Geometry and Dynamics of Singular Symplectic manifolds

Eva Miranda

UPC & CRM & Observatoire de Paris

Henan University, Day 8
https://web.mat.upc.edu/eva.miranda/coursHenan.htm

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

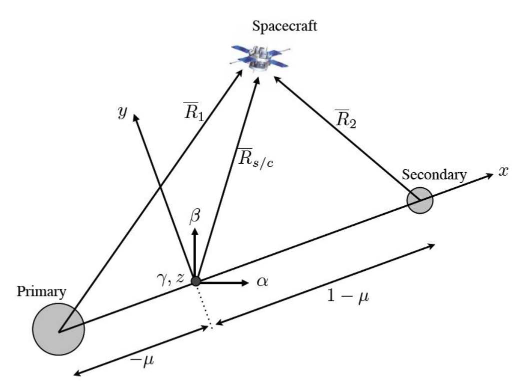
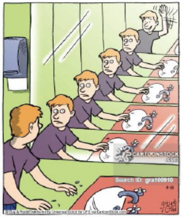
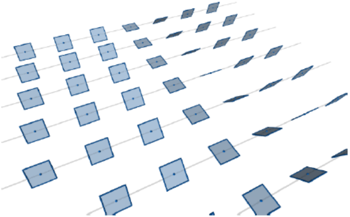


Figure: Circular 3-body problem





b-Symplectic	b-Contact
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

b-forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- We can introduce **b-contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$.

Attacking the b^m-Weinstein's conjecture

Theorem (M-Oms)

Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.

Proof.

1. $\alpha = u\frac{dz}{z^m} + \beta$
2. The restriction on Z of the 2-form $\Theta = ud\beta + \beta \wedge du$ is symplectic and the Reeb vector field is Hamiltonian.
3. u is non-constant on Z.
4. R_α is Hamiltonian on Z for $-u$,
5. $u^{-1}(p)$ where p regular is a circle,
6. R_α periodic on $u^{-1}(p)$.

□

New families of periodic orbits

Contact geometry of RPC3BP revisited

In rotating coordinates: $H(q,p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$

- Symplectic polar coordinates: $(r, \alpha, P_r, P_\alpha)$.
- McGehee change of coordinates: $r = \frac{2}{x^2}$.

b^3-symplectic form: $-4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha$.

Theorem

After the McGehee change, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y \omega)$ for $c > 0$ are b^3-contact manifolds. The critical set is a cylinder and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.

New families of periodic orbits

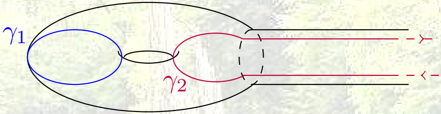
Proof.

- On the critical set, Hamiltonian $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H) = P_r^2 - P_\alpha = \frac{1}{2}\frac{P_r^2}{2} + c > 0$;
- b^3-contact form $\alpha = (P_r \frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$ with $Z = \{(x, \alpha, P_r, P_\alpha) | x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}$;
- $R_\alpha|_Z = X_{P_r}$ and the cylinder is foliated by periodic orbits.

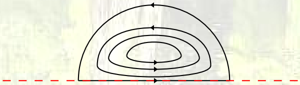
□

The singular Weinstein conjecture re-loaded

A true singular Weinstein structures should also admit singular orbits as below:



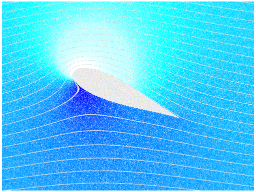
Or,



Singular Weinstein conjecture

Let (M, α) be a compact b-contact manifold with critical hypersurface Z. Then there exists always a Reeb orbit $\gamma : \mathbb{R} \to M \setminus Z$ such that $\lim_{t \to \pm\infty} \gamma(t) = p_{\pm} \in Z$ and $R_{\alpha}(p_{\pm}) = 0$ (singular periodic orbit).

Incompressible fluids on Riemannian manifolds



Classical Euler equations on $\mathbb{R}^3$:

$$\begin{cases} \dfrac{\partial X}{\partial t} + (X \cdot \nabla) X = -\nabla P \\ \operatorname{div} X = 0 \end{cases}$$

The evolution of an **inviscid and incompressible fluid flow** on a Riemannian n-dimensional manifold (M, g) is described by the Euler equations:

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla P \,, \qquad X = 0$$

- X is the velocity field of the fluid: a non-autonomous vector field on M.
- P is the inner pressure of the fluid: a time-dependent scalar function on M.

Incompressible fluids on Riemannian manifolds

If X does not depend on time, it is a steady or stationary Euler flow: it models a fluid flow in equilibrium. The equations can be written as:

$$\nabla_X X = -\nabla P\,, \qquad X = 0\,,$$

$$\iff \quad i_X d\alpha = -dB\,, \qquad d\iota_X \mu = 0\,, \qquad \alpha(\cdot) := g(X, \cdot)$$

where $B := P + \frac{1}{2}||X||^2$ is the Bernoulli function.

Beltrami fields:

$\operatorname{curl} X = fX,$ with $f \in C^\infty(M) \qquad X = 0.$

Example (Hopf fields on S^3 and ABC fields on T^3)

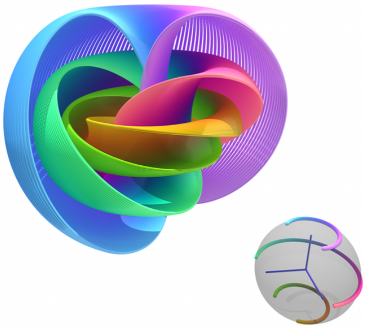
- The Hopf fields $u_1 = (-y, x, \xi, -z)$ and $u_2 = (-y, x, -\xi, z)$ are Beltrami fields on S^3.
- The ABC flows
  $(\dot{x}, \dot{y}, \dot{z}) = (A\sin z + C\cos y, B\sin x + A\cos z, C\sin y + B\cos x)$,
  $((x, y, z) \in (\mathbb{R}/2\pi\mathbb{Z})^3)$ are Beltrami.

The Hopf fibration as a Reeb flow

$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2+|v|^2 = 1\}, \alpha = \frac{1}{2}(ud\overline{u} - \overline{u}du + vd\overline{v} - \overline{v}dv)$.

The orbits of the Reeb vector field form the Hopf fibration! Why?

$$R_\alpha = iu\frac{\partial}{\partial u} - i\overline{u}\frac{\partial}{\partial \overline{u}} + iv\frac{\partial}{\partial v} - i\overline{v}\frac{\partial}{\partial \overline{v}}$$



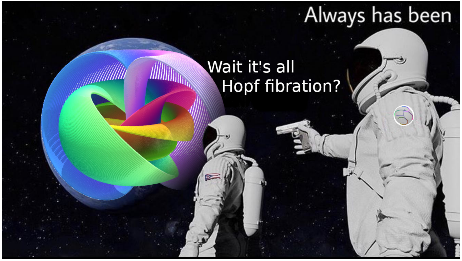
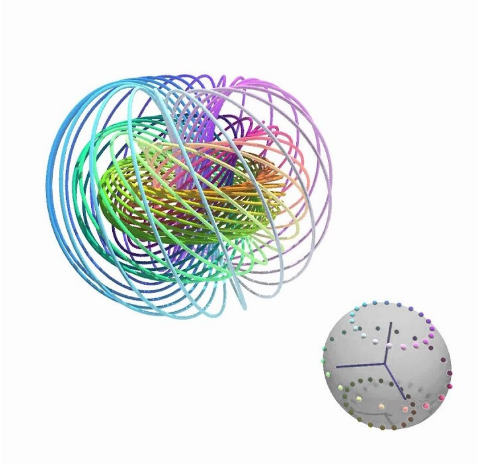


Figure: Pictures by Niles Johnson

Déja vu?

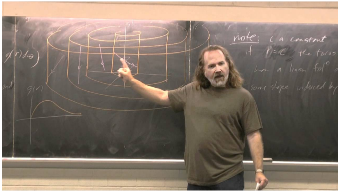
$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ can be endowed with Hopf coordinates $(z_1, z_2) = (\cos s \exp i\phi_1, \sin s \exp i\phi_2)$, $s \in [0, \pi/2]$, $\phi_{1,2} \in [0, 2\pi)$. The Hopf field $R := \partial_{\phi_1} + \partial_{\phi_2}$ is a steady Euler flow (Beltrami) with respect to the round metric.



The magic mirror

In terms of $\alpha = \iota_X g$ and μ (volume form) the **stationary Euler equations** read

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$



- Etnyre-Ghrist:
  {Rotational non singular Beltrami v.f.} $\rightleftarrows$ {Reeb v.f. reparametrized}
- With Cardona and Peralta-Salas we have extended this picture to manifolds with cylindrical ends to get singular contact structures.
- CMPP: The Beltrami/contact correspondence works in higher dimensions.

Let's prove it!

- The Beltrami equation $\leftrightsquigarrow$ $d\alpha = f\iota_X\mu$. Since $f > 0$ and X does not vanish $\rightsquigarrow$ $\alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$.
- X satisfies $\iota_X(d\alpha) = \iota_X\iota_X\mu = 0$ so $X \in \ker d\alpha$ $\leftrightsquigarrow$ it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.

A magic mirror

- Weinstein conjecture for Reeb vector fields $\rightsquigarrow$ **periodic orbits for Beltrami vector fields**
- Uhlenbeck's genericity properties of eigenfunctions of Laplacian $\rightsquigarrow$ **existence of singular periodic orbits** (M-Oms-Peralta)

Escape orbits and Singular orbits

Singular periodic orbits are a particular case of *escape orbits* γ, $\gamma \subset M \setminus Z$ such that $\lim_{t\to\infty} \gamma(t) = p$ where p is an equilibrium point in Z (respectively $\lim_{t\to-\infty} \gamma(t) = p$).

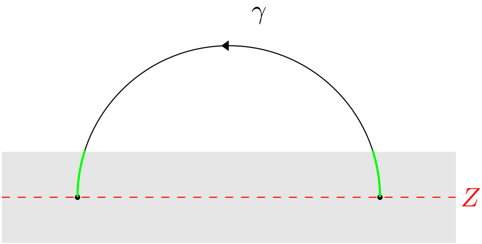


Figure: Singular periodic orbit vs. Escape orbits (in green)

A magic mirror

b-Beltrami vector fields to the rescue

b-Beltrami vector field X $\text{curl}X = \lambda X$

Theorem (Cardona-M.-Peralta-Salas)

- *Any rotational Beltrami field non-vanishing as a section of* ^bTM *on* M *is a Reeb vector field (up to rescaling) for some* b*-contact form on* M*.*
- *Given a* b*-contact form* α *with Reeb vector field* X *then any nonzero rescaling of* X *is a rotational Beltrami field for some* b*-metric and* b*-volume form on* M*.*

Practical tip

X is a Beltrami vector field on (M, g) $\leftrightsquigarrow$ the Reeb vector field associated to the b-contact form $\alpha = g(X, \cdot)$ is given by $\frac{1}{\|X\|^2}X$.

True inspiration comes in a hat...

For regular Beltrami fields, there cannot exist surfaces invariant by Hamiltonian vector fields. However for singular vector fields....

We need a super(wo)man

Escape orbits and Singular orbits

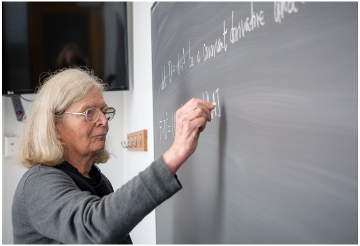
Exact b-metric $\leftrightsquigarrow$ Melrose b-contact forms:

$$g = \frac{dz^2}{z^2} + \pi^* h \tag{1}$$

with h Riemannian metric on Z.

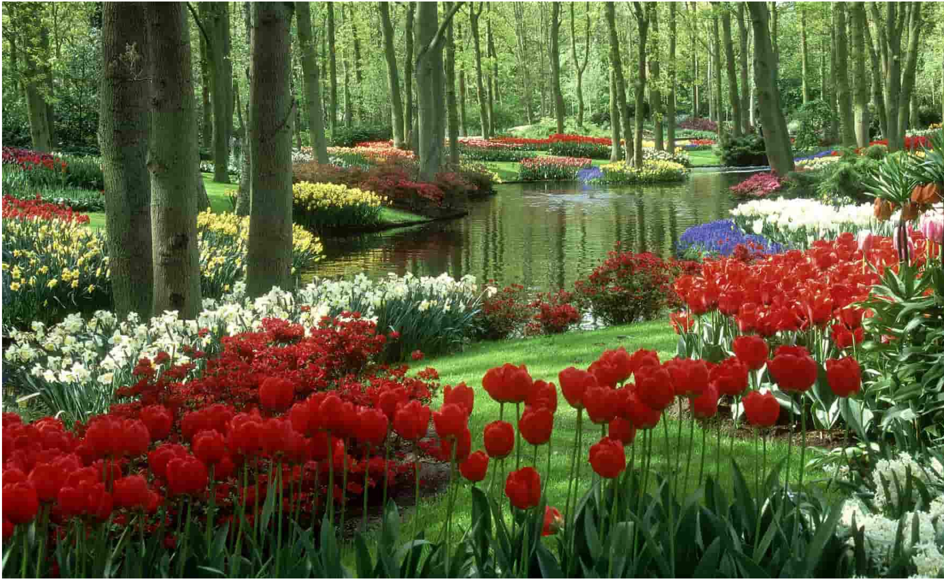
Theorem (M-Oms-Peralta, "lockdown theorem")

There exists at least $2 + b_1(Z)$ escape orbits for Reeb vector fields of generic Melrose b-contact forms on (M, Z).



Proof: The Beltrami equation $\rightsquigarrow$ the Hamiltonian function associated to (R, Z) is an eigenfunction of the induced Laplacian on Z $\rightsquigarrow$ (Uhlenbeck) generically Morse and non-zero critical values.

A garden of singular orbits



A garden of singular orbits

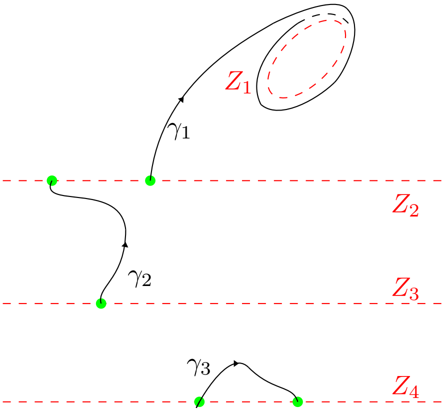


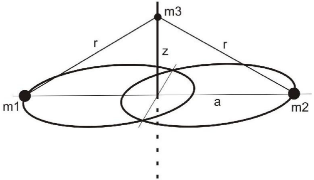
Figure: Different types of escape and singular periodic orbits: γ_1 is a generalized singular periodic orbit, γ_2, γ_3 are singular periodic orbits

Generalized singular periodic orbits

Definition

An orbit $\gamma : \mathbf{R} \to M \setminus Z$ of a b-Beltrami field X is a generalized singular periodic orbit if there exist $t_1 < t_2 < \cdots < t_k \to \infty$ such that $\gamma(t_k) \to p_+ \in Z$ and $t_{-1} > t_{-2} > \cdots > t_{-k} \to -\infty$ such that $\gamma(t_{-k}) \to p_- \in Z$, as $k \to \infty$.

p_+ and p_- may be contained in different components and are not necessarily zeros of X.



This includes oscillatory motions:orbits $(q(t), p(t))$ in the phase space $T^*\mathbb{R}^n$ such that $\limsup_{t\to\pm\infty} \|q(t)\| = \infty$ and $\liminf_{t\to\pm\infty} \|q(t)\| < \infty$.

A more symmetric case

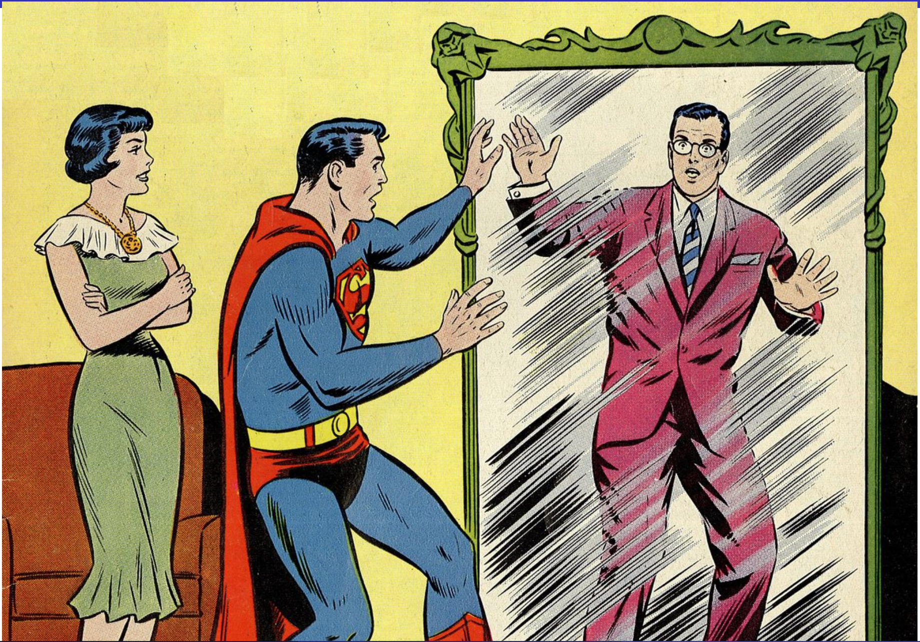
For $g = \frac{dz^2}{z^2} + dx^2 + dy^2$, we can prove more.

Theorem (M-Oms-Peralta Salas)

When g is semi-locally as above and X a generic asymptotically symmetric b-Beltrami vector field, X has a generalized singular periodic orbit. Moreover, it has a singular periodic orbit or at least 4 escape orbits.

In the case of $(\mathbb{T}^3, \alpha = C \cos y dx + B \sin x dy + (C \sin y + B \cos x)\frac{dz}{\sin z})$ for $|B| \neq |C|$, the singular Weinstein conjecture is satisfied.

What about the restricted three body problem?

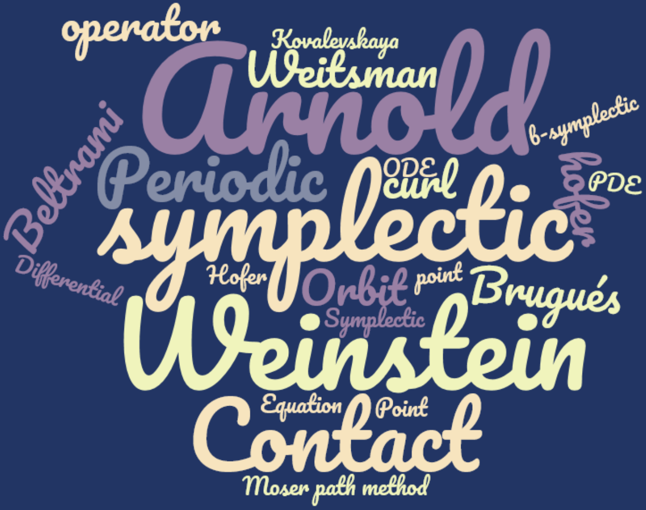


Final questions

- Can we prove existence of singular Weinstein orbits for generic b-contact forms?
- Extend the apparatus of variational calculus to extend the action functional to this set-up.

$$\mathcal{A}_\alpha(\gamma) = \int_\gamma \alpha$$

- Find higher dimensional applications to celestial mechanics (for instance, escape orbits 5-body problem).


operator
Kovalevskaya
Weitsman
Arnold
Beltrami
b-symplectic
Periodic
ODE
curl
hofer
PDE
symplectic
Differential
Hofer
Orbit
point
Brugués
Symplectic
Weinstein
Equation
Point
Contact
Moser path method


SUMMARY

(Singular) symplectic manifolds

b^m -Symplectic

Symplectic

Folded symplectic

Obstruction theory via cohomology

Theorem (Guillemin-M-Pires)

For a compact b-symplectic manifold (M, Z) we have $H^1(Z) \neq \{0\}$ and consequently ${}^bH^2(M) \neq \{0\}$.

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Obstruction theory via cohomology

Theorem (Marcut-Osorno, and Oms)

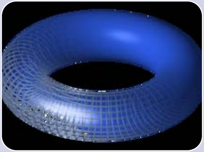
Let (M^{2n}, ω) be an orientable b-symplectic manifold with compact critical hypersurface Z, then there exists an element $c \in H^2(M)$ such that $c^{n-1} \neq 0$.

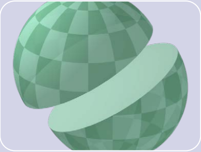
$\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$	symplectic	*bona fide* log-symplectic
$m > 1$, $n > 0$	✗	✓
$m > 1$, $n = 0$	✗	✗
$m = 1$, $n > 0$	✓	✓
$m = 1$, $n = 0$	✓	✗
$m = 0$, $n > 0$	✗	✗

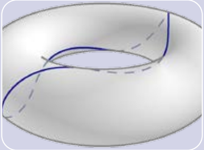
Theorem (Cavalcanti)

If a compact oriented manifold M^{2n} , with $n > 1$, admits a b-symplectic structure then there are classes $a, b \in H^2(M, \mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2 = 0$.

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

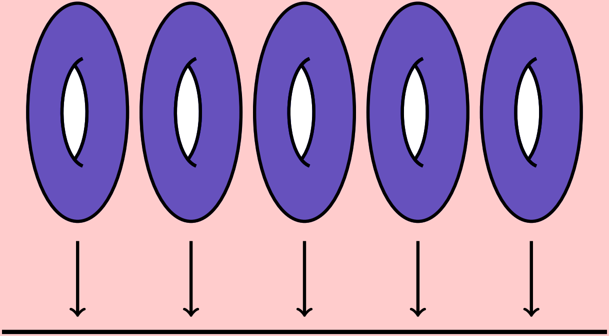
Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

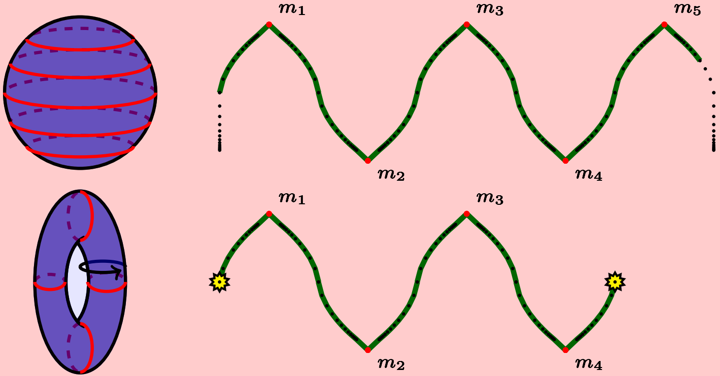
Liouville torus and integrable systems



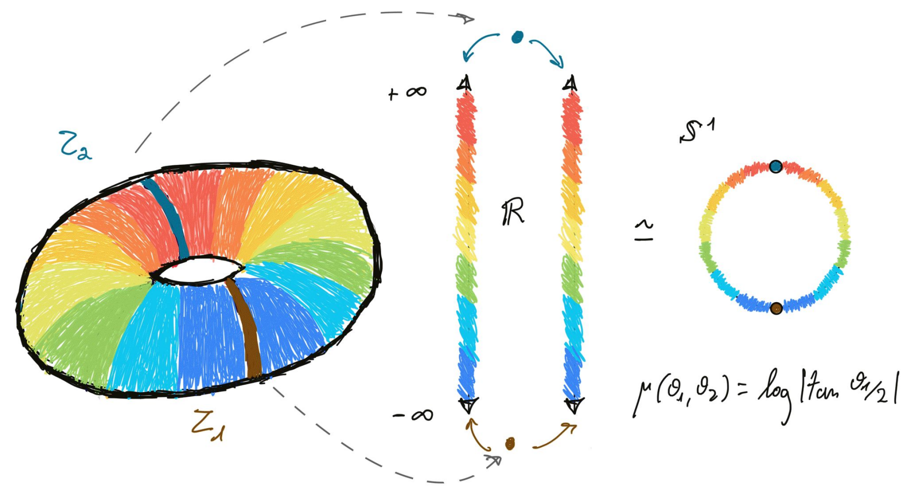
KAM theory $\rightsquigarrow$ "some" of the Liouville torus survive under perturbations of the integrable system.

b-surfaces and their moment map

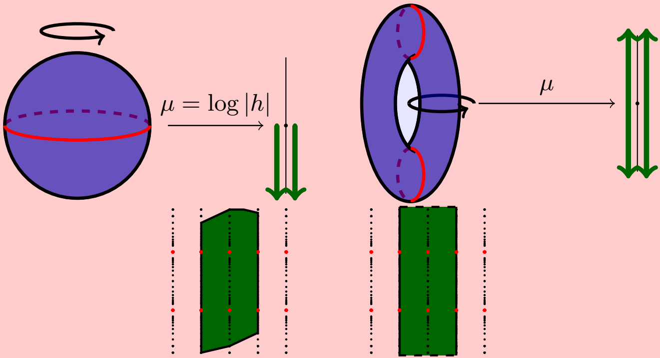
A toric b-surface is defined by a smooth map $f : S \longrightarrow {}^b\mathbb{R}$ or $f : S \longrightarrow {}^b\mathbb{S}^1$ (a posteriori the moment map).



A picture done by a student of this class



A b-Delzant theorem



Guillemin-M.-Pires-Scott

There is a one-to-one correspondence between b-toric manifolds and b-Delzant polytopes. Toric b-manifolds are either:

- ${}^b\mathbb{T}^2 \times X$ (X a toric symplectic manifold of dimension $(2n-2)$).
- obtained from ${}^b\mathbb{S}^2 \times X$ via symplectic cutting.

Periodic orbits and applications