Modular vector fields in b-symplectic manifolds

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Problem 1 (I)

Let (M^n, Π) an orientable, connected Poisson manifold. Then, we know that $\Omega^n(M) \cong \mathcal{C}^\infty(M)$. We define the **modular vector field** as

$$\begin{array}{rccc} X_\Pi^\Omega : & \mathcal{C}^\infty(M) & \longrightarrow & \mathcal{C}^\infty(M) \\ & f & \longmapsto & \frac{\mathcal{L}_{u_f}\Omega}{\Omega} \end{array},$$

or, more formally, $X_\Pi^\Omega(f)$ is the only function such that $\mathcal{L}_{u_f}\Omega = X_\Pi^\Omega(f)\Omega$. Here, u_f denotes the Hamiltonian vector field of f, this means, such that $u_f(g) = \{f, g\}$.

Problem 1 (II)

a) Show that X_Π^Ω is a well defined derivation.

b) Show that, for any $H \in \mathcal{C}^\infty(M)$ nowhere vanishing,

$$X_\Pi^{H\Omega} = X_\Pi^\Omega - u_{\log|H|}.$$

c) Let (M^{2m}, ω) a symplectic manifold. Show that the modular vector field $X_{\omega^{-1}}^\Omega$ is a Hamiltonian vector field.
(Hint: Compute the modular vector field in local Darboux coordinates and use the previous part of the exercise to get the global result)

d) Compute the modular vector field for the *b*-Poisson manifold $(\mathbb{R}^2, \{\cdot,\cdot\})$, where $\{x, y\} = y$.

Problem 1 (III)

Show that X_Π^Ω is a well defined derivation (i.e. vector field)

$\{\cdot,\cdot\}$ is a biderivation, so

$$u_{\alpha f+\beta g} = \alpha u_f + \beta u_g \; ; \; u_{fg} = g u_f + f u_g .$$

This implies that

$$X_\Pi^\Omega(\alpha f + \beta g) = \alpha X_\Pi^\Omega(f) + \beta X_\Pi^\Omega(g),$$

$$X_\Pi^\Omega(fg) = g X_\Pi^\Omega(f) + f X_\Pi^\Omega(g)$$

Problem 1 (IV)

Show that, for any $H \in \mathcal{C}^\infty(M)$ nowhere vanishing, $X_\Pi^{H\Omega} = X_\Pi^{\Omega} - u_{\log|H|}$.

First of all,

$$u_{\log|H|}(f) = \{\log|H|, f\} = -\{f, \log|H|\} = -u_f(\log|H|) = -\frac{1}{H}u_f(H).$$

Then,

$$X_\Pi^{H\Omega}(f)\Omega = \frac{1}{H}X_\Pi^{H\Omega}(f)H\Omega = \frac{1}{H}\mathcal{L}_{u_f}(H\Omega) =$$
$$\mathcal{L}_{u_f}(\Omega) + \frac{1}{H}u_f(H)\Omega = \left(X_\Pi^{\Omega}(f) - u_{\log|H|}(f)\right)\Omega$$

Problem 1 (V)

Let (M^{2m}, ω) a symplectic manifold. Show that the modular vector field $X^{\Omega}_{\omega^{-1}}$ is a Hamiltonian vector field.

- ω^m is a volume form, and therefore $\Omega = F\omega^m$ for some $F \in \mathcal{C}^\infty(M)$ nowhere vanishing. Thus,

$$X^{\Omega}_{\omega^{-1}} = X^{\omega^m}_{\omega^{-1}} - u_{\log|F|}.$$

- Question: given $f \in \mathcal{C}^\infty(M)$, can we compute

$$\mathcal{L}_{u_f}\omega^m?$$

Problem 1 (VI)

Short answer: Yes.

The Hamiltonian vector field is symplectic, so it preserves the volume:

$$\mathcal{L}_{u_f}\omega^m = 0.$$

Problem 1 (VII)

Long answer: Wait, but can you compute it?
Also yes.
Taking Darboux coordinates,

$$\omega = \sum_{i=1}^{m} dx_i \wedge dy_i$$

$$u_f = \sum_{i=1}^{m} \left(\frac{\partial f}{\partial x_i} \frac{\partial}{\partial y_i} - \frac{\partial f}{\partial y_i} \frac{\partial}{\partial x_i} \right)$$

$$\omega^m = dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m.$$

Using Cartan's formula,

$$\mathcal{L}_{u_f} \omega^m = d(i_{u_f} \omega^m)$$

Problem 1 (VIII)

Computing,

$$i_{u_f}\omega^m =$$

$$= \sum_{i=1}^{m} -\frac{\partial f}{\partial y_i} dx_1 \wedge dy_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_k \wedge dy_m -$$
$$-\frac{\partial f}{\partial x_i} dx_1 \wedge dy_1 \wedge \cdots \wedge \widehat{dy_i} \wedge \cdots \wedge dx_k \wedge dy_m$$

Thus

$$d(i_{u_f}\omega^m) = \sum_{i=1}^{m} \left(\frac{\partial^2 f}{\partial y_i \partial x_i} - \frac{\partial^2 f}{\partial x_i \partial y_i} \right) dx_1 \wedge dy_1 \wedge \ldots \wedge dx_m \wedge dy_m = 0$$

Problem 1 (IX)

In conclusion,

$$X_{\omega^{-1}}^{\omega^m} = 0,$$

and therefore

$$X_{\omega^{-1}}^{F\omega^m} = -u_{\log|F|},$$

which is Hamiltonian.

Problem 1 (X)

Compute the modular vector field for the b-Poisson manifold $(\mathbb{R}^2, \{\cdot, \cdot\})$, where $\{x, y\} = y$.

We take $\Omega = dx \wedge dy$.

$$\{f, g\} = y\frac{\partial f}{\partial x}\frac{\partial g}{\partial y} - y\frac{\partial f}{\partial y}\frac{\partial g}{\partial x},$$

so

$$u_f = y\frac{\partial f}{\partial x}\frac{\partial}{\partial y} - y\frac{\partial f}{\partial y}\frac{\partial}{\partial x}$$

Problem 1 (XI)

As in the last part we compute

$$X_\pi^\Omega(f)\Omega = \mathcal{L}_{u_f}\Omega = d\left(i_{u_f}(dx \wedge dy)\right) =$$
$$= d\left(y\left(-\frac{\partial f}{\partial x}dx - \frac{\partial f}{\partial y}dy\right)\right) =$$
$$= -dy \wedge \left(\frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy\right) = \frac{\partial f}{\partial x}dx \wedge dy$$
$$\Rightarrow X_\pi^\Omega(f) = \frac{\partial f}{\partial x}$$

Thus, $X_\pi^\Omega = \frac{\partial}{\partial x}$.

Which is not a Hamiltonian vector field!

Problem (XII)

In the general case,

$$\omega = \frac{dz}{z} \wedge dt + \sum_{i=1}^{n-1} dx_i \wedge dy_i,$$

then the modular vector field is

$$X^{\Omega}_{\omega^{-1}} = \frac{\partial}{\partial t}.$$

This vector field is

1. A b-vector field (tangent to $Z = \{z = 0\}$).
2. Transverse to the leaves of the symplectic foliation within Z.