Geometry and Dynamics of Singular Symplectic manifolds

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Modular vector field

Definition

Given a Poisson manifold (M, Π) and a volume form Ω, the modular vector field X_Π^Ω associated to the pair (Π, Ω) is the derivation given by the mapping

$$f \mapsto \frac{L_{X_f}\Omega}{\Omega}$$

1. $L_{X_\Pi^\Omega}(\Pi) = 0$ and $L_{X_\Pi^\Omega}(\Omega) = 0$.
2. $X^{H\Omega} = X^\Omega - X_{log(H)}$. $\rightsquigarrow$ its first cohomology class in Poisson cohomology does not depend on Ω.
3. Examples of unimodular (vanishing modular class) Poisson manifolds: symplectic manifolds.
4. In the case of b-Poisson manifolds in dimension 2, $\{x, y\} = y$ and the modular vector field is $\frac{\partial}{\partial x}$.

Modular vector fields

Modular vector field for Darboux form

The modular vector field of a local b-Poisson manifold with local normal form,

$$\Pi = y_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

with respect to the volume form $\Omega = \sum_i dx_i \wedge dy_i$ is,

$$X^{\Omega} = \frac{\partial}{\partial x_1}.$$

The modular vector field of a b-Poisson manifold is tangent to the critical set Z and is transverse to the symplectic leaves of the induced symplectic foliation on Z.

Induced Poisson structures

Induced Poisson structures

a b-Poisson structure Π on M^{2n} induces a regular corank 1 Poisson structure on Z.

Given a Poisson manifold Z with codimension 1 symplectic foliation $\mathcal{L}$,

1. Does $(Z, \Pi_{\mathcal{L}})$ extend to a b-Poisson structure on a neighbourhood of Z in M?
2. If so to what extent is this structure unique?

Invariants: Dimension 2

Radko classified b-Poisson structures on compact oriented surfaces giving a list of invariants:

- Geometrical: The topology of S and the curves γ_i where Π vanishes.
- Dynamical: The periods of the "modular vector field" along γ_i.
- Measure: The regularized Liouville volume of S, $V_h^\epsilon(\Pi) = \int_{|h|>\epsilon} \omega_\Pi$ for h a function vanishing linearly on the curves $\gamma_1, \ldots, \gamma_n$.

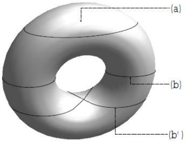

Figure: Two admissible vanishing curves (a) and (b) for Π; the ones in (b') is not admissible.

Singular forms

- A vector field v is a **b-vector field** if $v_p \in T_pZ$ for all $p \in Z$. The **b-tangent bundle** ${}^bTM$ is defined by

$$\Gamma(U, {}^bTM) = \left\{ \begin{array}{c} \text{b-vector fields} \\ \text{on } (U, U \cap Z) \end{array} \right\}$$

- The **b-cotangent bundle** ${}^bT^*M$ is $({}^bTM)^*$. Sections of $\Lambda^p({}^bT^*M)$ are **b-forms**, ${}^b\Omega^p(M)$.The standard differential extends to

$$d : {}^b\Omega^p(M) \to {}^b\Omega^{p+1}(M)$$

- A **b-symplectic form** is a closed, non-degenerate, b-form of degree 2.
- This dual point of view, allows to prove a b-Darboux theorem and semilocal forms via an adaptation of Moser's path method because we can play the same tricks as in the symplectic case.

Space for notes

Example

$M = \mathbb{T}^4$ and $Z = \mathbb{T}^3 \times \{0\}$. Consider on Z the codimension 1 foliation given by $\theta_3 = a\theta_1 + b\theta_2 + k$, with rationally independent $a, b \in \mathbb{R}$. Then take

$$\begin{aligned} h &= \log(\sin\theta_4), \\ \alpha &= \frac{a}{a^2+b^2+1}\, d\theta_1 + \frac{b}{a^2+b^2+1}\, d\theta_2 - \frac{1}{a^2+b^2+1}\, d\theta_3, \\ \omega &= d\theta_1 \wedge d\theta_2 + b\, d\theta_1 \wedge d\theta_3 - a\, d\theta_2 \wedge d\theta_3, \end{aligned}$$

The 2-form $\omega_\Pi = dh \wedge \alpha + \omega$ defines a b-symplectic form in a neighbourhood of Z, which can be extended to M.

Geometrical invariants

Theorem (Mazzeo-Melrose)

The b-cohomology groups of a compact M are computable by

$$^{b}H^{*}(M) \cong H^{*}(M) \oplus H^{*-1}(Z).$$

Indeed,

Theorem (Guillemin-M.-Pires)

$$^{b}H^{*}(M) \cong H^{*}_{\Pi}(M)$$

Geometrical invariants

Corollary (Radko)

Let (M, Z) be a compact connected two-dimensional b-symplectic manifold, where M is of genus g and Z a union of n curves on M. Then the Poisson cohomology of M is given by

$$H^0_\Pi(M) = \mathbb{R}$$
$$H^1_\Pi(M) = \mathbb{R}^{n+2g}$$
$$H^2_\Pi(M) = \mathbb{R}^{n+1}.$$

Obstruction theory via cohomology

Theorem (Guillemin-M-Pires)

For a compact b-symplectic manifold (M, Z) we have $H^1(Z) \neq \{0\}$ and consequently ${}^bH^2(M) \neq \{0\}$.

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Obstruction theory via cohomology

Theorem (Marcut-Osorno, and Oms)

Let (M^{2n}, ω) be an orientable b-symplectic manifold with compact critical hypersurface Z, then there exists an element $c \in H^2(M)$ such that $c^{n-1} \neq 0$.

$\#m\mathbb{C}P^2\#n\overline{\mathbb{C}P^2}$	symplectic	*bona fide* log-symplectic
$m > 1,\ n > 0$	✗	✓
$m > 1,\ n = 0$	✗	✗
$m = 1,\ n > 0$	✓	✓
$m = 1,\ n = 0$	✓	✗
$m = 0,\ n > 0$	✗	✗

Theorem (Cavalcanti)

If a compact oriented manifold M^{2n} , with $n > 1$, admits a b-symplectic structure then there are classes $a, b \in H^2(M, \mathbb{R})$ such that $a^{n-1}b \neq 0$ and $b^2 = 0$.

Theorem

For all $p \in Z$, there exists a Darboux coordinate system $x_1, y_1, \ldots, x_n, y_n$ centered at p such that Z is defined by $x_1 = 0$ and

$$\Pi = x_1 \frac{\partial}{\partial x_1} \wedge \frac{\partial}{\partial y_1} + \sum_{i=2}^{n} \frac{\partial}{\partial x_i} \wedge \frac{\partial}{\partial y_i}$$

Space for notes

b-Moser

Theorem (Guillemin-M-Pires)

Let ω_0 and ω_1 be two b-symplectic forms on (M, Z). If $\omega_0|_Z = \omega_1|_Z$, then there exist neighborhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_{01}$ such that $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.

An alternative statement is the following:

Theorem (Guillemin-M-Pires)

Let ω_0 and ω_1 be two b-symplectic forms on (M, Z). If they induce on Z the same restriction of the Poisson structure and their modular vector fields differ on Z by a Hamiltonian vector field, then there exist neighborhoods U_0, U_1 of Z in M and a diffeomorphism $\gamma : U_0 \to U_1$ such that $\gamma|_Z = id_Z$ and $\gamma^*\omega_1 = \omega_0$.

b-Moser (cont)

Theorem (Guillemin-M-Pires)

Let ω be a b-symplectic form on (M, Z) and $p \in Z$. Then we can find a coordinate chart $(U, x_1, y_1, \ldots, x_n, y_n)$ centered at p such that on U the hypersurface Z is locally defined by $y_1 = 0$ and

$$\omega = dx_1 \wedge \frac{dy_1}{y_1} + \sum_{i=2}^{n} dx_i \wedge dy_i.$$

Theorem (Guillemin-M-Pires)

Suppose that ω_t, for $0 \leq t \leq 1$, is a smooth family of b-symplectic forms on (M, Z) joining ω_0 and ω_1 and such that the b-cohomology class $[\omega_t]$ does not depend on t. Then, there exists a family of diffeomorphisms $\gamma_t : M \to M$, for $0 \leq t \leq 1$ such that γ_t leaves Z invariant and $\gamma_t^* \omega_t = \omega_0$.

Space for notes

Applications: Radko's theorem

Theorem (Radko)

The set of curves, modular periods and regularized Liouville volume completely determines, up to Poisson diffeomorphisms, the Poisson b-structure on a compact oriented surface S.

This can be reproved using the Moser path method. The reformulation in the language of b-forms yields.

Corollary (Classification of b-symplectic surfaces à la Moser, Guillemin-M.-Pires)

Two b-symplectic forms ω_0 and ω_1 on an orientable compact surface are b-symplectomorphic if and only if $[\omega_0] = [\omega_1]$.

Indeed,

$$^{b}H^*(M) \cong H^*_{\Pi}(M)$$

Space for notes

The extension property

Theorem (Guillemin-Miranda-Pires)

Let (M^{2n+1}, Π_0) *be a compact corank-1 regular Poisson manifold with vanishing invariants then there exists an extension of* (M^{2n+1}, Π) *to a b-Poisson manifold* (U, Π)*. The extension is unique , up to isomorphism, among the extensions such that* $[v]$ *is the image of the modular class under the map:*

$$H^1_{Poisson}(U) \longrightarrow H^1_{Poisson}(M^{2n+1})$$

Construction

Given a Poisson vector field v on $(Z, \Pi_{\mathcal{L}})$ with v transverse to $\mathcal{L}$ chose $\alpha_Z \in \Omega^1(Z)$ and $\omega_Z \in \omega^2(Z)$ such that:

1. $\iota_v \alpha_Z = 1.$
2. $\iota_v \omega_Z = 0.$
3. α_Z is a defining one form for the symplectic foliation.
4. ω_Z restricts to the induced symplectic form on each symplectic leaf.

Construction

Now consider $p : U \longrightarrow Z$ a tubular neighbourhood of Z in U and let

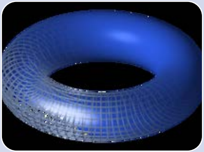
$$\omega = p^*(\alpha_Z) \wedge \frac{df}{f} + p^*(\omega_Z)$$

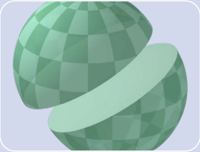
with f a defining function for Z.
Then one can check that this is a semi-local extension to Poisson b-manifold and that the modular vector field of ω restricted to Z is v
Uniqueness relies strongly on a relative Moser's theorem for our b-Poisson manifolds. In order to talk about this, we need to introduce b-cohomology.

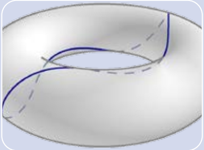
(Singular) symplectic manifolds

- b^m -Symplectic
- Symplectic
- Folded symplectic

Déjà-vu...







Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle coordinates

b-Symplectic manifolds

- Darboux theorem
- Delzant and convexity theorems
- Action-Angle theorem

Folded symplectic manifolds

- Darboux theorem (Martinet)
- Delzant-type theorems (Cannas da Silva-Guillemin-Pires)
- Action-agle theorem (M-Cardona)

Examples and counterexamples

Orientable Surface	CP^2	S^4
• Is symplectic • Is folded symplectic • (orientable or not) is b-symplectic	• Is symplectic • Is folded symplectic • Is not b-symplectic	• Is not symplectic • Is not b-symplectic • Is folded-symplectic

Desingularizing b^m-symplectic structures

Theorem (Guillemin-M.-Weitsman)

Given a b^m-symplectic structure ω on a compact manifold (M^{2n}, Z):

- *If $m = 2k$, there exists a family of symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z and for which the family of bivector fields $(\omega_\epsilon)^{-1}$ converges in the C^{2k-1}-topology to the Poisson structure ω^{-1} as $\epsilon \to 0$.*
- *If $m = 2k+1$, there exists a family of folded symplectic forms ω_ϵ which coincide with the b^m-symplectic form ω outside an ϵ-neighbourhood of Z.*

In particular:

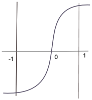
- Any b^{2k}-symplectic manifold admits a symplectic structure.
- Any b^{2k+1}-symplectic manifold admits a folded symplectic structure.
- The converse is not true: S^4 admits a folded symplectic structure but no b-symplectic structure.

Sketch of the proof: $m = 2k$

General principle: If you do not like something, just change it!

$$\omega = \frac{dx}{x^{2k}} \wedge \left(\sum_{i=0}^{2k-1} \alpha_i x^i\right) + \beta \tag{1}$$

- $f \in \mathcal{C}^\infty(\mathbb{R})$ odd function s.t. $f'(x) > 0$ for $x \in [-1, 1]$,



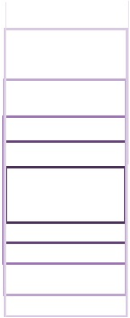
and such that outside $[-1, 1]$,

$$f(x) = \begin{cases} \frac{-1}{(2k-1)x^{2k-1}} - 2 & \text{for} \quad x < -1 \\ \frac{-1}{(2k-1)x^{2k-1}} + 2 & \text{for} \quad x > 1 \end{cases}$$

- Re-scale on ϵ.
- Replace $\frac{dx}{x^{2k}}$ by df_ϵ to obtain $\omega_\epsilon = df_\epsilon \wedge (\sum_{i=0}^{2k-1} \alpha_i x^i) + \beta$ which is symplectic.

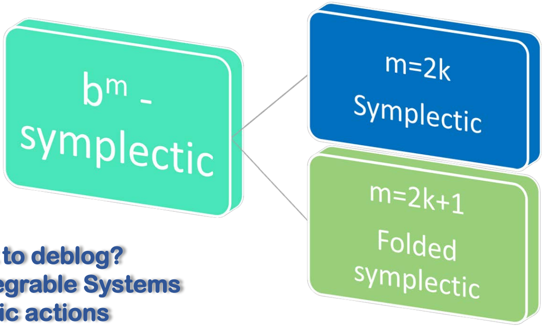
Applications of desingularization

- Convexity for $\mathbb{T}^k$-actions.



- Delzant theorem and Delzant-type theorem for semitoric systems (bolytopes).
- Applications to KAM.
- Periodic orbits of problems in celestial mechanics and applications to stability.

Desingularizing everything...



What to deblog?

- **Integrable Systems**
- **Toric actions**
- **Forms..**

The restricted 3-body problem

- Simplified version of the general 3-body problem. One of the bodies has negligible mass.
- The other two bodies move independently of it following Kepler's laws for the 2-body problem.
- After doing a change to Mc Gehee coordinates ($r = \frac{2}{x^2}, \quad x \in \mathbf{R}^+,$) the symplectic structure becomes a singular object $\omega = -\frac{4}{x^3} dx \wedge dy + d\alpha \wedge dG.$ for $x > 0$

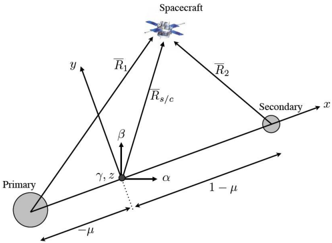


Figure: Circular 3-body problem

Applications of desingularization

- Periodic orbits away from critical set.
- KAM theory.
- Action-angle coordinates with limitations.