Geometry and Dynamics of Singular Symplectic manifolds

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Space for notes

Obstruction theory via cohomology

Theorem (Guillemin-M-Pires)

For a compact b-symplectic manifold (M, Z) we have $H^1(Z) \neq \{0\}$ and consequently ${}^bH^2(M) \neq \{0\}$.

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Obstruction theory via cohomology

Theorem (Marcut-Osorno, and Oms)

Let (M^{2n}, ω) be an orientable b-symplectic manifold with compact critical hypersurface Z, then there exists an element $c \in H^2(M)$ such that $c^{n-1} \neq 0$.

$\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$	symplectic	*bona fide* log-symplectic
$m > 1$, $n > 0$	✗	✓
$m > 1$, $n = 0$	✗	✗
$m = 1$, $n > 0$	✓	✓
$m = 1$, $n = 0$	✓	✗
$m = 0$, $n > 0$	✗	✗

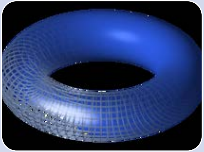
Theorem (Cavalcanti)

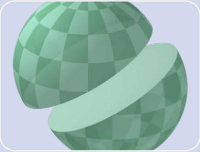
If a compact oriented manifold M^{2n} , with $n > 1$, admits a b-symplectic structure then there are classes $a, b \in H^2(M, \mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2 = 0$.

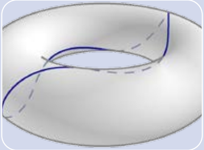
(Singular) symplectic manifolds

- b^m -Symplectic
- Symplectic
- Folded symplectic

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

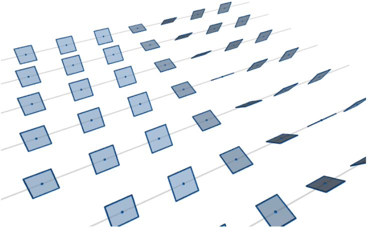
- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k + 1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

In particular:

- Any b^{2k}-symplectic manifold admits a symplectic structure.
- Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.
- The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

Space for notes

The Symplectic/Contact mirror



Symplectic	**Contact**
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Darboux theorem $\omega = \sum_{i=1}^{2n} dx_i \wedge dy_i$	$\alpha = dx_0 - \sum_{i=1}^{2n} x_i dy_i$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

An example

The kernel of a 1-form α on M^{2n+1} is a contact structure whenever $\alpha \wedge (d\alpha)^n$ is a volume form $\Leftrightarrow d\alpha|_\xi$ is non-degenerate.

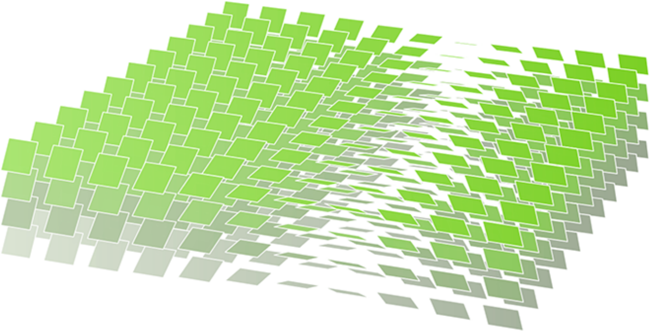


Figure: Standard contact structure on $\mathbb{R}^3$

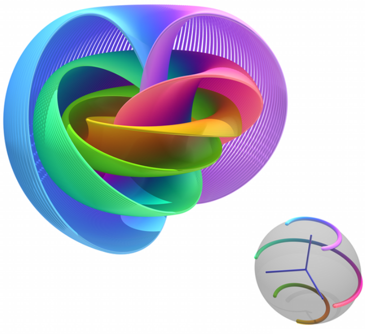
$$\alpha = dz - ydx \quad \xi = \ker \alpha = \mathrm{Span} = \{\frac{\partial}{\partial \mathrm{y}}, \mathrm{y}\frac{\partial}{\partial \mathrm{z}} + \frac{\partial}{\partial \mathrm{x}}\} \; d\alpha = -dy \wedge dx = dx \wedge dy$$

$$\Rightarrow \; \alpha \wedge d\alpha \; = \; dx \wedge dy \wedge dz$$

The Hopf fibration as a Reeb flow

$S^3 := \{(u,v) \in \mathbb{C}^2 \mid |u|^2 + |v|^2 = 1\}, \alpha = \frac{1}{2}(ud\overline{u} - \overline{u}du + vd\overline{v} - \overline{v}dv)$.
The orbits of the Reeb vector field form the Hopf fibration! Why?

$$R_\alpha = iu\frac{\partial}{\partial u} - i\overline{u}\frac{\partial}{\partial \overline{u}} + iv\frac{\partial}{\partial v} - i\overline{v}\frac{\partial}{\partial \overline{v}}$$



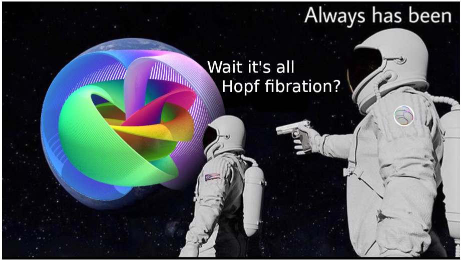
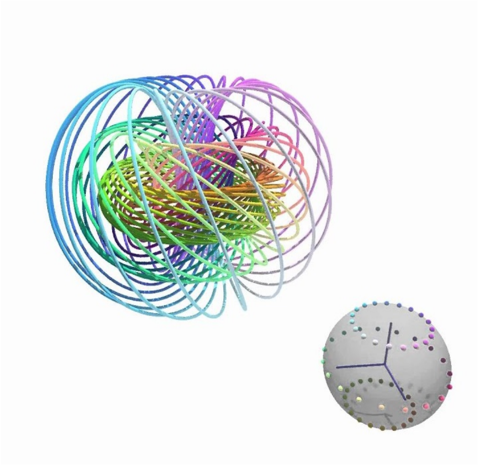


Figure: Pictures by Niles Johnson

Déja vu?

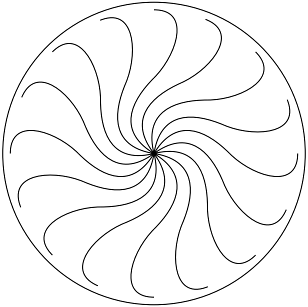
$\mathbb{S}^3 = \{(z_1, z_2) \in \mathbb{C}^2 : |z_1|^2 + |z_2|^2 = 1\}$ can be endowed with Hopf coordinates $(z_1, z_2) = (\cos s \exp i\phi_1, \sin s \exp i\phi_2)$, $s \in [0, \pi/2]$, $\phi_{1,2} \in [0, 2\pi)$. The Hopf field $R := \partial_{\phi_1} + \partial_{\phi_2}$ is a steady Euler flow (Beltrami) with respect to the round metric.



Overtwisted contact structures

Definition

$(M^3, \xi = \ker \alpha)$ is *overtwisted* if there exists D^2 s.t. $TD \cap \xi$ defines a 1-dimensional foliation given by



A contact manifold that is not overtwisted is called *tight*.

Rigidity and Flexibility in contact geometry

S^3 admits non-equivalent contact structures (dichotomy).

1. $\chi_{std} = \ker(dz + r^2 d\theta)$ in cylindrical coordinates (tight standard structure).
2. $\chi_{OT} = \ker(\cos rdz + r \sin rd\theta)$ (overtwisted contact structure).

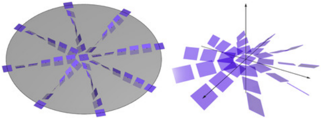


Figure: An overtwisted and a tight structure

Theorem (Eliashberg)

Any almost contact homotopy class on a closed 3-manifold contains a unique (up to isotopy) overtwisted contact structure.

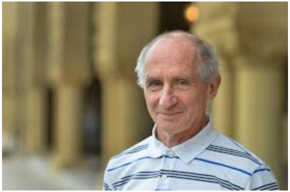
Existence of contact structures

All 3-dimensional manifolds are contact (Martinet-Lutz) and in higher dimensions:

Theorem (Borman-Eliashberg-Murphy)

Any almost contact closed manifold is contact.

The almost contact condition is a formal condition and h-principle is the key ingredient of the proof.

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.

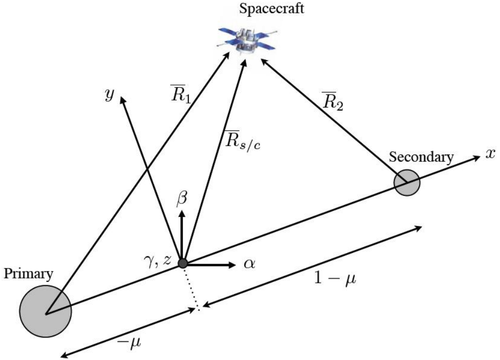


Figure: Circular 3-body problem

An example from Celestial Mechanics: Planar restricted 3-body problem

- The time-dependent self-potential of the small body is $U(q,t) = \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|}$, with $q_E = q_E(t)$ the position of the planet with mass $1-\mu$ at time t and $q_M = q_M(t)$ the position of the one with mass μ.
- The Hamiltonian of the system is $H(q,p,t) = p^2/2 - U(q,t), \quad (q,p) \in \mathbb{R}^2 \times \mathbb{R}^2$, where $p = \dot{q}$ is the momentum of the planet.
- Consider the canonical change $(X, Y, P_X, P_Y) \mapsto (r, \alpha, P_r =: y, P_\alpha =: G)$.
- Introduce **Mc Gehee coordinates** (x, α, y, G), where $r = \frac{2}{x^2}, \quad x \in \mathbb{R}^+$, can be then extended to infinity ($x = 0$).
- The symplectic structure becomes a singular object

$$-\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$$

which extends to a b^3-symplectic structure on $\mathbb{R} \times \mathbb{T} \times \mathbb{R}^2$.

Symplectic and contact geometry of these systems

(b^m-symplectic)

$$\omega = \frac{\mathbf{1}}{\mathbf{x_1^m}}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

or (m-folded)

$$\omega = \mathbf{x_1^m}\mathbf{dx_1} \wedge \mathbf{dy_1} + \sum_{\mathbf{i \geq 2}} \mathbf{dx_i} \wedge \mathbf{dy_i}$$

Contact Geometry

The restriction to $H = ct$ induces a contact structure whenever there exists a *Liouville vector field* is transverse to it. This contact structure may admit singularities.

How are these singularities?

Restricted planar circular 3-body problem

- Time-dependent potential: $U(q,t) = \frac{1-\mu}{|q-q_E(t)|} + \frac{\mu}{|q-q_M(t)|}$
- Time-dependent Hamiltonian:
  $H(q,p,t) = \frac{|p|^2}{2} - U(q,t), \quad (q,p) \in \mathbf{R}^2 \setminus \{q_E, q_M\} \times \mathbb{R}^2$
- Rotating coordinates $\rightsquigarrow$ Time independent Hamiltonian
  $H(q,p) = \frac{p^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1 q_2 - p_2 q_1$

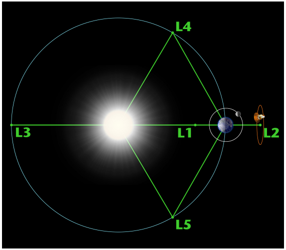
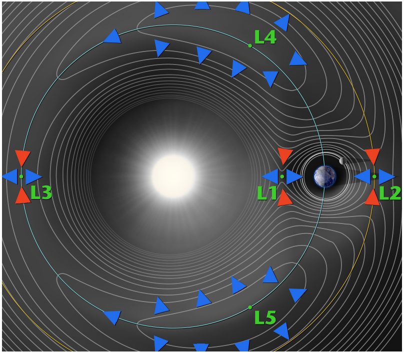


Figure: Lagrange points (Source: NASA/WMAP Science Team)

- H has 5 critical points: L_i Lagrange points $(H(L_1) \leq \cdots \leq H(L_5))$
- Periodic orbits of X_H? Perturbative methods (dynamical systems) or.... contact geometry!

Topology of the circular restricted 3-body problem



- For low energy levels $c \in \mathbb{R}$, $\Sigma_c = H^{-1}(c)$ has 3 connected components: Σ_c^E (the satellite stays close to the earth), Σ_c^M (to the moon), or it is far away.
- On the axis between earth and moon there is a critical point of the energy (L_1, the first Lagrange point). If $c > H(L_1)$, (the satellite can cross from the region around the earth to the region around the moon)$\rightsquigarrow$ there are two connected components, one bounded $\Sigma_c^{E,M}$ and an unbounded one.

Moser regularization of the restricted 3-body problem

- To deal with the singularities of the Kepler problem, Moser (1970) introduced a regularization procedure. This can be applied to the planar circular restricted 3-body problem.
- Via Moser's regularization Σ_c^E and Σ_c^M can be compactified to $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ diffeomorphic to $\mathbb{R}P^3$.
- Moser's regularization $\overline{\Sigma}_c^{E,M}$ is diffeomorphic to $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Contact Geometry of the restricted 3-body problem

Theorem (Albers-Frauenfelder-Van Koert-Paternain)

For $c < H(L_1)$ both connected components $\overline{\Sigma}_c^E$ and $\overline{\Sigma}_c^M$ admit a compatible contact form λ. Moreover, there exists $\epsilon > 0$ such that if $c \in (H(L_1), H(L_1) + \epsilon)$ the same assertion holds true for $\overline{\Sigma}_c^{E,M}$.

Corollary (Albers-Frauenfelder-Van Koert-Paternain)

For $c < H(L_1)$ the contact structures $\left(\overline{\Sigma}_c^E, \ker\lambda\right)$ and $\left(\overline{\Sigma}_c^M, \ker\lambda\right)$ coincide with the tight $\mathbb{R}P^3$ and for $c \in (H(L_1), H(L_1) + \epsilon)$ the contact structure $\left(\overline{\Sigma}_c^{E,M}, \ker\lambda\right)$ coincides with the tight $\mathbb{R}P^3 \# \mathbb{R}P^3$.

Contact geometry of the restricted 3-body problem

Weinstein's conjecture 

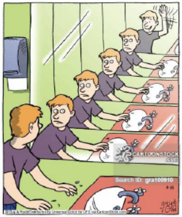
The Reeb vector field of a contact compact manifold admits at least one periodic orbit.

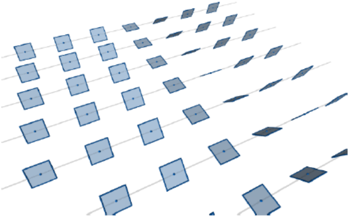
Taubes proved it in dimension 3 for regular contact structures. Application:

Theorem (Albers-Frauenfelder-Van Koert-Paternain)

For any value $c < H(L_1)$, the regularized planar circular restricted three body problem has a closed orbit with energy c.

- **What if we consider the b^3-symplectic model?**
- **Does this contact structure have singularities?**
- **Can we still prove the existence of periodic orbits?**
- **Can we localize these periodic orbits with respect to the line at infinity?**





b-**Symplectic**	*b*-**Contact**
$\dim M = 2n$	$\dim M = 2n+1$
2-form ω, non-degenerate $d\omega = 0$	1-form α, $\alpha \wedge (d\alpha)^n \neq 0$
Hamiltonian $\iota_{X_H}\omega = -dH$	Reeb $\alpha(R) = 1$, $\iota_R d\alpha = 0$ Ham. $\begin{cases} \iota_{X_H}\alpha = H \\ \iota_{X_H} d\alpha = -dH + R(H)\alpha. \end{cases}$

b-forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The b-**tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are b-**forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- We can introduce **b-contact structures on a manifold** M^{2n+1} as b-forms of degree 1 for which $\alpha \wedge (d\alpha)^n \neq 0$.

Attacking the b^m-Weinstein's conjecture

Theorem (M-Oms)

Let (M, α) be a 3-dimensional b^m-contact manifold and assume the critical hypersurface Z to be closed. Then there exists infinitely many periodic Reeb orbits on Z.

Contact geometry of RPC3BP revisited

In rotating coordinates: $H(q,p) = \frac{|p|^2}{2} - \frac{1-\mu}{|q-q_E|} + \frac{\mu}{|q-q_M|} + p_1q_2 - p_2q_1$

- Symplectic polar coordinates: $(r, \alpha, P_r, P_\alpha)$.
- McGehee change of coordinates: $r = \frac{2}{x^2}$.

b^3-symplectic form: $-4\frac{dx}{x^3} \wedge dP_r + d\alpha \wedge dP_\alpha$.

Theorem

After the McGehee change, the Liouville vector field $Y = p\frac{\partial}{\partial p}$ is a b^3-vector field that is everywhere transverse to Σ_c for $c > 0$ and the level-sets $(\Sigma_c, \iota_Y\omega)$ for $c > 0$ are b^3-contact manifolds. The critical set is a cylinder and the Reeb vector field admits infinitely many non-trivial periodic orbits on the critical set.

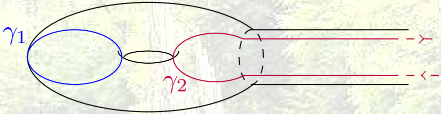
Proof.

- On the critical set, Hamiltonian $H = \frac{1}{2}P_r^2 - P_\alpha$, so that $Y(H) = P_r^2 - P_\alpha = \frac{1}{2}\frac{P_r^2}{2} + c > 0$;
- b^3-contact form $\alpha = (P_r \frac{dx}{x^3} + P_\alpha d\alpha)|_{H=c}$ with $Z = \{(x, \alpha, P_r, P_\alpha) | x = 0, \frac{1}{2}P_r^2 - P_\alpha = c\}$;
- $R_\alpha|_Z = X_{P_r}$ and the cylinder is foliated by periodic orbits.

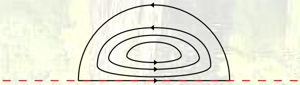
□

The singular Weinstein conjecture re-loaded

A true singular Weinstein structures should also admit singular orbits as below:



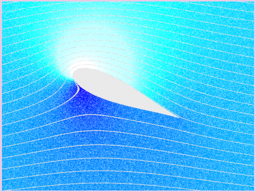
Or,



Singular Weinstein conjecture

Let (M, α) be a compact b-contact manifold with critical hypersurface Z. Then there exists always a Reeb orbit $\gamma : \mathbb{R} \to M \setminus Z$ such that $\lim_{t\to\pm\infty} \gamma(t) = p_{\pm} \in Z$ and $R_{\alpha}(p_{\pm}) = 0$ (singular periodic orbit).

Incompressible fluids on Riemannian manifolds



Classical Euler's equations on $\mathbb{R}^3$:

$$\begin{cases} \frac{\partial X}{\partial t} + (X \cdot \nabla) X = -\nabla P \\ \operatorname{div} X = 0 \end{cases}$$

The evolution of an **inviscid and incompressible fluid flow** on a Riemannian n-dimensional manifold (M, g) is described by the Euler equations:

$$\frac{\partial X}{\partial t} + \nabla_X X = -\nabla P \,, \qquad X = 0$$

- X is the velocity field of the fluid: a non-autonomous vector field on M.
- P is the inner pressure of the fluid: a time-dependent scalar function on M.

Incompressible fluids on Riemannian manifolds

If X does not depend on time, it is a steady or stationary Euler flow: it models a fluid flow in equilibrium. The equations can be written as:

$$\nabla_X X = -\nabla P\,, \qquad X = 0\,,$$

$$\iff \; i_X d\alpha = -dB\,, \qquad d\iota_X \mu = 0\,, \qquad \alpha(\cdot) := g(X,\cdot)$$

where $B := P + \frac{1}{2}||X||^2$ is the Bernoulli function.

Beltrami fields:

$\operatorname{curl} X = fX,$ with $f \in C^\infty(M) \qquad X = 0.$

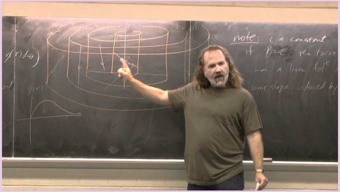
Example (Hopf fields on S^3 and ABC fields on T^3)

- The Hopf fields $u_1 = (-y, x, \xi, -z)$ and $u_2 = (-y, x, -\xi, z)$ are Beltrami fields on S^3.
- The ABC flows
  $(\dot{x}, \dot{y}, \dot{z}) = (A\sin z + C\cos y, B\sin x + A\cos z, C\sin y + B\cos x)$,
  $((x,y,z) \in (\mathbb{R}/2\pi\mathbb{Z})^3)$ are Beltrami.

Recall: The magic mirror

In terms of $\alpha = \iota_X g$ and μ (volume form) the **stationary Euler equations** read

$$\begin{cases} \iota_X d\alpha = -dB \\ d\iota_X \mu = 0 \end{cases}$$



- Etnyre-Ghrist:
  {Rotational non singular Beltrami v.f.} $\rightleftarrows$ {Reeb v.f. reparametrized}
- With Cardona and Peralta-Salas we have extended this picture to manifolds with cylindrical ends to get singular contact structures.
- CMPP: The Beltrami/contact correspondence works in higher dimensions.

Let's prove it!

- The Beltrami equation $\leftrightsquigarrow$ $d\alpha = f\iota_X\mu$. Since $f > 0$ and X does not vanish $\rightsquigarrow$ $\alpha \wedge d\alpha = f\alpha \wedge \iota_X\mu > 0$.
- X satisfies $\iota_X(d\alpha) = \iota_X\iota_X\mu = 0$ so $X \in \ker d\alpha$ $\leftrightsquigarrow$ it is a reparametrization of the Reeb vector field by the function $\alpha(X) = g(X, X)$.