathcal{C}^\infty(M)$ and the differential is that of Chevalley and Eilenberg [3].

Using an adaptation of a lemma of Conn [5] valid in the case $\mathfrak{g}$ is semisimple of compact type and a weak version of Sobolev lemma, we can prove that this Chevalley-Eilenberg is locally exact via homotopy operators that are tame. We then associate a non-linear complex to this complex and apply Hamilton's statement of Nash-Moser theorem to prove that the new complex is exact.

Exactness of the non-linear complex complex gives $\mu_1 = \mu_0 \circ \Phi$ where Φ is the time-1-map of a Hamiltonian vector field. So indeed Φ is not only a Poisson diffeomorphism but also a Hamiltonian diffeomorphism.

In the case M is not compact but $M = B_R$ is a ball centered at the origin of $\mathbb{R}^n$, we can prove the following result contained in [10]. The proof uses an iterative method inspired by Newton's method explained in [8] to define the equivalence of the nearby Hamiltonian actions.

Theorem 2.2. *Let $\mu_0 : B_R \longrightarrow \mathfrak{g}^*$ and $\mu_1 : B_R \longrightarrow \mathfrak{g}^*$ be two C^1-close moment maps with $\mathfrak{g}$ semisimple of compact type, then there exists a Poisson diffeomorphism $\Phi : B_{R/2} \longrightarrow B_{R/2}$ such that $\mu_1 = \mu_0 \circ \Phi$.*

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